Young and Rough Equations in Infinite Dimension

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Joint works with Luca Lorenzi (Parma) and Gianmario Tessitore (Bicocca)

The equation

We consider the following differential equation in a Banach space E

$$dY(t) = AY(t)dt + \sigma(Y(t))dX(t), \quad t \in (0,1], \; Y(0) = y_0 \in E \qquad \text{(RDE)}$$

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$$Y(t) = e^{tA}y_0 + \int_0^t e^{(t-s)A}\sigma(Y(s))dX(s), \qquad t \in [0,1] \tag{MILD}$$

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$\mathbb{P}$-a.s. $\omega \in \Omega$:

- W_ω has not finite 1-variation
- W_ω (and $\sigma(Z_\omega)$) α-Hölder continuous for every $\alpha < \frac{1}{2}$
- W_ω has finite p-variation for every $p > 2$

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Negative result (Lyons '91): $\nexists$ a Banach space $\mathcal{B} \subseteq C([0,1])$ such that

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- $|I_Y(f,g)(s,t) - f(s)(g(t) - g(s))| \le C_{\alpha+\beta}[f]_\alpha[g]_\beta|t-s|^{\alpha+\beta}$
- $I_Y(f,g)(s,t) = \lim_{|P|\to 0} \sum_{[u,v]\in P} f(u)(g(v) - g(u))$, $0 \le s < t \le 1$, P partition of $[s,t]$

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Consequences: let $\mathcal{B} = C^\alpha([0,1])$, $\alpha \in \left(\frac{1}{2}, 1\right)$.

- I extends to a continuous map $I_Y : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$: Young Integral
- What about the case $\alpha \leq \frac{1}{2}$?

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$\alpha \leq \frac{1}{2} \Rightarrow \int_s^t X(r)dX(r)$ defines $\int_s^t F(X(r))dX(r)$

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$\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right] \Rightarrow \mathbb{X}(s,t)$ possible choice for "$\displaystyle\int_s^t (X(r)-X(s))dX(r)$": infinitely many!

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⋆ If $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$ then $R(s,t) = F'(X(s))\mathbb{X}(s,t) + O(|t-s|^{3\alpha})$ and the limit exists:

$$\lim_{|P(s,t)| \to 0} \sum_{[u,v] \in P(s,t)} \left[F(X(u))(X(v) - X(u)) + F'(X(u))\mathbb{X}(u,v)\right]$$

Lyons '98: $\boldsymbol{X} = (X, \mathbb{X})$ α-Hölder rough path, $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$, $F \in C_b^2(\mathbb{R})$. The rough integral

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$$\left|I_R(F(X), \boldsymbol{X})(s,t) - F(X(s))(X(t) - X(s)) - F'(X(s))\mathbb{X}(s,t)\right| \le C(F, \boldsymbol{X})|t-s|^{3\alpha}$$

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Lyons solves a rough differential equation of the form

$$dY(t) = \sigma(Y(t))dX(t), \; Y(0) = y_0 \text{ (i.e., } Y(t) = y_0 + I_R(\sigma(Y), \boldsymbol{X})(0,t))$$

but the concept of solution and the techniques exploited are complicated and involved. Another (but equivalent) definition of solution and proof of the existence and uniqueness of solutions can be obtained by means of *Sewing Lemma* and *Controlled Rough Paths*, introduced in '04 and generalized to the mild situation in '10 by Gubinelli.

Sewing Lemma

Let $N \in \mathbb{N}$, $\alpha, \beta \geq 0$, $[0,1]^N_< := \{(t_1, \ldots, t_N) \in [0,1]^N : 0 \leq t_1 \leq \ldots \leq t_N \leq 1\}$

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⋆ $C^\beta([0,1]^N_<)$: functions $A : [0,1]^N_< \to \mathbb{R}$ such that $\displaystyle\sup_{0 \leq t_1 < \ldots < t_n \leq 1} \frac{|A(t_1, \ldots, t_N)|}{|t_N - t_1|^\beta} < \infty$

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$$(\delta_1 A)(s,t) = A(t) - A(s), \qquad (\delta_2 A)(s,t,u) = A(s,u) - A(s,t) - A(t,u)$$

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It $f \in C^\alpha, g \in C^\beta$, $\alpha + \beta > 1$, then $IA = I_Y(f,g)$

Controlled Rough Paths

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Mild Integration: the Young Case

$A : D(A) \subseteq E \to E$ generates a semigroup $(e^{tA}) \subseteq \mathscr{L}(E)$, $X \in C^\alpha$, $\alpha \in \left(\frac{1}{2}, 1\right)$

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Mild Integration: the Young Case

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$$(\delta_{S,1}B)(s,t) := B(t) - e^{(t-s)A}B(s), \quad (\delta_{S,2}B)(s,t,u) = B(s,u) - B(t,u) - e^{(u-t)A}B(s,t)$$

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If (e^{tA}) is analytic, then $\|e^{rA} - Id\|_{\mathscr{L}(D_A(\eta,\infty),E)} \le C r^{\eta}$, $\eta \in [0,1]$

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$$C^{\alpha}_{-\gamma}((0,1]^N_<; E_\eta) := \left\{ B : (0,1]^N_< \to E_\eta : \sup_{0 \leq t_1 < \ldots < t_N \leq 1} (t_1)^\gamma \frac{\|B(t_1, \ldots, t_N)\|_\eta}{|t_N - t_1|^\alpha} < \infty \right\}$$

Mild Singular Sewing Lemma (A-Lorenzi-Tessitore '24, Neamtu-Hocquet '24): Let $\alpha \in \left(\frac{1}{2}, 1\right)$, $\eta > 1 - \alpha$, $\gamma < \alpha$ and $B \in C^{\alpha}_{-\gamma}((0,1]^2_<; E_\eta)$ s. t. $\delta_{S,2} B \in C^{\alpha+\eta}_{-\gamma}((0,1]^3_<; E)$.

Then, $I_S B$, defined on $(0,1]$, extends to $[0,1]$ with values in E_η and

$$\|\delta_{S,1} I_S B(s,t)\|_{E_{\eta+\varepsilon}} \leq C(\varepsilon, B)|t-s|^{\alpha-\gamma-\varepsilon}, \ \varepsilon \in [0, \alpha - \gamma)$$

Theorem (A-Lorenzi-Tessitore '24): if $X \in C^\alpha$, $\alpha \in \left(\frac{1}{2}, 1\right)$, $\eta > 1 - \alpha$, $y_0 \in E_\theta$ with $\theta \in [0, \eta]$ and $2\eta - \theta < \alpha$, σ smooth.

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$\mathcal{B} = \{f \in \dots : R^f_S \in C^{\alpha+\eta}([0,1]^2_<;E_\eta),\ f' \in C^{\eta}([0,1]^2_<;E_\eta)\}$

THANK YOU FOR YOUR ATTENTION!!