Anomalous Diffusivity and Universality for stationary SPDEs at the Critical dimension

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Physically Relevant (stationary) SPDES: Examples

① Random Growth

$$\partial_t h = \frac{1}{2}\Delta h + \lambda \nabla h \cdot Q \nabla h + \xi \qquad \text{(KPZ)}$$

② Turbulent incompressible Fluids

$$\partial_t v = \frac{1}{2}\Delta v - \lambda v \cdot \nabla v + \nabla p + \nabla^{\perp} \vec{\xi}, \qquad \nabla \cdot v = 0 \qquad \text{(SNS)}$$

③ Driven Diffusive systems

$$\partial_t \eta = \frac{1}{2}\Delta \eta + \lambda (w \cdot \nabla) \eta^2 + \nabla \cdot \vec{\xi} \qquad \text{(SBE)}$$

- $\vec{\xi}$ d-dimensional space-time white noise
- $Q \in \mathbb{R}^d \times \mathbb{R}^d$, $w \in \mathbb{R}^d$
- $\lambda > 0$ coupling constant

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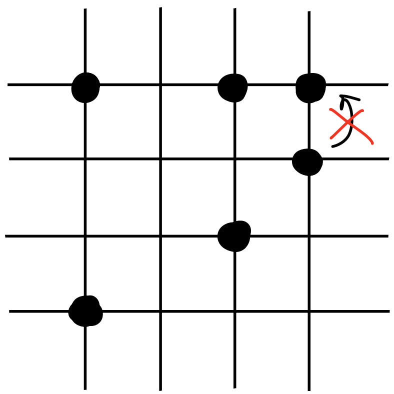
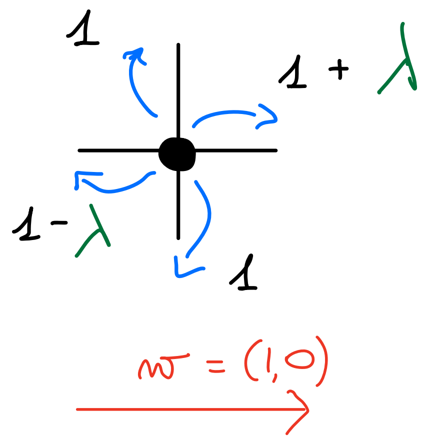
Motivation: Stochastic Burgers Equation

▷ Mesoscopic model for Driven Diffusive Systems
[van Beijren, Kutner, Spohn, PRL '85]

ASEP:





✓ One conserved quantity : # particles

✓ Diffusive : SRW

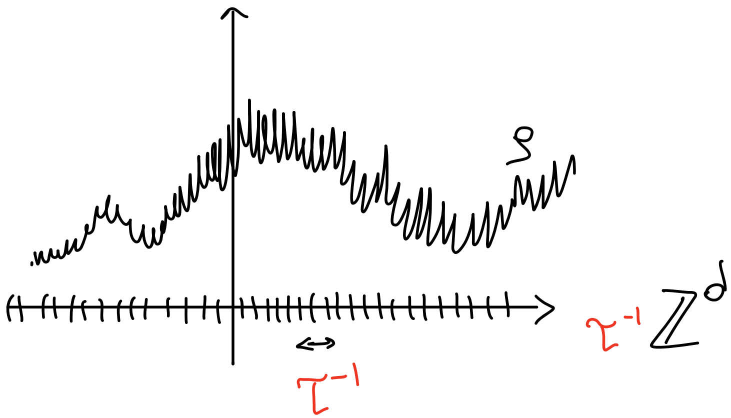
✓ Driven : Drift (w, λ)

✓ Invariant measure : $\bigotimes_{x \in \mathbb{Z}^d} \mathrm{Ber}(\rho)$

Derivation

(SBE: $\partial_t \eta = \frac{1}{2}\Delta\eta + \lambda (w \cdot \nabla)\eta^2 + \nabla \cdot \vec{\xi}$)

— MICRO (ϱ)

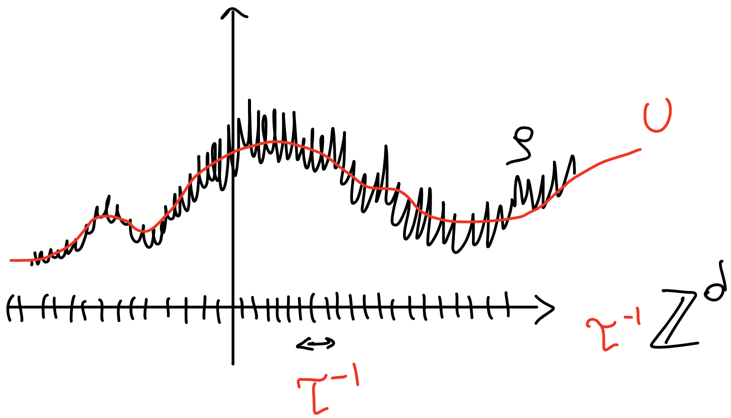


Derivation

$$\left(\text{SBE:} \quad \partial_t \eta = \frac{1}{2}\Delta\eta + \lambda\,(w\cdot\nabla)\,\eta^2 + \nabla\cdot\vec{\xi} \right)$$

— MICRO (ϱ)

— MACRO (U)



$$\partial_t U = (w\cdot\nabla)\,F(U)$$

HYDRODYNAMIC EQUATION

$\rightsquigarrow$ model dependent

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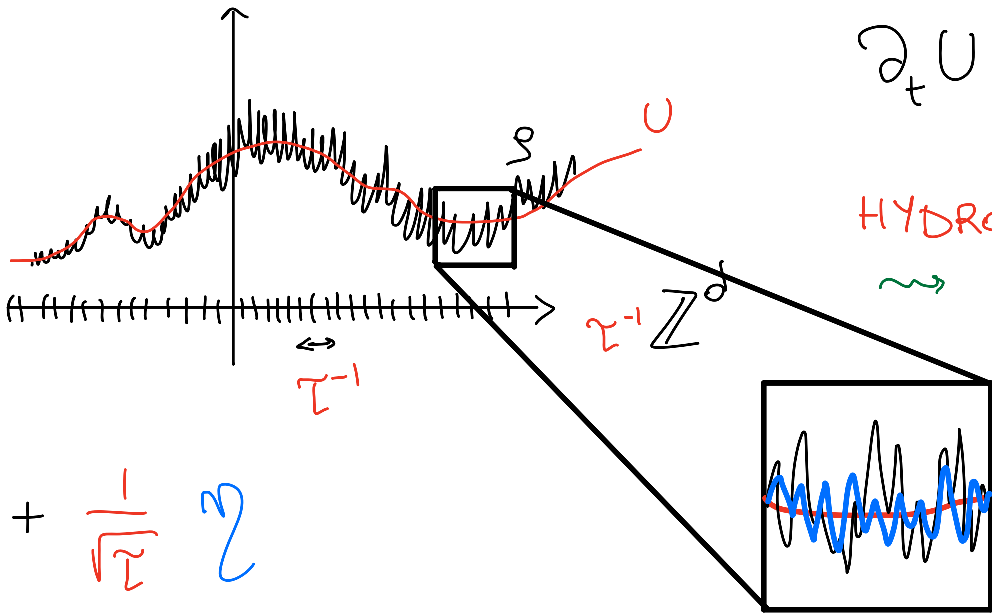
— MESO (η)



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$$\rho \approx U + \frac{1}{\sqrt{\tau}}\,\eta$$

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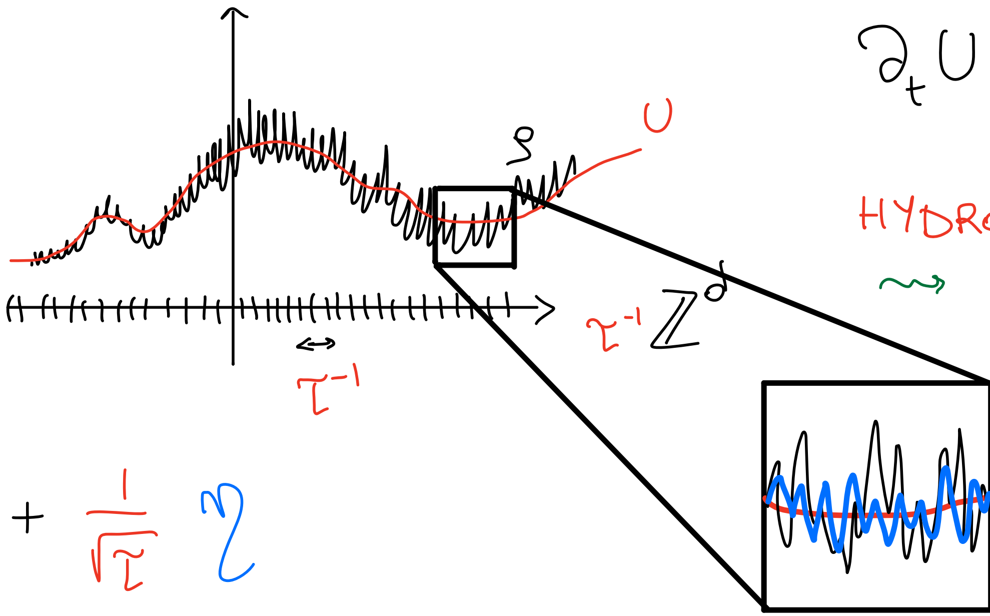

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Noise :
- Gaussian (Fluctuations)
- White in time (Markovian)
- Divergence form (Loc. Cons. Law)

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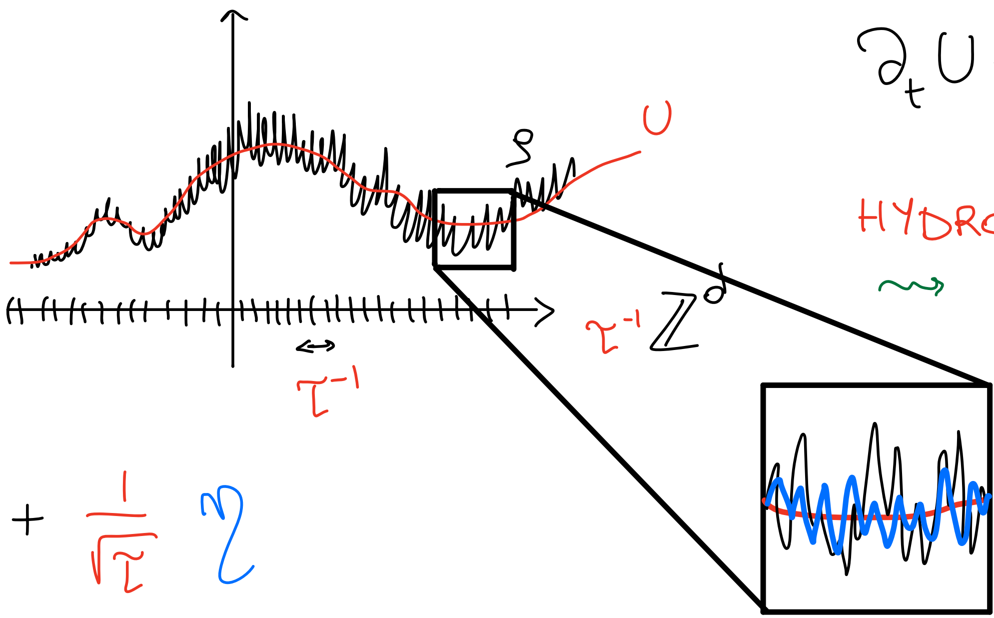

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$$\partial_t \eta = \underbrace{\frac{1}{2}\Delta\eta}_{\text{Diffusion}} + \underbrace{(w\cdot\nabla)\left[F(0) + F'(0)\eta + \frac{F''(0)}{2}\eta^2 + \dots\right]}_{\text{Drift (non-lin)}} + \underbrace{\nabla\cdot\vec{\xi}}_{\text{Noise}}$$

Noise :
- Gaussian (Fluctuations)
- White in time (Markovian)
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Goal & Conjectures

ROTATION INVARIANCE

$w = e_1$

$$\partial_t \eta = \frac{1}{2} \Delta \eta + \lambda\, \partial_1 \eta^2 \quad + \nabla \cdot \vec{\xi} \qquad \text{(SBE)}$$

- How does the non-linearity affect the behaviour of SBE?
- What are its scaling exponents?
- What is its large-scale behaviour?

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Heuristics:
$$\eta_\tau(t,x) := \tau^{d/4}\,\eta(\tau t, \sqrt{\tau}\,x)$$

$$\partial_t \eta = \frac{1}{2}\Delta\eta + \underbrace{\lambda\,\tau^{-\frac{d-2}{4}}}_{\lambda_\tau}\partial_1 \eta^2 + \nabla\cdot\vec{\xi}$$

$d=1$	$\lambda_\tau \uparrow \infty$	RELEVANT	SUB-CRITICAL
$d\geq 3$	$\lambda_\tau \downarrow 0$	IRRELEVANT	SUPER-CRITICAL
$d=2$	$\lambda_\tau = \lambda$	MARGINAL	CRITICAL

Challenge & Regularization

$$\partial_t \eta = \frac{1}{2} \Delta \eta + \lambda \, \partial_1 \eta^2 + \nabla \cdot \vec{\xi} \qquad \text{(SBE)}$$

PROBLEM : SBE is ILL-POSED $\forall d$

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PROBLEM : SBE is ILL-POSED $\forall d$

REGULARISATION : $\varrho \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $\int \varrho(x)\,dx = 1$

- NOISE : $\vec{\xi} \longmapsto \varrho * \vec{\xi} = (\varrho * \xi_1, \ldots \varrho * \xi_d) \in \mathcal{C}^\infty$
- NON-LINEARITY : $\eta^2 \longmapsto \varrho^{*2} * \eta^2$

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Lemma [CMT '24]

$$\partial_t \eta_1 = \frac{1}{2}\Delta \eta_1 + \lambda \partial_1 \varrho^{*2} * \eta_1^2 + \nabla \cdot \vec{\xi}_1$$

✓ Globally well-posed & Markov

✓ Invariant measure : $\mu_1 = \mu * \varrho$ $\qquad \mu$, Gaussian white noise

Remark :

$$\eta_\tau(t,x) = \tau^{d/4}\,\eta(\tau t, \sqrt{\tau}\, x) \rightsquigarrow \varrho_\tau \longrightarrow \delta_0$$

A good observable : DIFFUSIVITY MATRIX

$$\partial_t \eta_1 = \frac{1}{2}\Delta \eta_1 + \lambda\, \partial_1\, \varrho^{*2} * \eta_1^2 + \nabla \cdot \varrho * \vec{\xi}\,, \qquad \eta_1(0,\cdot) = \mu_1$$

$$\boxed{D_{i,j}(t) := \frac{1}{t}\int_{\mathbb{R}^d} x_i x_j\, \mathbb{E}\left[\eta_1(t,x)\,\eta_1(0,0)\right] dx}$$

- Relevance : $\lambda = 0 \implies D(t) = \mathcal{O}(1)$ ($\rightsquigarrow$ Diffusive)

as $t \uparrow \infty$ $\quad D(t) = \begin{cases} \uparrow & (\rightsquigarrow \text{Superdiffusive}) \\ \downarrow & (\rightsquigarrow \text{Subdiffusive}) \end{cases}$

- Scaling Exponents : $\boxed{\eta^{\tau}(t,x) = \tau^{d/4}\, r_{\tau}\, \eta\left(\tau t, \tau^{1/2} R_{\tau}^{-1/2} x\right)}$

Known Results : Out of Criticality

$d=1$
SUB-CRITICAL

✓ SBE well-posed even with $\rho = \delta_0$

⇝ [Bertini-Cancrini '96, Hairer '13, Gubinelli-Perkowski '16, Gonçalves-Jara-Gubinelli-Perkowski '13-'18, Kuppiainen '17, Chandra-Ferdinand '23, ...]

✓ Superdiffusive, $D(t) \sim C\, t^{1/3}$

⇝ [Balasz-Quastel-Seppäläinen '11, Gu-Komorowski '22-'24]

✓ Scaling limit, KPZ Fixed Point NOT GAUSSIAN!

⇝ [Quastel-Sarkar '23, Virag '20]

$d \geq 3$
SUPER-CRITICAL

✓ SBE ILL-posed ⇝ RS / PC / ES / RG DON'T APPLY!

[A solution theory is NOT expected]

✓ Diffusive, $D(t) \sim C$

⇝ SBE [C.-Gubinelli-Toninelli '24] ASEP [Esposito-Marra-Yau '94, Landim-Yau '96]

✓ Scaling limit, SHE GAUSSIAN

⇝ SBE [C.-Gubinelli-Toninelli '24] ASEP [Chang-Landim-Olla '01, Landim-Olla-Varadhan '04]

Known Results : At Criticality

$d=2$

CRITICAL

✓ SBE ILL-posed ⇝ RS / PC / ES / RG DON'T APPLY!

[A solution theory is NOT expected]

✓ Superdiffusive, Conj: $D(t) \sim (\log t)^{2/3}$

⇝ [van Beijeren-Kutner-Spohn '85]

$$(\log\log\log t)^{-3-\delta}(\log t)^{2/3} \lesssim D(t) \lesssim (\log t)^{2/3}(\log\log\log t)^{3+\delta}$$

(In Tauberian sense)

⇝ SBE [De Gasperi-Haunschmid '24]

ASEP [Yau '04]

✓ Scaling Limit, OPEN!!

Main Results: Logaritmic Superdiffusivity

$$(\mathrm{SBE}_1)\quad \partial_t \eta_1 = \frac{1}{2}\Delta \eta_1 + \lambda\, \partial_1\, \varrho^{*2} * \eta_1^2 + \nabla \cdot \varrho * \vec{\xi}, \qquad \eta_1(0,\cdot) = \mu_1(\cdot)$$

Theorem [C.- Moulard-Toninelli '24]

$$\boxed{D(t) = \begin{pmatrix} c_{\mathrm{eff}}\, \lambda^{4/3} (\log t)^{2/3} (1+o(1)) & 0 \\ 0 & 1 \end{pmatrix}}$$

- [vBKS85]'s conjecture is PROVED.
- FIRST EXACT result for CRITICAL off-equilibrium models
- SCALING EXPONENTS

$$R_\tau = \begin{pmatrix} (\log \tau)^{-2/3} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\eta_\tau(t,x) = \tau^{1/2} (\log \tau)^{1/6}\, \eta\left(\tau t, \tau^{1/2} R_\tau^{-1/2} x\right)}$$

Main Results: Superdiffusive CLT for SBE

$$\eta_\tau(t,x) := \tau^{1/2} (\log \tau)^{1/6} \eta_1\left(\tau t, \tau^{1/2} R_\tau^{-1/2} x\right), \quad R_\tau := \begin{pmatrix} (\log\tau)^{-2/3} & 0 \\ 0 & 1 \end{pmatrix}$$

Theorem [C.-Moulard-Toninelli '24]

$$\eta_\tau \xRightarrow{\text{fdd}} \eta^{\text{eff}} \quad \text{as } \tau \uparrow \infty$$

where

$$\partial_t \eta^{\text{eff}} = \frac{1}{2} \nabla \cdot D_{\text{eff}} \nabla \eta^{\text{eff}} + \nabla \cdot \sqrt{D_{\text{eff}}}\, \vec{\xi}, \quad D_{\text{eff}} = \begin{pmatrix} C_{\text{eff}} \lambda^{4/3} & 0 \\ 0 & 1 \end{pmatrix}$$

✓ FIRST CRITICAL SPDE with large scales FULLY deter.

✓ The microscopic diffusivity in e_1 vanishes!

✓ The constant $\lambda^{4/3}$ is dimension independent

Main Results: Superdiffusive CLT for DDS

$$\partial_t \eta_1^F = \frac{1}{2}\Delta \eta_1^F + \partial_1 \varrho^{*2} F(\eta^F) + \nabla\cdot \vec{\xi}_1$$

Theorem [C.-Klose-Moulard, '25+]

Assuming $F \in \mathcal{C}^\infty$ and $\int F'(x) e^{-\frac{x^2}{2}} dx = 0$,

$$\eta_\tau^F \xRightarrow{\text{fdd}} \eta^{\text{eff},F} \quad \text{as } \tau \uparrow \infty$$

where

$$\partial_t \eta^{\text{eff},F} = \frac{1}{2}\nabla\cdot D_{\text{eff}} \nabla \eta^{\text{eff},F} + \nabla\cdot \sqrt{D_{\text{eff}}}\,\vec{\xi}, \quad D_{\text{eff}} = \begin{pmatrix} \mathcal{C}_{\text{eff}} |c_2(F)|^{4/3} & 0 \\ 0 & 1 \end{pmatrix}$$

✓ JUSTIFIES & CORRECTS Physics derivation:

$$\eta_\tau^F \approx \eta_\tau \quad \text{with } \lambda = c_2(F)$$

How does η^{eff} emerge? $\quad (F(x) = x^2)$

- Show: $\quad \eta_\tau \approx \eta^{\text{eff}}_\tau$

$$\partial_t \eta^{\text{eff}}_\tau = \frac{1}{2} \nabla \cdot D^\tau_{\text{eff}} \nabla \eta^{\text{eff}}_\tau + \nabla \cdot \sqrt{D^\tau_{\text{eff}}}\, \vec{\xi}^\tau$$

and D^τ_{eff} non-local operator s.t.

$$\widehat{D^\tau_{\text{eff}}}(k) = \begin{pmatrix} \frac{1}{(\log \tau)^{2/3}} + g^\tau\left(L^\tau\left(\frac{1}{2}|R_\tau k|^2\right)\right) & 0 \\ 0 & 1 \end{pmatrix} \qquad L^\tau(x) = \frac{\lambda^2}{\log \tau} \log\left(1 + \frac{\tau}{x}\right)$$

$$\boxed{\dot{g}^\tau = \frac{1}{\sqrt{\frac{1}{|\log \tau|^{2/3}} + g^\tau}} \qquad g^\tau(0) = 0} \rightsquigarrow g^\tau(x) = \left(\frac{3}{2} x + \frac{1}{\log \tau}\right)^{2/3} - \frac{1}{(\log \tau)^{2/3}}$$

$\longrightarrow$ FLOW EQUATION (RG) for g^τ

Other statistical mechanics systems at Criticality

* 2d SHE & Directed Polymers $\partial_t u^\tau = \frac{1}{2}\Delta u^\tau + \frac{\lambda}{\sqrt{\log \tau}} u^\tau \xi^\tau$

Phase Transition [Caravenna-Sun-Zygouras '14]

- $\lambda < \hat{\lambda} = \sqrt{2\pi}$ Gaussian [Caravenna-Sun-Zyg., Cosco-Zeitouni, ...]
- $\lambda = \hat{\lambda}$ Stochastic Heat Flow [Caravenna-Sun-Zygouras, Tsai]

* 2d KPZ Equation

$$\partial_t h^\tau = \frac{1}{2}\Delta h^\tau + \frac{\lambda}{\sqrt{\log \tau}} |\nabla h^\tau|^2 + \xi^\tau$$

Gaussian [Chatterjee-Dunlap '20, Gu '21, Caravenna-Sun-Zygouras '22]

* 2d Allen-Cahn with critical initial Datum

$$\partial_t \phi_\tau = \frac{1}{2}\phi_\tau + m\,\phi_\tau - \phi_\tau^3, \quad \phi_\tau(0,\cdot) = \frac{\lambda}{\sqrt{\log \tau}}\mu_\tau$$

[Gabriel-Rosati-Zygouras '23]

* Diffusion in the Curl of 2d Gaussian Free Field

$$dX_t = \lambda \,\mathrm{curl}\, \zeta(X_t)\,dt + dB_t$$

Superdiffusive CLT [C.-Haunschmid-Toninelli '21, Chatzigeorgiou-Morfe-Otto-Wang '23, Armstrong-Bou Rabee-Kuusi '24]

* 4D Ising & ϕ^4_4 models Gaussian [Aizenmann-Duminil-Copin '22]

Conclusions

* Identified the Large-scale behaviour of a CRITICAL SPDE beyond pathwise approaches.

* Introduced Novel tools for out-of-equilibrium, irreversible statistical mechanics stationary models.

SPDEs

$$\partial_t h = \frac{1}{2}\Delta h + \lambda\left[(\partial_1 h)^2 - (\partial_2 h)^2\right] + \xi \qquad (\text{AKPZ})$$

⇝ Weak Coupling J/W Erhard,
⇝ Strong Coupling

$$\partial_t v = \frac{1}{2}\Delta v - \lambda\, v\cdot\nabla v + \nabla p - \nabla^\perp \xi, \quad \nabla\cdot v = 0 \qquad (\text{SNS})$$

⇝ Weak Coupling [C.-Kiedrowski]

(SELF-) INTERACTING DIFFUSIONS

$$dX_t = \lambda\, \mathrm{curl}\, \zeta(X_t)\, dt + dB_t \qquad (\Delta\text{CGFF})$$

$$dX_t = -\lambda^2 \int_0^t \nabla V(X_t - X_s)\, ds\, dt + dB_t \qquad (\text{SRBP})$$

⇝ Weak Coupling [C.-Giles '24]

IPS

Asymmetric Simple Exclusion Process
Lattice Gas Models

Grazie!