Pathwise Wong-Zakai convergence and CLT for the stochastic Landau-Lifschitz-Gilbert equation

Emanuela Gussetti

Bielefeld University - CRC 1283

University of Milano-Bicocca, September 2025

The deterministic model

The Landau-Lifshitz-Gibert equation (LLG) describes

- magnetization **M** of a ferromagnetic material (iron, cobalt, nickel),
- **M** is a vector field that describes the distribution of magnetic moment for volume unit.
- magnetic moment: magnetic strength and orientation of a "magnet".

The deterministic model

Let $\mathbf{M} : [0, T] \times \mathbb{T}^d \to \mathbb{S}^2$, for $d = 1, 2, 3$ that satisfies

$$\frac{\partial \mathbf{M}}{\partial t} = \lambda_1 \mathbf{M} \times H - \lambda_2 \mathbf{M} \times (\mathbf{M} \times H)$$

and the saturation condition $|\mathbf{M}(0, x)|_{\mathbb{R}^3} = 1$ holds.

- $\mathbb{S}^2$ denotes the unit sphere of $\mathbb{R}^3$,
- $\lambda_1 \neq 0$, $\lambda_2 > 0$ are constants,
- $\times$ denotes the cross product,
- $H = -\nabla_{\mathbf{M}} \mathcal{E}$, where $\mathcal{E}$ is the total energy.

The deterministic model

Let $\mathbf{M} : [0, T] \times \mathbb{T}^d \to \mathbb{S}^2$, for $d = 1, 2, 3$ that satisfies

$$\frac{\partial \mathbf{M}}{\partial t} = \lambda_1 \mathbf{M} \times \Delta \mathbf{M} - \lambda_2 \mathbf{M} \times (\mathbf{M} \times \Delta \mathbf{M})$$

and the saturation condition $|\mathbf{M}(0, x)|_{\mathbb{R}^3} = 1$ holds.

- $\mathbb{S}^2$ denotes the unit sphere of $\mathbb{R}^3$,
- $\lambda_1 \neq 0$, $\lambda_2 > 0$ are constants,
- $\times$ denotes the cross product,
- $\Delta \mathbf{M}$ comes from the exchange energy.

The stochastic model

Let $\mathbf{M} : [0, T] \times \mathbb{T}^d \to \mathbb{S}^2$, for $d = 1, 2, 3$ that satisfies

$$\frac{\partial \mathbf{M}}{\partial t} = \lambda_1 \mathbf{M} \times (\Delta \mathbf{M} + \dot{W}) - \lambda_2 \mathbf{M} \times (\mathbf{M} \times \Delta \mathbf{M})$$

- $\Delta \mathbf{M}$ comes from the exchange energy,
- $\dot{W}$ represents the thermal fluctuations and it can be also a white noise in time with a smooth spatial dependence.

Intuition behind rough paths

W Brownian motion: every path $1/2-$-Hölder **continuous**

Intuition behind rough paths

W Brownian motion: every path $1/2-$-Hölder **continuous** and a.e. **not differentiable**.

Intuition behind rough paths

W Brownian motion: every path $1/2-$-Hölder **continuous** and a.e. **not differentiable**.
Question: How to define

$$\int_0^t \mathbf{M}_r dW_r?$$

If $\mathbf{M} \in C^\beta([0,T];\mathbb{R})$ and $W \in C^\alpha([0,T];\mathbb{R})$, so that $\alpha + \beta > 1$ then we can employ **Young's integration**.
But we expect $M \in C^{1/2-}$ **as well as** $W \in C^{1/2-}$**!**

Intuition behind rough paths

T. Lyons came up with rough paths (more analytical construction). Start by considering X smooth and $f \in C_b^2(\mathbb{R})$ and consider the 'toy integral'

$$\int_0^t f(X_r)dX_r\,.$$

From Taylor's expansion, where $\delta X_{s,t} := X_t - X_s$

$$\begin{aligned}\int_s^t f(X_r)dX_r &= \int_s^t f(X_s)dX_r + \int_s^t f'(X_s)\delta X_{s,r}dX_r + \int_s^t o(|\delta X_{s,r}|^2)dr \\ &= f(X_s)\delta X_{s,t} + f'(X_s)\int_s^t \delta X_{s,r}dX_r + \int_s^t o(|\delta X_{s,r}|^2)dr \\ &=: f(X_s)X_{s,t} + f'(X_s)\mathbb{X}_{s,t} + X^\natural_{s,t}\end{aligned}$$

The couple $(X, \mathbb{X})$ is a **rough path** lift of X.

Brownian motion

Consider W Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. How to define a rough path from W?

Brownian motion

Consider W Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. How to define a rough path from W?
The first iterated integral is an increment, namely

$$W_{s,t} := W_t - W_s\,.$$

Brownian motion

Consider W Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. How to define a rough path from W?

The first iterated integral is an increment, namely

$$W_{s,t} := W_t - W_s \,.$$

The second iterated integral: **we need probability!**

$$\mathbb{W}^{Str}_{s,t}(\omega) := \int_s^t W_{s,r} \circ_{Str} dW_r(\omega) \qquad \forall \omega \in \Omega_{Str} \subset \Omega$$

so that $\mathbb{P}(\Omega_{Str}) = 1$,

Brownian motion

Consider W Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. How to define a rough path from W?
The first iterated integral is an increment, namely

$$W_{s,t} := W_t - W_s \,.$$

The second iterated integral: **we need probability!**

$$\mathbb{W}^{Str}_{s,t}(\omega) := \int_s^t W_{s,r} \circ_{Str} dW_r(\omega) \qquad \forall \omega \in \Omega_{Str} \subset \Omega$$

so that $\mathbb{P}(\Omega_{Str}) = 1$, **OR**

$$\mathbb{W}^{Ito}_{s,t}(\omega) := \int_s^t W_{s,r} \circ_{Ito} dW_r(\omega) \qquad \forall \omega \in \Omega_{Ito} \subset \Omega$$

$\mathbb{P}(\Omega_{Ito}) = 1$.

Brownian motion

Consider W Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. How to define a rough path from W?

The first iterated integral is an increment, namely

$$W_{s,t} := W_t - W_s \,.$$

The second iterated integral: **we need probability!**

$$\mathbb{W}^{Str}_{s,t}(\omega) := \int_s^t W_{s,r} \circ_{Str} dW_r(\omega) \qquad \forall \omega \in \Omega_{Str} \subset \Omega$$

so that $\mathbb{P}(\Omega_{Str}) = 1$, **OR**

$$\mathbb{W}^{Ito}_{s,t}(\omega) := \int_s^t W_{s,r} \circ_{Ito} dW_r(\omega) \qquad \forall \omega \in \Omega_{Ito} \subset \Omega$$

$\mathbb{P}(\Omega_{Ito}) = 1$.Thus we have two rough path lifts of W defined $\mathbb{P}$-a.s.

Count the regularity...

We choose the Stratonovich lift $(W, \mathbb{W}^{\mathrm{Str}})$. Hence

$$\int_s^t f(W_r) \circ \mathrm{d}W_r = f(W_s) W_{s,t} + f'(W_s) \mathbb{W}^{\mathrm{Str}}_{s,t} + f^{\natural}{}_{s,t} \, ,$$

where

- W is C^{α}-Hölder, with $\alpha \in [1/3, 1/2)$,

Count the regularity...

We choose the Stratonovich lift $(W, \mathbb{W}^{\mathrm{Str}})$. Hence

$$\int_s^t f(W_r) \circ \mathrm{d}W_r = f(W_s) W_{s,t} + f'(W_s) \mathbb{W}^{\mathrm{Str}}_{s,t} + f^{\natural}{}_{s,t} ,$$

where

- W is C^α-Hölder, with $\alpha \in [1/3, 1/2)$,
- $\mathbb{W}^{\mathrm{Str}}$ is $C^{2\alpha}$-Hölder [i.e. $2\alpha \in [2/3, 1)$,
- $f^{\natural}$ is of $C^{3\alpha}$-Hölder [i.e. $3\alpha \in (1, 3/2)$, that is **constants**].

"Spherical" rough path

Consider $w \in C^\alpha([0,T]; L^2(\mathbb{T}^d;\mathbb{R}^3))$ for $\alpha \in [1/3, 1/2)$.
AIM: construct rough path corresponding to $w \times \mathbf{M}$

$$W_{s,t} := -(w_t - w_s) \times \cdot \equiv \begin{pmatrix} 0 & W^3_{s,t} & -W^2_{s,t} \\ -W^3_{s,t} & 0 & W^1_{s,t} \\ W^2_{s,t} & -W^1_{s,t} & 0 \end{pmatrix}$$

$$\mathbb{W}_{s,t} := \iint_{s<r_1<r<t} dw_r \times (dw_{r_1} \times \cdot) = W_{s,t} W_{s,t} + L_{s,t} \, .$$

where $W_{s,t}, L_{s,t}$ is antisymmetric.
Note that the first iterated integral can be reshaped into a matrix.

Definition of Solution to the Stochastic LLG

We say that $\mathbf{M} : [0, T] \times \mathbb{T}^1 \to \mathbb{S}^2$ is a solution to the sLLG if

1. $\mathbf{M}(t, x) \in \mathbb{S}^2$ for a.e. $[0, T] \times \mathbb{T}^1$,
2. $\mathbf{M} \in C(H^1) \cap L^2(H^2)$,
3. if there exists a two index map $\mathbf{M}^\natural$ so that
   - the equality holds in H^{-1}

$$\mathbf{M}_t - \mathbf{M}_s = \int_s^t (\Delta \mathbf{M} + \mathbf{M}|\nabla \mathbf{M}|^2 + \mathbf{M} \times \Delta \mathbf{M}) dr + W_{s,t}\mathbf{M}_s + \mathbb{W}_{s,t}\mathbf{M}_s + \mathbf{M}^\natural_{s,t},$$

   - $\mathbf{M}^\natural \in C^{3\alpha}([0, T]^2; H^{-1})$, with $3\alpha > 1$.

Applications

The main feature of the rough paths approach is the **continuity of the Itô-Lyons map**, i.e. let $\mathbf{M} = \Phi((W, \mathbb{W}))$, then

$$\begin{aligned} \Phi : &\text{Rough Paths} \longrightarrow \text{Banach} \\ &(W, \mathbb{W}) \mapsto \mathbf{M} . \end{aligned}$$

Applications

The main feature of the rough paths approach is the **continuity of the Itô-Lyons map**, i.e. let $\mathbf{M} = \Phi((W, \mathbb{W}))$, then

$$\begin{aligned}\Phi : &\text{Rough Paths} \longrightarrow \text{Banach} \\ &(W, \mathbb{W}) \mapsto \mathbf{M}\,.\end{aligned}$$

The immediate consequences are:

- Wong Zakai convergence result,
- Support theorem,
- Large Deviations principle

Technical issues

- **Product formula** (like Itô) to consider $\mathbf{M}^2$,

Technical issues

- **Product formula** (like Itô) to consider $\mathbf{M}^2$,
- Estimate of the remainder $\mathbf{M}^{2,\natural}$ corresponding to $\mathbf{M}^2$ (through Sewing Lemma),
- Estimate of the remainder $\nabla\mathbf{M}^{2,\natural}$ corresponding to $(\nabla\mathbf{M})^2$
The estimate of the remainder depends on the drifts.

Uniqueness

Uniqueness: only on $\mathbb{T}^1$: also in the deterministic case! We prove uniqueness:

- We consider two possible solutions in $\mathbf{M}_1, \mathbf{M}_2 \in L^4(H^1)$ and let $z := \mathbf{M}_1 - \mathbf{M}_2$. Objective: $\|z\|_{L^4(L^2)} = 0$.
- Consider the equation for $z \otimes z$: the noise term vanishes!
- Same proof as in the deterministic case: here we use an **interpolation inequality** that **holds only in 1D**.

Nonuniqueness in more dimensions in the deterministic case.

Existence through compactness

Consider and denote by

- $(G^n, \mathbb{G}^n)$ approximation of $(G, \mathbb{G})$

Existence through compactness

Consider and denote by

- $(G^n, \mathbb{G}^n)$ approximation of $(G, \mathbb{G})$
- $\mathbf{M^n}$ the solution to the sLLG driven by $(G^n, \mathbb{G}^n)$ (but more regular)

Existence through compactness

Consider and denote by

- $(G^n, \mathbb{G}^n)$ approximation of $(G, \mathbb{G})$
- $\mathbf{M^n}$ the solution to the sLLG driven by $(G^n, \mathbb{G}^n)$ (but more regular)
- $(\mathbf{M^n})_n$ is uniformly bounded in $L^\infty(H^1) \cap L^2(H^2) \cap \mathcal{V}^p(L^2)$ for $\mathbf{M}^0 \in H^1$

Existence through compactness

Consider and denote by

- $(G^n, \mathbb{G}^n)$ approximation of $(G, \mathbb{G})$
- $\mathbf{M^n}$ the solution to the sLLG driven by $(G^n, \mathbb{G}^n)$ (but more regular)
- $(\mathbf{M^n})_n$ is uniformly bounded in $L^\infty(H^1) \cap L^2(H^2) \cap \mathcal{V}^p(L^2)$ for $\mathbf{M}^0 \in H^1$
  in $L^\infty(H^k) \cap L^2(H^{K+1}) \cap \mathcal{V}^p(H^{k-1})$ for $\mathbf{M}^0 \in H^k$

Existence through compactness

Consider and denote by

- $(G^n, \mathbb{G}^n)$ approximation of $(G, \mathbb{G})$
- $\mathbf{M^n}$ the solution to the sLLG driven by $(G^n, \mathbb{G}^n)$ (but more regular)
- $(\mathbf{M^n})_n$ is uniformly bounded in $L^\infty(H^1) \cap L^2(H^2) \cap \mathcal{V}^p(L^2)$ for $\mathbf{M}^0 \in H^1$
  in $L^\infty(H^k) \cap L^2(H^{K+1}) \cap \mathcal{V}^p(H^{k-1})$ for $\mathbf{M}^0 \in H^k$
- Aubin-Lions lemma implies **strong convergence** of a subsequence in $L^2(H^1)$

Existence through compactness

Consider and denote by

- $(G^n, \mathbb{G}^n)$ approximation of $(G, \mathbb{G})$
- $\mathbf{M^n}$ the solution to the sLLG driven by $(G^n, \mathbb{G}^n)$ (but more regular)
- $(\mathbf{M^n})_n$ is uniformly bounded in $L^\infty(H^1) \cap L^2(H^2) \cap \mathcal{V}^p(L^2)$ for $\mathbf{M}^0 \in H^1$
  in $L^\infty(H^k) \cap L^2(H^{K+1}) \cap \mathcal{V}^p(H^{k-1})$ for $\mathbf{M}^0 \in H^k$
- Aubin-Lions lemma implies **strong convergence** of a subsequence in $L^2(H^1)$
  in $L^2(H^{k+1})$
- identification of the limit in the non linearities.

Wong Zakai convergence

What is the Wong Zakai convergence?

$$(G^n, \mathbb{G}^n) \to^{p-var} (G, \mathbb{G}) \implies \mathbf{M}^n \to^{\mathbf{X}} \mathbf{M}$$

in some space $\mathbf{X}$.

Wong Zakai convergence

What is the Wong Zakai convergence?

$$(G^n, \mathbb{G}^n) \to^{p-var} (G, \mathbb{G}) \implies \mathbf{M}^n \to^{\mathbf{X}} \mathbf{M}$$

in some space $\mathbf{X}$.

Compactness implies the Wong Zakai convergence:

$$\mathbf{M}^{\mathbf{n}} \to \mathbf{M} \quad in\, \mathbf{X} = L^2(H^1)\,.$$

Wong Zakai convergence

What is the Wong Zakai convergence?

$$(G^n, \mathbb{G}^n) \to^{p-var} (G, \mathbb{G}) \implies \mathbf{M}^n \to^{\mathbf{X}} \mathbf{M}$$

in some space **X**.
Compactness implies the Wong Zakai convergence:

$$\mathbf{M^n} \to \mathbf{M} \quad in\ \mathbf{X} = L^2(H^1)\,.$$

Aim: better convergence of $(\mathbf{M^n})_n$ to the solution M in 1D.

Wong Zakai convergence

What is the Wong Zakai convergence?

$$(G^n, \mathbb{G}^n) \to^{p-var} (G, \mathbb{G}) \implies \mathbf{M}^n \to^{\mathbf{X}} \mathbf{M}$$

in some space $\mathbf{X}$.
Compactness implies the Wong Zakai convergence:

$$\mathbf{M^n} \to \mathbf{M} \quad in\ \mathbf{X} = L^2(H^1)\,.$$

Aim: better convergence of $(\mathbf{M^n})_n$ to the solution M in 1D.
But we want to achieve a stronger **Wong Zakai convergence:**

$$\mathbf{M^n} \to \mathbf{M} \quad in\ \mathbf{X} = L^\infty(H^1) \cap L^2(H^2)\,.$$

Wong Zakai convergence

Idea:

- $(\mathbf{M}^n)_n$ sequence of solutions to the sLLG driven by $(G^n, \mathbb{G}^n)$,
- $\mathbf{M}$ be the limit solution of $(\mathbf{M}^n)_n$ to the sLLG driven by $(G, \mathbb{G})$,

Consider $\nabla(\mathbf{M}^n - \mathbf{M})^2$, then we need to estimate $\nabla(\mathbf{M}^n - \mathbf{M})^{2,\natural}$.

$$\lim_{n \to +\infty} \sup_{t \in [0,T]} \|\mathbf{M}^n - \mathbf{M}\|_{H^1}^2 + \int_0^T \|\Delta(\mathbf{M}^n - \mathbf{M})\|_{L^2}^2 = 0$$

and the Itô-Lyons map is continuous.

$$(G, \mathbb{G}) \longrightarrow \mathbf{M}$$

Technical lemma

Let $z := \mathbf{M^n} - \mathbf{M}$ Let $(\partial_x z)^{\natural,2,i,j} : [0,T]^2 \to L^1(\mathbb{T}^1)$ be a two index map

$$(\partial_x z)^{\natural,2,i,j}_{s,t} := \delta(\partial_x z)^i(\partial_x z)^j_{s,t} - \mathcal{D}^{i,j} - \mathcal{N}$$

where

- $\mathcal{D}^{i,j}$ is the drift,
- $\mathcal{N} = \mathcal{N}((G,\mathbb{G}),(\partial_x G,\partial_x\mathbb{G}),(G^n,\mathbb{G}^n),(\partial_x G^n,\partial_x\mathbb{G}^n))$.

After applying the sewing lemma,

$$
\begin{aligned}
\|(\partial_x z)^{\natural,2}_{s,t}\|_{L^1} \leq\ & C\omega^{2/p}\big(\omega^{1/p}_{(G^n-G)} + \omega^{1/p}_{(\partial_x(G^n-G))} + \omega^{1/p}_{(\mathbb{G}^n-\mathbb{G})} + \omega^{1/p}_{\partial_x(\mathbb{G}^n-\mathbb{G})}\big) \\
& + C\big((\omega^{1/p}+\omega^{2/p})(\mathcal{D}(\partial_x z, z) + \mathcal{D}(z,\partial_x z) + \mathcal{D}(z,z)\big) \\
& + C\omega^{3/p}(\|\partial_x z\|_{L^\infty L^2} + \|z\|_{L^\infty L^2})\big).
\end{aligned}
$$

In particular

$$
\begin{aligned}
\|z\|^2_{L^\infty(H^1)\cap L^2(H^2)} \lesssim\ & \exp(C(\mathbf{M}^0,\mathbf{M}^{n,0})t)\|z^0\|^2_{H^1} \\
& + C(\mathbf{M}^0,\mathbf{M}^{n,0})[\omega^{1/p}_{G^n-G} + \omega^{2/p}_{\mathbb{G}^n-\mathbb{G}}]\,.
\end{aligned}
$$

We actually proved:

1. Lipschitz continuity of Itô-Lyons map,
2. Large deviation principle and support theorem (in the higher order regularity).

We actually proved:

1. Lipschitz continuity of Itô-Lyons map,
2. Large deviation principle and support theorem (in the higher order regularity).

Other applications are:

- Feller Property associated to the semigroup.

We actually proved:

1. Lipschitz continuity of Itô-Lyons map,
2. Large deviation principle and support theorem (in the higher order regularity).

Other applications are:

- Feller Property associated to the semigroup.
- Pathwise Central Limit theorem and a moderate deviation principle.

CLT and SPDEs

Let $\epsilon > 0$ and let $u^\epsilon : [0, T] \to \mathbb{R}$ be the unique solution to

$$\delta u^\epsilon_{s,t} = \int_s^t b(u^\epsilon_r) dr + \sqrt{\epsilon} \int_s^t u^\epsilon_r \circ dW_r \,,$$

with initial condition $u^0 \in \mathbb{R}$.

CLT and SPDEs

Let $\epsilon > 0$ and let $u^\epsilon : [0, T] \to \mathbb{R}$ be the unique solution to

$$\delta u^\epsilon_{s,t} = \int_s^t b(u^\epsilon_r) dr + \sqrt{\epsilon} \int_s^t u^\epsilon_r \circ dW_r \,,$$

with initial condition $u^0 \in \mathbb{R}$.
Let $\bar{u}$ be the unique solution to the associated deterministic equation

$$\delta \bar{u}_{s,t} = \int_s^t b(\bar{u}_r) dr \,,$$

with $\bar{u}_0 = u^0$.

CLT and SPDEs

We study the **limit for** $\epsilon \to 0$ of

$$X^\epsilon := \frac{u^\epsilon - \bar{u}}{\sqrt{\epsilon}}\,.$$

CLT and SPDEs

We study the **limit for** $\epsilon \to 0$ of

$$X^\epsilon := \frac{u^\epsilon - \bar{u}}{\sqrt{\epsilon}} .$$

From the known CLT results for SDEs with Itô integral, we expect $\lim_{\epsilon \to 0} X^\epsilon = X$ in $L^2(\Omega)$, where X is the unique solution to

$$\delta X_{s,t} = \int_s^t b'(\bar{u}_r) X_r dr + \int_s^t \bar{u}_r dW_r ,$$

CLT and SPDEs

We study the **limit for** $\epsilon \to 0$ of

$$X^\epsilon := \frac{u^\epsilon - \bar{u}}{\sqrt{\epsilon}} .$$

From the known CLT results for SDEs with Itô integral, we expect $\lim_{\epsilon\to 0} X^\epsilon = X$ in $L^2(\Omega)$, where X is the unique solution to

$$\delta X_{s,t} = \int_s^t b'(\bar{u}_r) X_r dr + \int_s^t \bar{u}_r dW_r ,$$

which is a **linear equation with additive noise**.

CLT and SPDEs

Observations:

- The CLT linearises the equation

CLT and SPDEs

Observations:

- The CLT linearises the equation
- The convergence occurs in $L^2(\Omega)$: already better than the mere convergence in law.

CLT and SPDEs

Observations:

- The CLT linearises the equation
- The convergence occurs in $L^2(\Omega)$: already better than the mere convergence in law.

Questions:

- **Why** the CLT linearises?
- Can we get a better convergence?

Differentiating the Itô-Lyons map (Tudor and Quian)

We have established that **there exists a continuous map**

$$\Phi : \mathcal{RP}^p(\mathbb{R}) \to C([0,T];\mathbb{R})\,.$$

Let us differentiate it! Recall...

$$X^\epsilon = \frac{\Phi(\tau_{\sqrt{\epsilon}}\mathbf{W}) - \Phi(\mathbf{0})}{\sqrt{\epsilon}}\,.$$

Prolems:

- $\mathcal{RP}^p(\mathbb{R})$ not a vectorial space: define the **increment**.

Differentiating the Itô-Lyons map (Tudor and Quian)

We have established that **there exists a continuous map**

$$\Phi : \mathcal{RP}^p(\mathbb{R}) \to C([0,T];\mathbb{R})\,.$$

Let us differentiate it! Recall...

$$X^\epsilon = \frac{\Phi(\tau_{\sqrt{\epsilon}}\mathbf{W}) - \Phi(\mathbf{0})}{\sqrt{\epsilon}}\,.$$

Prolems:

- $\mathcal{RP}^p(\mathbb{R})$ not a vectorial space: define the **increment**.
- $\mathcal{RP}^p(\mathbb{R})$ not Banach....**notion of derivative**.

Back to our example

Modulo some rules of calculus (chain rule and differentiation of the stochastic integral), we can differentiate $u = \Phi(\mathbf{G})$ in $\mathbf{H}$ in the direction $\mathbf{W}$. Recall

$$\delta\Phi(\mathbf{G})_{s,t} = \int_s^t b(\Phi(\mathbf{G})_r)dr + G_{s,t}\Phi(\mathbf{G})_s + \mathbb{G}_{s,t}\Phi(\mathbf{G}) + \Phi^{\natural}_{s,t}\,.$$

Denote by $Y := D\Phi[\mathbf{H}](\mathbf{W})$, then Y needs to be a solution to

$$\begin{aligned}\delta Y_{s,t} = &\int_s^t b'(\Phi(\mathbf{H})_r)Y_r dr + W_{s,t}\Phi(\mathbf{H})_s + H_{s,t}Y_s \\ &+ ([HW] + [WH])_{s,t}\Phi(H)_s + \mathbf{H}_{s,t}Y_s + Y^{\natural}_{s,t}\,.\end{aligned}$$

This is **consistent with the usual theory for CLT**, indeed

$$\delta Y_{s,t} = \int_s^t b'(\Phi(\mathbf{H})_r) Y_r dr + W_{s,t}\Phi(\mathbf{H})_s + H_{s,t} Y_s + ([HW]+[WH])_{s,t}\Phi(H)_s + \mathbf{H}_{s,t} Y_s + Y^{\natural}_{s,t}\,.$$

evaluated in $H = 0$ becomes

$$\delta Y_{s,t} = \int_s^t b'(\Phi(\mathbf{0})_r) Y_r dr + W_{s,t}\Phi(\mathbf{0})_s + Y^{\natural}_{s,t}\,.$$

Comments on the path-wise CLT

- **The CLT linearises because it is a derivative!**

Comments on the path-wise CLT

- **The CLT linearises because it is a derivative!**
- No moments on the initial condition are needed,

Comments on the path-wise CLT

- **The CLT linearises because it is a derivative!**
- No moments on the initial condition are needed,
- Path-wise convergence implies convergence in law, provided Y is the unique solution.

Comments on the path-wise CLT

- **The CLT linearises because it is a derivative!**
- No moments on the initial condition are needed,
- Path-wise convergence implies convergence in law, provided Y is the unique solution.
- The limit is Gaussian, if the RP is Gaussian.
- One gets a path-wise speed of convergence: it is possible to derive a moderate deviation principle.

On increments and possible directions

Given two RP $\mathbf{X} \equiv (X, \mathbb{X}), \mathbf{Y} \equiv (Y, \mathbb{Y}) \in \mathcal{RP}^p(\mathbb{R})$, the direct sum of the components

$$\mathbf{X} + \mathbf{Y} := (X + Y, \mathbb{X} + \mathbb{Y}) \,. \tag{1}$$

is not a rough paths. But a if we define

$$\{\mathbf{X} + \mathbf{Y}\} := (X + Y, \mathbb{X} + \mathbb{Y} + [XY] + [YX]) \,, \tag{2}$$

this is in the space! How to build the mixed integrals $[XY], [YX]$?

On increments and possible directions

Denote the Young integral of X against Y by $\mathcal{I}^Y(X,Y)$ and denote by

$$[XY] := \mathcal{I}^Y(X,Y)\,, \quad [YX] := \mathcal{I}^Y(Y,X)\,,$$

and assume that for a.e. $\omega \in \Omega$, the couple $\mathbf{Z}(\omega) = (Z(\omega), \mathbb{Z}(\omega))$ defined by

$$Z_{s,t}(\omega) \equiv (X_{s,t}(\omega), Y_{s,t}(\omega)), \quad \mathbb{Z}_{s,t}(\omega) \equiv \begin{pmatrix} \mathbb{X}_{s,t}(\omega) & [XY]_{s,t}(\omega) \\ [YX]_{s,t}(\omega) & \mathbb{Y}_{s,t}(\omega) \end{pmatrix}\,,$$

It is possible to go beyond the complementary Young regularity paths!

On increments and possible directions

One would like to have:

1. **Measurability** of the joint lift:

On increments and possible directions

One would like to have:

1. **Measurability** of the joint lift:possible! E.g. see joint lifts on semimartingales.

On increments and possible directions

One would like to have:

1. **Measurability** of the joint lift:possible! E.g. see joint lifts on semimartingales.
2. **Some continuity properties** of $\mathbf{Z}$ with respect to X, Y.

Thank you for your attention!!

Bibliography:

- A pathwise stochastic Landau-Lifshitz-Gilbert equation with application to large deviations.
  E. G., A. Hocquet.
  Journal of Functional Analysis 285 (9), 110094, 2023.
- Pathwise central limit theorem and moderate deviations via rough paths for SPDEs with multiplicative noise.
  E. G.
  Electronic Journal of Probability 30, 1-51, 2025.