Rough Uhlenbeck Compactness and Additive Functions

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Motivation

- Yang-Mills measure on $M = [0,1]^2$ *formally* is the Gibbs-type measure

$$\mu(\mathrm{d}A) \text{“} = \text{”} \frac{1}{Z} \exp\left(-\|F^A\|_{L^2}^2\right) \mathrm{d}A$$

- $A = (A_1, A_2) : M \to \mathfrak{g} \times \mathfrak{g}$ is connection 1-form
- F^A is curvature 2-form $\to$ additional *geometric* aspects
- **Goal:** Study properties of the measure via rough Uhlenbeck compactness

Notation

- $G = U(N)$ space of $N \times N$ unitary matrices
- $\mathfrak{g} = \mathfrak{u}(N)$ space of $N \times N$ skew-Hermitian matrices
- For any $g \in G$ and $Y \in \mathfrak{g}$ we define $\mathrm{Ad}_g Y := gYg^{-1}$
- $A \in \Omega^1(M; \mathfrak{g})$ is *connection* 1-*form*, i.e. $A = (A_1, A_2) : M \to \mathfrak{g} \times \mathfrak{g}$
- *Curvature* of A denoted by F^A is defined by

$$F^A := \partial_1 A_2 - \partial_2 A_1 + [A_1, A_2]$$

- For 1-form A, we recall

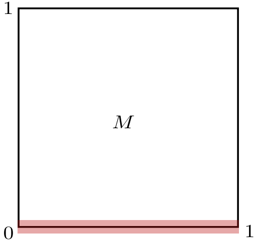
$$\mathrm{div}(A) = \partial_1 A_1 + \partial_2 A_2$$

Gauge transformations

- *Gauge transformation* $g \in C^\infty(M;G)$ acts on $A \in \Omega^1(M;\mathfrak{g})$ by $A^g = (A_1^g, A_2^g)$ where
$$A_i^g = gA_i g^{-1} - \partial_i g g^{-1}$$
- Gauge transformation defines equivalence relation:
$$A \sim \bar{A} \iff \exists\, g \in C^\infty(M;G) \text{ s.t. } A^g = \bar{A}$$
- *gauge orbit space* $\mathcal{A} := \Omega^1(M;\mathfrak{g})/C^\infty(M;G)$ is the relevant physical space

Axial gauge

- Each equivalence class $[A]_\sim$ has an axial gauge representative
- 1-form A is in *axial gauge* if $A_2 = 0$ everywhere and $A_1 = 0$ on horizontal axis

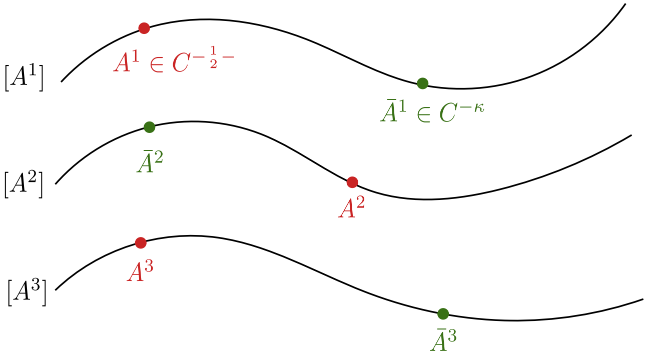


- Yang-Mills measure under axial gauge is then 2D white noise:

$$\mu(\mathrm{d}A) \text{``}=\text{''} \frac{1}{Z} \exp\left(-\|\partial_2 A_1\|_{L^2}^2\right) \mathrm{d}A$$

Main question

- **Fact:** $\partial_2 A_1 \in C^{-1^-}$ a.s. (2D W.N.); *stochastic argument* $\implies$ $A_1 \in C^{-\frac{1}{2}^-}$
- **Main question:** Can we study a representative with better regularity than the axial gauge?



Main result: rough Uhlenbeck compactness

Theorem (Chevyrev-Klose-M., '23^+)

For any $\kappa > 0$, there exists $\delta > 0$ and $C > 0$ with the following property: if a sequence $(A_n)_{n\geq 0} \subset \Omega^1(M;\mathfrak{g})$ of 1-forms in ***axial gauge*** *satisfies*

$$\|A_n\| \leq \delta, \tag{1}$$

then there exists a sequence of gauge transformations $(g_n)_{n\geq 0} \subset C^\infty(M;G)$ such that

$$\|A_n^{g_n}\|_{C^{-\kappa}} \leq C\|A_n\|.$$

In particular, $A_n^{g_n}$ has a convergent subsequence in $C^{-\kappa-\varepsilon}$ for any $\varepsilon > 0$. Finally, the map $A \mapsto A^g$ is continuous.

- **Related work:** Uhlenbeck (1982) and Chevyrev (2019)

Corollary of rough Uhlenbeck compactness

Corollary (Chevyrev-Klose-M., '23^+)

YM measure on $M^\varepsilon = [0,\varepsilon]^2$ for $\varepsilon > 0$ sufficiently small, can be gauge fixed to C^{0^-} distribution and approximated by smooth functions.

- The smallness of domain M^ε is needed to ensure $\|A_n\| \leq \delta$

Proof technique

Step 1

- Find g such that $\operatorname{div} A^g = 0$
- $\implies$ *Elliptic* SPDE
- Solve with regularity structures
- Norm of solution is bounded by "norm" of model

$\longrightarrow$

Step 2

- Define rough additive functions
- Bound model "norm" by "norm" for additive functions
- Advantage: easier to control

Singular Elliptic SPDE

- Recall $A_i^g = gA_ig^{-1} - \partial_i g g^{-1} = \mathrm{Ad}_g A_i - \partial_i g g^{-1}$
- $\mathrm{div} A^g = 0$ implies $A^g = \nabla^\perp w$ where $\nabla^\perp = (\partial_2, -\partial_1)$
- We set $H = \mathrm{Ad}_g$
- We solve for (w, H) in the system of formal equations (with suitable B.C.)

$$\begin{cases} \Delta w & = \partial_2(HA_1) + [\partial_1 w, \partial_2 w] + [\partial_1 w, HA_1] \\ \Delta He_i & = -\partial_1[He_i, HA_1] + \partial_1[He_i, \partial_2 w] - \partial_2[He_i, \partial_1 w], \quad i = 1, ..., \dim \mathfrak{g} \end{cases}$$

- Use regularity structures to obtain solution under smallness assumption akin to Gerencser–Hairer '19
- Identify H: show that $\exists\, g \in C^\infty(M; G)$ s.t., actually, $H = \mathrm{Ad}_g$
- One obtains

$$\|A^g\|_{C^{-\kappa}} \lesssim \|w\|_{C^{1-\kappa}} \lesssim \|Z\|_{\text{model}}$$

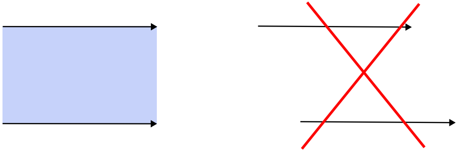
Rough additive functions

- For a line ℓ we define

$$A(\ell) = \int_\ell A$$

- Fix $\alpha \in (\frac{1}{3}, \frac{1}{2})$
- For A in axial gauge, we define

$$|A|_\alpha := \sup_{\ell \sim \bar{\ell}} \frac{|A(\ell) - A(\bar{\ell})|}{\text{Area}(\ell, \bar{\ell})^\alpha}$$



Rough additive functions

- We define ℓ_t for $t \in [0,1]$ to be the t-th portion of the line
- Iterated integral of A denoted by $\mathbb{A}$

$$\mathbb{A}(\ell, \bar{\ell}) = \int_0^1 A(\ell_t) \otimes \mathrm{d}A(\bar{\ell}_t).$$

- We define

$$\|\mathbb{A}\|_\alpha := \sup_{\ell \sim \bar{\ell} \sim \ell'} \frac{|\mathbb{A}(\ell, \ell') - \mathbb{A}(\bar{\ell}, \ell')|}{|\ell'|^\alpha \operatorname{Area}(\ell, \bar{\ell})^\alpha}$$

- The pair $\mathbf{A} = (A, \mathbb{A})$ is *rough additive function* with "norm"

$$\|\mathbf{A}\|_\alpha := |A|_\alpha + \|\mathbb{A}\|_\alpha,$$

and metric $\left\|\mathbf{A}, \bar{\mathbf{A}}\right\|_\alpha := |A - \bar{A}|_\alpha + \|\mathbb{A} - \bar{\mathbb{A}}\|_\alpha$

Next steps

- What about $M = [0,1]^2$? Gluing/patching solutions in small boxes
- Arbitrary manifolds M, e.g. for starters $M = \mathbb{T}^2$ ($\to$ no axial gauge)
- Non-trivial principal G-bundle?
- **Open problem.** Three dimensions

Thank you for your attention!