The critical 2d Stochastic Heat Flow

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Statistical Mechanics & Stochastic PDEs

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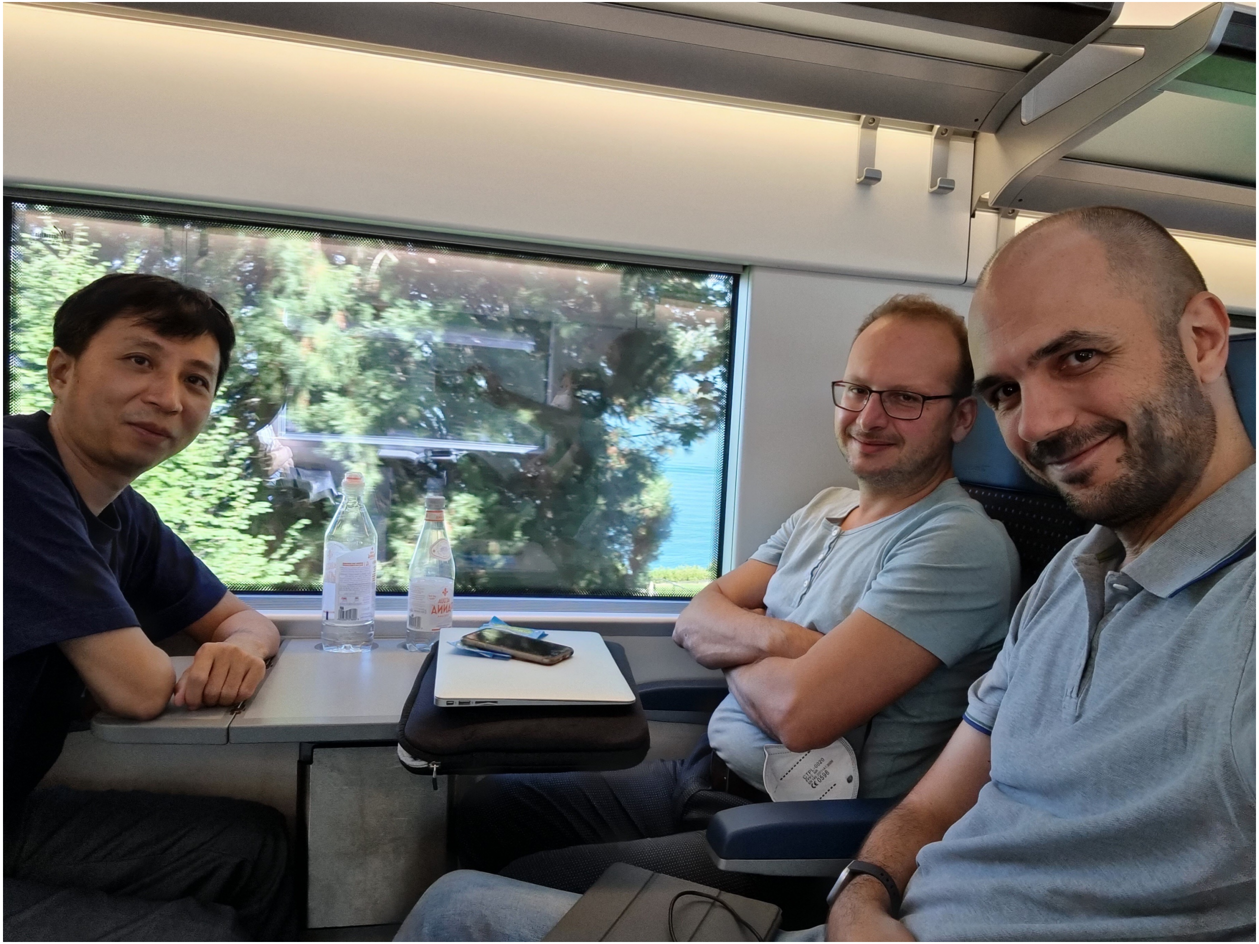


Plan of the course

Lecture 1. Sub-critical regime [Caravenna]

Basic tools • Gaussian behavior

Lecture 2. Critical regime I [Sun]

Coarse-graining • Universality

Lecture 3. Critical regime II [Zygouras]

Moment estimates • Key features

References

[CSZ 23] F. Caravenna, R. Sun, N. Zygouras
THE CRITICAL 2D STOCHASTIC HEAT FLOW
Invent. Math. 233 (2023), 325-460

[CSZ 23+] Ann. Probab. (to appear)
[CSZ 20] Ann. Probab. 48 (2020)
[CSZ 19b] Commun. Math. Phys. 372 (2019)
[CSZ 19a] Electron. J. Probab. 24 (2019)
[CSZ 17b] Ann. Appl. Probab. 27 (2017)
[CSZ 17a] J. Eur. Math. Soc. 19 (2017)

Introduction and Motivation

THE STOCHASTIC HEAT EQUATION

(MULTIPLICATIVE!)

$$(\text{SHE}) \quad \begin{cases} \partial_t u(t,x) = \Delta u(t,x) + \beta\, \xi(t,x)\, u(t,x) & t>0,\ x\in\mathbb{R}^d \\ u(0,x) \equiv 1 \end{cases}$$

- $\beta > 0$ coupling constant
- $\xi(t,x)$ "space-time white noise" (δ-correlated Gaussian)

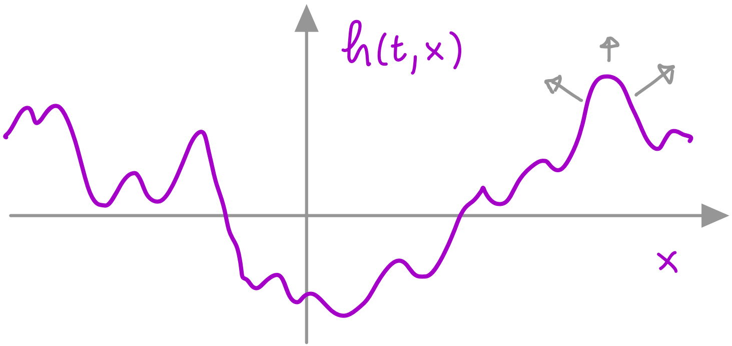
GOAL: Construct the solution $u(t,x)$ for $d=2$

THE KARDAR-PARISI-ZHANG EQUATION

[PRL 1986]

COLE-HOPF TRANSFORMATION: $h(t,x) := \log U(t,x)$ solves (formally)

$$\text{(KPZ)} \qquad \partial_t h(t,x) = \underbrace{\Delta h(t,x)}_{\text{SMOOTHING}} + \underbrace{|\nabla h(t,x)|^2}_{\perp\ \text{GROWTH}} + \beta \underbrace{\xi(t,x)}_{\text{NOISE}}$$



SHE can help us make sense of KPZ

WHITE NOISE

Formally: $(\xi(t,x))_{t\geq 0,\, x\in\mathbb{R}^d}$ centered Gaussian

$$\mathbb{E}[\xi(t,x)\,\xi(s,y)] = \delta(t-s)\,\delta(x-y)$$ "i.i.d. in space-time"

Rigorously: $\left(\langle \xi, \varphi\rangle = \text{"}\int \varphi(t,x)\,\xi(t,x)\,dt\,dx\text{"}\right)_{\varphi\in C_c^\infty}$ centred Gaussian

$$\mathbb{E}[\langle \xi,\varphi\rangle\langle \xi,\psi\rangle] = \langle \varphi,\psi\rangle = \int \varphi(t,x)\,\psi(t,x)\,dt\,dx$$

$\varphi \mapsto \langle \xi,\varphi\rangle$ random distribution (not a function, nor a measure)

$(d=0)$ $\quad \xi(t) = \frac{d}{dt}B(t)$ "derivative of BM"

Singularity

(SHE) and (KPZ) are ill-defined due to singular products

$$\xi(t,x)\; u(t,x) \qquad\qquad |\nabla h(t,x)|^2$$

$\xi(t,x)$ is a distribution $\rightsquigarrow$ $u(t,x)$ and $h(t,x)$ expected to be:

- non-smooth functions $(d=1)$
- genuine distributions $(d \geq 2)$

Henceforth we focus on (SHE)

THE ROLE OF DIMENSION

Space-time blow up: $\tilde{u}(t,x) := u(\varepsilon^2 t, \varepsilon x)$ solves

$$\partial_t \tilde{u}(t,x) = \Delta \tilde{u}(t,x) + \beta\, \varepsilon^{\frac{2-d}{2}}\, \tilde{\xi}(t,x)\, \tilde{u}(t,x)$$

As $\varepsilon \downarrow 0$ the noise formally $\begin{cases} \text{vanishes} & (d<2) \\ \text{stays constant} & (d=2) \\ \text{diverges} & (d>2) \end{cases}$

$d=2$ is CRITICAL DIMENSION for SHE

Dimension $d=1$

(1980s) $u(t,x)$ well-posed by stochastic integration (Ito-Walsh)

(2010s) Robust solution theories for "sub-critical" singular PDEs

- REGULARITY STRUCTURES [Hairer]
- PARACONTROLLED CALCULUS [Gubinelli, Imkeller, Perkowski]
- ENERGY SOLUTIONS [Goncalves, Jara]
- RENORMALIZATION [Kupiainen]

Dimensions $d \geq 3$

(2015- ...) Results by several authors

Magnen, Unterberger, Chatterjee, Dunlap, Gu, Ryzhik, Zeitouni, Comets, Cosco, Mukherjee, Lygkonis, Zygouras, Nakajima, Nakashima, Junk, ...

DISCRETIZED WHITE NOISE

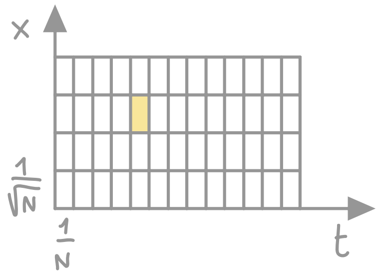
(alternative to mollification or Fourier cutoff $\rightsquigarrow$ Fabio's course)

Discrete approximation via i.i.d. random variables

$$\big(X(k,z) \big)_{k \in \mathbb{N},\, z \in \mathbb{Z}^d} \qquad \mathbb{E}[X] = 0 \qquad \mathbb{E}[X^2] = 1$$

For $(t,x) \sim \left(\frac{k}{N}, \frac{z}{\sqrt{N}}\right) \in \frac{\mathbb{N}}{N} \times \frac{\mathbb{Z}^d}{\sqrt{N}}$

$$\left(\frac{k}{N}, \frac{z}{\sqrt{N}}\right) \leq (t,x) \leq \left(\frac{k+1}{N}, \frac{z+1}{\sqrt{N}}\right)$$



$$\xi_N(t,x) := \underbrace{N^{\frac{1}{2}+\frac{d}{4}}}_{(\mathrm{vol})^{-\frac{1}{2}}} \boxed{X(k,z)} \xrightarrow[N\to\infty]{} \xi(t,x)$$

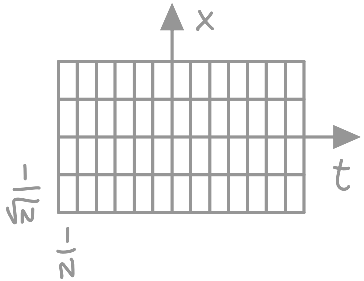
Ex. $\displaystyle\int \xi_N(t,x)\, \varphi(t,x)\, dt\, dx \xrightarrow{d} \mathcal{N}\left(0, \|\varphi\|_{L^2}^2\right)$

$\varphi = \mathbb{1}_A \quad A$ rectangle

REGULARIZING SHE VIA DISCRETIZATION

We fix $d=2$ and restrict to the lattice

$$(t,x) = \left(\frac{k}{N}, \frac{z}{\sqrt{N}}\right) \in \frac{\mathbb{N}}{N} \times \frac{\mathbb{Z}^2}{\sqrt{N}}$$



Discretized SHE

$$\boxed{\partial_t^N u_N(t,x) = \Delta^N u_N(t,x) + \beta \cdot N \cdot X(k+1, z) \langle u_N(t,x) \rangle}$$

DISCR. DERIVATIVE

$$N\left\{u\left(t+\tfrac{1}{N},x\right) - u(t,x)\right\}$$

DISCR. LAPLACIAN

$$\frac{N}{4} \sum_{x' \sim x} \left\{u(t,x') - u(t,x)\right\}$$

I.I.D. RVs

$$\mathbb{E}[X]=0 \quad \mathbb{E}[X^2]=1$$

$$\frac{1}{4} \sum_{x' \sim x} u(t,x')$$

Well-defined solution $u_N(t,x) \geq 0$ (if $\beta X \geq -1$) $[u_N(0,\cdot) \equiv 1]$

CONVERGENCE ?

Can we hope that $u_N(t,x) \xrightarrow[N\to\infty]{} \mathcal{U}(t,x)$? (non-trivial limit)

YES ! But...

1. Convergence as (random) distributions :

$$\int_{\mathbb{R}^2} \varphi(x)\, u_N(t,x)\, dx \xrightarrow[N\to\infty]{d} \int_{\mathbb{R}^2} \varphi(x)\, \underbrace{\mathcal{U}(t,dx)}_{\text{random measure on } \mathbb{R}^2}$$

2. Rescale coupling $\boxed{\beta = \beta_N = O\left(\frac{1}{\sqrt{\log N}}\right) \xrightarrow[N\to\infty]{} 0}$ WHY?!

Main Theorem (Lectures 2 + 3) [CSZ 23]

Rescale $\beta \sim \frac{\sqrt{\pi}}{\sqrt{\log N}}$ (critical!)

$$\beta = \frac{\sqrt{\pi}}{\sqrt{\log N}}\left(1 + \frac{\vartheta}{\log N}\right) \qquad \vartheta \in \mathbb{R}$$

Then $u_N(t,x)$ converges to a unique & non-trivial limit:

$$\left(u_N(t,x)\,dx\right)_{t\geq 0} \xrightarrow[N\to\infty]{\text{f.d.d.}} \mathcal{U}^{\vartheta} = \left(\mathcal{U}^{\vartheta}_t(dx)\right)_{t\geq 0}$$

$\mathcal{U}^{\vartheta}$ is a stochastic process of random measures on $\mathbb{R}^2$:

critical 2d STOCHASTIC HEAT FLOW (SHF)

PLAN OF THIS LECTURE

I. DIRECTED POLYMERS & PHASE TRANSITION

Statistical mechanics • Disorder strength

II. SUB-CRITICAL RESULTS

One-point distribution • Gaussian fluctuations

III. TOOLS & PROOFS

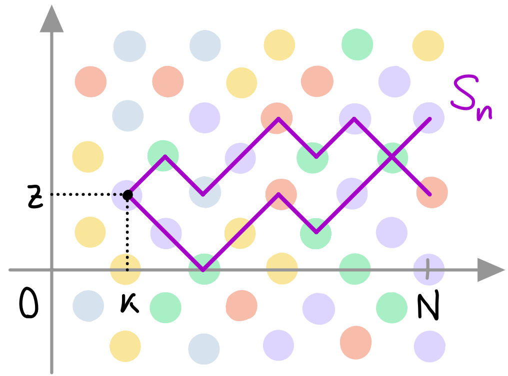
Polynomial chaos • Variance computations

I. Directed Polymers & Phase Transition

Directed Polymer in Random Environment

- $(S_n)_{n\geq 0}$ simple random walk on $\mathbb{Z}^2$

$$P\left(S_n - S_{n-1} = \begin{matrix}(\pm 1, 0)\\ (0, \pm 1)\end{matrix}\right) = \frac{1}{4}$$



- $(\omega(n,z))_{n\in\mathbb{N},\, z\in\mathbb{Z}^2}$ i.i.d. environment

$$\mathbb{E}[\omega] = 0 \qquad \mathbb{E}[\omega^2] = 1 \qquad \lambda(\beta) = \log \mathbb{E}[e^{\beta\omega}] < \infty$$

$$\longrightarrow \mathbb{E}[Z_N] = 1$$

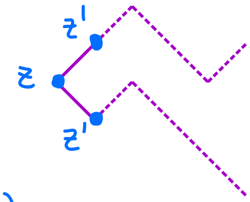
Partition Functions

$$Z_N^\omega(k,z) := E\left[e^{\underbrace{\sum_{n=k}^{N} \beta\,\omega(n,S_n) - \lambda(\beta)}_{H_N(S,\omega)}} \,\middle|\, S_k = z \right]$$

Feynman-Kac formula

Discretized SHE solution $u_N(t,x) = Z_N^{\omega}\left(N(1-t), \sqrt{N}\,x\right)$

Proof. Markov property of SRW:



$$Z_N^{\omega}(k,z) = e^{\beta\,\omega(k,z)-\lambda(\beta)}\ \frac{1}{4}\sum_{z'\sim z} Z_N^{\omega}(k+1,z')$$

$-Z_N(k+1,z)$ $\qquad$ $1+\{e^{\cdots}-1\}$

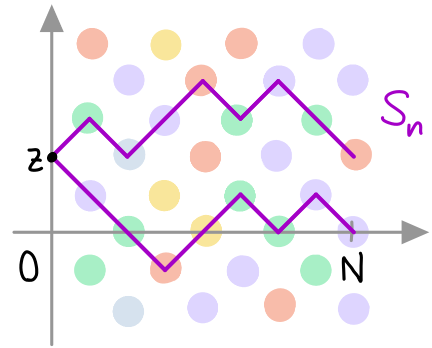
$$-\partial_{k,k+1} Z_N^{\omega}(k,z) = \Delta_z Z_N^{\omega}(k+1,z) + \underbrace{\left\{e^{\beta\omega(k,z)-\lambda(\beta)}-1\right\}}_{\text{i.i.d. } \beta X(k,z)\ \geq -1} Z_N^{\omega}(k+1,z)$$

(time-reversed) discretized SHE

SCALING LIMITS OF PARTITION FUNCTIONS

Henceforth we fix $t=1$ and we focus on partition functions:

$$Z_N^\omega(z) := \mathbb{E}\Big[e^{\overbrace{\sum_{n=1}^{N} \beta\,\omega(n,S_n)-\lambda(\beta)}^{H_N(S,\omega)}} \,\Big|\, S_0 = z \Big]$$



Correlated random field over $z \in \mathbb{Z}^2$

Goal: scaling limit of $Z_N^\omega(x\sqrt{N}) = U_N(1,x)$ for $x \in \mathbb{R}^2$

$\lfloor x\sqrt{N} \rfloor$

By construction $\mathbb{E}[Z_N^\omega] = 1$. Second moment?

Second moment & phase transition

Disorder strength $\beta = \beta_N = \dfrac{\hat\beta_N \sqrt{\pi}}{\sqrt{\log N}}$ $\qquad \beta \longleftrightarrow \hat\beta$

Theorem 1.

$$\mathbb{V}\mathrm{ar}\left[Z_N(x\sqrt{N}) \right] \begin{cases} \to \dfrac{\hat\beta^2}{1-\hat\beta^2} < \infty & \text{if } \hat\beta_N \to \hat\beta < 1 \\ \sim c \log N \to \infty & \text{if } \hat\beta_N \to \hat\beta = 1 \\ \gg \log N \to \infty & \text{if } \hat\beta_N \to \hat\beta > 1 \end{cases}$$

For $x \neq y \in \mathbb{R}^2$:

$$\mathbb{C}\mathrm{ov}\left[Z_N(x\sqrt{N}), Z_N(y\sqrt{N}) \right] \begin{cases} \sim \dfrac{\hat\beta^2}{1-\hat\beta^2} \dfrac{\boxed{K(x-y)}}{\log N} \to 0 & \hat\beta < 1 \\ \to \boxed{K(x-y)} < \infty & \hat\beta = 1 \\ \to \infty & \hat\beta > 1 \end{cases}$$

$\boxed{\sim \log \dfrac{1}{|x-y|^2}}$

II. SUB-CRITICAL RESULTS

SUB-CRITICAL REGIME

There is a phase transition for $\beta = \frac{\hat{\beta}\sqrt{\pi}}{\sqrt{\log N}}$ with critical value $\hat{\beta}_c = 1$.

We focus in this lecture on the sub-critical regime $\hat{\beta} < 1$.

- One point: $\mathbb{E}[Z_N(x\sqrt{N})] = 1 \qquad \mathbb{V}\mathrm{AR}[Z_N(x\sqrt{N})] \longrightarrow \frac{\hat{\beta}^2}{1-\hat{\beta}^2} < \infty$

  Convergence in distribution $Z_N \xrightarrow{d}$?

- Average: $\underbrace{\langle Z_N(\cdot\sqrt{N}), \varphi \rangle}_{} = \int Z_N(x\sqrt{N})\,\varphi(x)\,dx \xrightarrow{d} \int \varphi(x)\,dx = \text{const.}$

  $\mathbb{V}\mathrm{AR}[\quad] = \iint \mathbb{C}\mathrm{OV}[Z_N(x\sqrt{N}), Z_N(y\sqrt{N})]\,\varphi(x)\,\varphi(y)\,dx\,dy \longrightarrow 0$

  Fluctuations of $\langle Z_N(\cdot\sqrt{N}), \varphi \rangle$?

LOG - NORMALITY

Rescale disorder strength $\beta = \frac{\hat{\beta}\sqrt{\pi}}{\sqrt{\log N}}$ with $\hat{\beta} < 1$. Fix $x \in \mathbb{R}^2$.

THEOREM 2.

- $Z_N(x\sqrt{N}) \xrightarrow[N\to\infty]{d} e^{\sigma Y - \frac{1}{2}\sigma^2}$ $\quad Y \sim N(0,1)$ $\quad \sigma^2 = \log \frac{\hat{\beta}^2}{1-\hat{\beta}^2}$

- Limiting Y's are independent (!) for distinct $x \in \mathbb{R}^2$

$\mathrm{Cov}[\cdot,\cdot] \to 0$

- If $\hat{\beta} \geq 1$ then $Z_N(x\sqrt{N}) \xrightarrow{d} 0$

EDWARDS-WILKINSON FLUCTUATIONS FOR Z

Rescale disorder strength $\beta = \frac{\hat{\beta}\sqrt{\pi}}{\sqrt{\log N}}$ with $\hat{\beta} < 1$. Fix $\varphi \in C_c(\mathbb{R}^2)$.

THEOREM 3.

$$\sqrt{\log N}\,\Big\{ \underbrace{\langle Z_N(\cdot\sqrt{N}), \varphi \rangle}_{\int Z_N(x\sqrt{N})\,\varphi(x)\,dx} - \mathbb{E}[\dots] \Big\} \xrightarrow{d} \overbrace{\langle v, \varphi \rangle}^{\text{“}\int v(x)\,\varphi(x)\,dx\text{”}} \sim \mathcal{N}\Big(0, \frac{\hat{\beta}^2}{1-\hat{\beta}^2}\,\underbrace{\sigma_\varphi^2}_{\iint \varphi(x)\,\varphi(y)\,K_{\hat{\beta}}(x-y)\,dx\,dy}\Big)$$

Limiting field $v(x) = v(1,x)$ solves the Edwards-Wilkinson equation:

$$\partial_t v(t,x) = \Delta v(t,x) + \frac{\hat{\beta}^2}{1-\hat{\beta}^2}\,\xi(t,x) \qquad \text{(ADDITIVE NOISE!)}$$

EDWARDS-WILKINSON FLUCTUATIONS FOR $\log Z$

Rescale disorder strength $\beta = \frac{\hat{\beta}\sqrt{\pi}}{\sqrt{\log N}}$ with $\hat{\beta} < 1$. Fix $\varphi \in C_c(\mathbb{R}^2)$.

THEOREM 4.

$$\sqrt{\log N}\,\Big\{ \underbrace{\langle \log Z_N(\cdot\sqrt{N}), \varphi \rangle}_{\int \log Z_N(x\sqrt{N})\,\varphi(x)\,dx} - \mathbb{E}[\ldots] \Big\} \xrightarrow{d} \overbrace{\langle v, \varphi \rangle}^{\text{“} \int v(x)\,\varphi(x)\,dx \text{”}} \sim \mathcal{N}\Big(0, \frac{\hat{\beta}^2}{1-\hat{\beta}^2}\, \underbrace{\sigma_\varphi^2}_{\iint \varphi(x)\,\varphi(y)\,K_{\hat{\beta}}(x-y)\,dx\,dy}\Big)$$

Limiting field $v(x) = v(1,x)$ solves the Edwards-Wilkinson equation:

$$\partial_t v(t,x) = \Delta v(t,x) + \frac{\hat{\beta}^2}{1-\hat{\beta}^2}\,\xi(t,x) \qquad \text{(ADDITIVE NOISE!)}$$

SOME REFERENCES

- THEOREM 2 (LOG-NORMALITY) [CSZ, AAP 17]

  [C., Cottini, EJP 22] [Cosco, Donadini, 23+]

- THEOREM 3 (E-W FLUCTUATIONS FOR Z) [CSZ, AAP 17]

- THEOREM 4 (E-W FLUCTUATIONS FOR $\log Z$) [CSZ, AOP 20]

  [Chatterjee, Dunlop, AOP 20] [Gu, SPDE 20]

  [Nakajima, Nakashima, EJP 23] [Dunlap, Gu, AOP 22] [Tao, SPDE 22]

  [Tao, 23+] [Dunlap, Graham, 23+]

III. Tools & Proofs

A SELECTION OF TOOLS AND PROOFS

We conclude this lecture illustrating :

- Polynomial chaos expansion
- Second moment computations $\rightsquigarrow$ Proof of THEOREM 1
- Hints on multi-scale structure $\rightsquigarrow$ Proof of THEOREM 2

Polynomial chaos expansion

$$\frac{e^{\lambda(2\beta)-2\lambda(\beta)}-1}{\beta^2} \sim \frac{e^{\beta^2}-1}{\beta^2}$$

- $$X(n,x) := \frac{e^{\beta\omega(n,x)-\lambda(\beta)}-1}{\beta} \qquad \mathbb{E}[X]=0 \qquad \mathbb{V}\mathrm{AR}[X] \sim 1$$

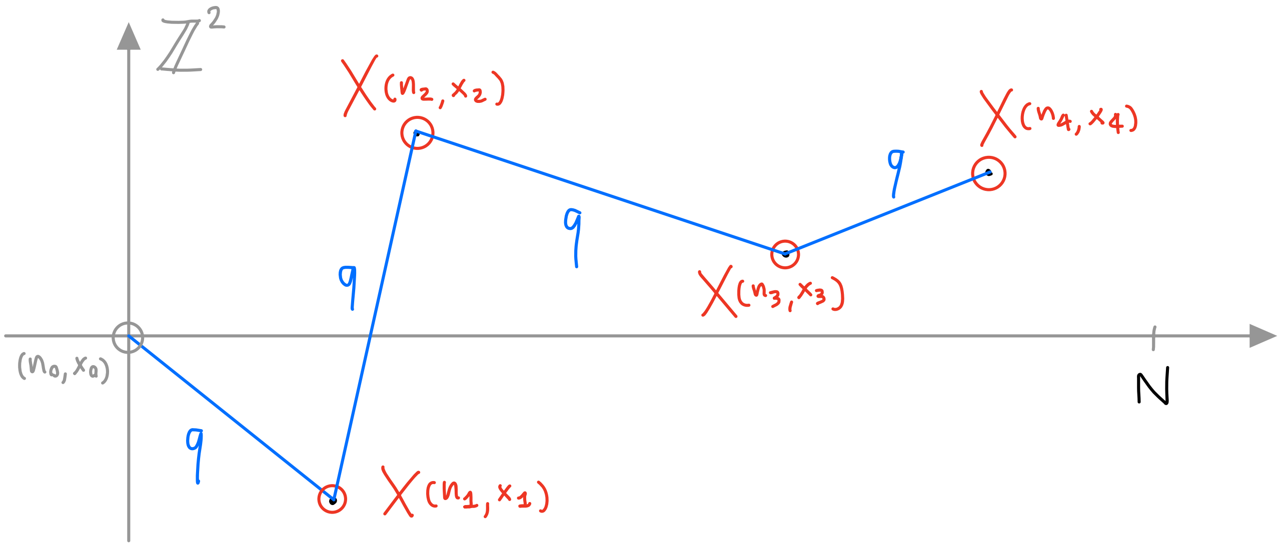
$$\mathbb{V}\mathrm{AR}[X] = \frac{e^{\lambda(2\beta)-2\lambda(\beta)}-1}{\beta^2}$$

- $$q_n(x) := P(S_n = x) \sim \frac{1}{n}\, g_{\frac{1}{2}}\!\left(\frac{x}{\sqrt{n}}\right) \cdot 2\,\mathbb{1}_{\{(n,x)\in\mathbb{Z}^3_{\mathrm{EVEN}}\}}$$

$$g_t(z) := \frac{1}{2\pi t}\, e^{-\frac{|z|^2}{2t}} \qquad \text{heat Kernel on } \mathbb{R}^2$$

$$Z_N(0) = 1 + \sum_{k=1}^{N} \beta^k \sum_{\substack{n_0=0<n_1<\dots<n_k\le N \\ x_0=0,\ x_1,\dots,x_k\in\mathbb{Z}^2}} \left(\prod_{i=1}^{k} q_{n_i-n_{i-1}}(x_i-x_{i-1})\right) \prod_{i=1}^{k} X(n_i,x_i)$$

$$Z_N(0) = 1 + \beta \sum_{n=1}^{N} \sum_{z \in \mathbb{Z}^2} q_n(z)\, X(n,z)$$
$$+ \beta^2 \sum_{0<n<n' \le N} \sum_{z,z' \in \mathbb{Z}^2} q_n(z)\, q_{n'-n}(z'-z)\, X(n,z)\, X(n',z') + \dots$$



Proof

$$Z_N(0) := E\left[e^{\sum_{n=1}^{N} \beta\omega(n,S_n)-\lambda(\beta)}\right]$$

$$= E\left[\prod_{n=1}^{N}\prod_{z\in\mathbb{Z}^d} e^{\{\beta\omega(n,z)-\lambda(\beta)\}\mathbb{1}_{\{S_n=z\}}}\right]$$

$$= E\left[\prod_{n=1}^{N}\prod_{z\in\mathbb{Z}^d}\left\{1+\left(e^{\beta\omega(n,z)-\lambda(\beta)}-1\right)\mathbb{1}_{\{S_n=z\}}\right\}\right]$$

$$= 1 + \sum_{(n,z)} \underbrace{\left(e^{\beta\omega(n,z)-\lambda(\beta)}-1\right)}_{\beta X(n,z)} \underbrace{P(S_n=z)}_{q_n(z)}$$

$$+ \sum_{(n,z)<(n',z')} \left(e^{\beta\omega(n,z)-\lambda(\beta)}-1\right)\left(e^{\beta\omega(n',z')-\lambda(\beta)}-1\right) P(S_n=z,\ S_{n'}=z')$$

KEY COMPUTATIONS

- $$\sum_{x\in\mathbb{Z}^2} q_n(x)^2 = \sum_{x\in\mathbb{Z}^2} q_n(x)\, q_n(-x) = q_{2n}(0) \sim \boxed{\frac{1}{\pi n}} \qquad (n\to\infty)$$

- $$R_N := \sum_{n=1}^{N} \sum_{x\in\mathbb{Z}^2} q_n(x)^2 \sim \sum_{n=1}^{N} \frac{1}{\pi n} \sim \boxed{\frac{\log N}{\pi}} + O(1)$$

Expected replica overlap $\dashrightarrow$ $\mathcal{L}_N$

$$R_N = \sum_{n=1}^{N} \sum_{x\in\mathbb{Z}^2} \underbrace{P(S_n = x)^2}_{P(S_n = x,\, S'_n = x)} = \sum_{n=1}^{N} P(S_n = S'_n) = E\Big[\overbrace{\sum_{n=1}^{N} \mathbb{1}_{\{S_n = S'_n\}}}^{\mathcal{L}_N}\Big]$$

INDEPENDENT RW's

PROOF OF THEOREM 1: VARIANCE COMPUTATION

Polynomial chaos: terms of different degrees ORTHOGONAL IN L^2

$$\mathbb{V}\mathrm{AR}[Z_N] \sim \sum_{k=1}^{N} (\beta^2)^k \sum_{\substack{0=n_0<n_1<\dots<n_k\leq N \\ x_0=0,\, x_1,\dots,x_k\in\mathbb{Z}^2}} \prod_{i=1}^{k} q_{n_i-n_{i-1}}(x_i-x_{i-1})^2$$

$$= \sum_{k=1}^{N} (\beta^2)^k \sum_{0=n_0<n_1<\dots<n_k\leq N} \prod_{i=1}^{k} \Bigg(\underbrace{\sum_{x\in\mathbb{Z}^2} q_{n_i-n_{i-1}}(x)^2}_{\sim \frac{1}{\pi(n_i-n_{i-1})}} \Bigg)$$

$$\sim \sum_{k=1}^{N} \left(\frac{\beta^2}{\pi}\right)^k \sum_{0=n_0<n_1<\dots<n_k\leq N} \prod_{i=1}^{k} \frac{1}{n_i-n_{i-1}}$$

We now enlarge the sum, allowing each increment $n_i - n_{i-1}$ to range freely in $\{1, 2, \dots, N\}$ → UPPER BOUND:

$$\mathbb{V}\mathrm{AR}[Z_N] \lesssim \sum_{k=1}^{N} \left(\frac{\hat{\beta}^2}{\pi}\right)^k \prod_{i=1}^{k} \left(\sum_{\underbrace{n}_{= n_i - n_{i-1}}=1}^{N} \frac{1}{n} \right)$$

$$\sim \sum_{k=1}^{N} \left(\frac{\hat{\beta}^2}{\pi}\right)^k (\log N)^k$$

$$\lesssim \sum_{k=1}^{N} (\hat{\beta}^2)^k \sim \boxed{\frac{\hat{\beta}^2}{1-\hat{\beta}^2}}$$

$$\boxed{\beta \sim \frac{\hat{\beta}\sqrt{\pi}}{\sqrt{\log N}}}$$

For a matching LOWER BOUND, we truncate the sum to $k \leq K$, then we restrict each increment to $1 \leq n_i - n_{i-1} \leq \frac{N}{K}$.

$$\mathbb{V}\mathrm{AR}[Z_N] \gtrsim \sum_{k=1}^{K} \left(\frac{\beta^2}{\pi}\right)^k \prod_{i=1}^{k} \Bigg(\sum_{\underbrace{n}_{= n_i - n_{i-1}} = 1}^{N/K} \frac{1}{n} \Bigg)$$

$$\sim \sum_{k=1}^{K} \left(\frac{\beta^2}{\pi}\right)^k \Big(\underbrace{\log \frac{N}{K}}_{\sim \log N \text{ as } N \to \infty} \Big)^k$$

$$\sim \sum_{k=1}^{K} (\hat{\beta}^2)^k \sim \boxed{\frac{\hat{\beta}^2}{1 - \hat{\beta}^2}}$$

$$\downarrow$$

$$K \to \infty$$

$$\boxed{\beta \sim \frac{\hat{\beta}\sqrt{\pi}}{\sqrt{\log N}}}$$

SECOND MOMENT: ALTERNATIVE APPROACH

$$\mathbb{E}\left[Z_N(\omega)^2\right] = \mathbb{E}\, E^{\otimes 2}\left[e^{\sum_{n=1}^{N} \beta\{\omega(n,S_n)+\omega(n,S_n')\} - 2\lambda(\beta)}\right]$$

$\mathbb{E}[e^{\beta\omega}] = e^{\lambda(\beta)}$

$$= E^{\otimes 2}\left[e^{\overbrace{\lambda(2\beta)-2\lambda(\beta)}^{\sim \beta^2}\ \overbrace{\sum_{i=1}^{N} \mathbb{1}_{\{S_i = S_i'\}}}^{\mathcal{L}_N}}\right]$$

$\sim \beta^2$ $\qquad$ $\mathcal{L}_N$ "REPLICA OVERLAP"

THEOREM: $\displaystyle \frac{\pi}{\log N}\,\mathcal{L}_N \xrightarrow[N\to\infty]{d} \mathrm{Exp}(1)$ $\qquad$ [Erdös-Taylor 60]

This explains the CRITICAL SCALING $\displaystyle \beta \sim \frac{\sqrt{\pi}}{\sqrt{\log N}}$

VARIANCE AND COVARIANCES

$$\mathbb{V}\mathrm{AR}[Z_N(0)] \sim \sum_{k=1}^{N} (\beta^2)^k \sum_{\substack{0=n_0<n_1<\dots<n_k\leq N \\ x=0,\ x_1,\dots,x_k\in\mathbb{Z}^2}} \prod_{i=1}^{k} q_{n_i-n_{i-1}}(x_i-x_{i-1})^2$$

$$\mathbb{C}\mathrm{OV}[Z_N(z), Z_N(z')] \sim \beta^2 \sum_{\substack{0=n_0<n_1\leq N \\ x_1\in\mathbb{Z}^2}} q_{n_1}(x_1-z)\, q_{n_1}(x_1-z') \cdot \Big\{ \underbrace{1+\mathbb{V}\mathrm{AR}[Z_{N-n_1}(x_1)]}_{\frac{\hat{\beta}^2}{1-\hat{\beta}^2}} \Big\}$$

$$\beta^2 \to \frac{\hat{\beta}^2\pi}{\log N}$$

$$\sum_{n_1=1}^{N} q_{2n_1}(z-z') \sim \frac{1}{\pi}\int_0^1 \frac{e^{-\frac{|z-z'|^2}{Nt}}}{t}\,dt =: \frac{1}{\pi}K\left(\frac{z-z'}{\sqrt{N}}\right) \sim \frac{1}{\pi}\log\frac{N}{|z-z'|^2}$$

SUB-CRITICAL COVARIANCES

Taking $z = x\sqrt{N}$, $z' = y\sqrt{N}$ we obtain

$$\mathbb{C}\mathrm{ov}\left[Z_N(x\sqrt{N}),\, Z_N(y\sqrt{N}) \right] \sim \frac{\hat{\beta}^2 \pi}{\log N} \cdot \frac{1}{\pi} K(x-y) \cdot \left\{ 1 + \frac{\hat{\beta}^2}{1-\hat{\beta}^2} \right\}$$

$$= \frac{\hat{\beta}^2}{1-\hat{\beta}^2} \frac{K(x-y)}{\log N}$$

$\sim \log \frac{1}{|x-y|}$

MULTI-SCALE STRUCTURE

[CSZ 23+]

Basic observation: $\displaystyle\sum_{n=1}^{N^{\alpha}} \frac{1}{n} \sim \log N^{\alpha} = \alpha \log N$

For $M \in \mathbb{N}$ define the scales $t_i = N^{\frac{i}{M}}$ for $i = 0, 1, \dots, M$

$1 = t_0 \ll t_1 \ll \dots \ll t_M = N$ as $N \to \infty$

Write $\displaystyle\frac{Z^{\omega}_{t_i}}{Z^{\omega}_{t_{i-1}}} = \frac{E\left[e^{H_{[0,t_i]}(S,\omega)}\right]}{E\left[e^{H_{[0,t_{i-1}]}(S,\omega)}\right]} = \tilde{E}_{t_{i-1}}\left[e^{H_{(t_{i-1},t_i]}(S,\omega)}\right]$

$H_{[0,t]} = H_{[0,s]} + H_{(s,t]}$

$\displaystyle\frac{d\tilde{P}_t}{dP} = \frac{e^{H_{[0,t]}(S,\omega)}}{E\left[e^{H_{[0,t]}}\right]}$

Proof of Theorem 1 (log-normality)

Sub-critical CLT: $\quad S_t$ under $\widetilde{P}_t \overset{d}{\approx} S_t$ under P (SRW)

[Gabriel, AIHP 23+]

$$\Longrightarrow \quad \frac{Z^{\omega}_{t_i}}{Z^{\omega}_{t_{i-1}}} \overset{d}{\approx} Y_i^{\omega} := E\left[e^{H_{(t_{i-1},t_i]}(S,\omega)}\right]$$

INDEPENDENT RVs $\quad \mathbb{E}[Y_i] = 1$

Then $\quad Z_N^{\omega} = \prod_{i=1}^{M} \frac{Z^{\omega}_{t_i}}{Z^{\omega}_{t_{i-1}}} \overset{d}{\approx} \prod_{i=1}^{M} Y_i^{\omega} = e^{\sum_{i=1}^{M} \log Y_i^{\omega}}$

Finally $\quad \log Y_i^{\omega} = \log\{1 + (Y_i^{\omega} - 1)\} \simeq (Y_i^{\omega} - 1) - \frac{1}{2}(Y_i^{\omega} - 1)^2$

$$\sum_{i=1}^{M} \log Y_i^{\omega} \xrightarrow{d} \mathcal{N}(0, \sigma^2) - \frac{1}{2}\sigma^2 \qquad \square$$

Thanks