The Critical 2d Stochastic Heat Flow

Lecture 2

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Critical 2d Stochastic Heat Flow (SHF)

Phase Transition on an Intermediate Disorder Scale

Recall the point-to-plane directed polymer partition functions

$$Z_N^{\beta_N}(z) = \mathbf{E}\Big[e^{\sum_{n=1}^N \{\beta_N \omega(n,S_n) - \lambda(\beta_N)\}} \,\Big|\, S_0 = z\Big].$$

A phase transition occurs on the intermediate disorder scale

$$\beta_N = \frac{\hat\beta}{\sqrt{R_N}} \sim \hat\beta \sqrt{\frac{\pi}{\log N}},$$

where $R_N = \mathbb{E}\Big[\sum_{n=1}^N 1_{\{S_n = S'_n\}}\Big]$ is the expected overlap between two independent SRW S and S' starting at 0.

More precisely, $\hat\beta_c = 1$, and

$$Z_N(0) \Rightarrow \begin{cases} 0 & \text{if } \hat\beta \geq \hat\beta_c, \\ e^{\sigma Y - \sigma^2/2} & \text{if } \hat\beta < \hat\beta_c, \end{cases}$$

where $Y \sim N(0,1)$ and $\sigma^2 = \log \frac{\hat\beta^2}{1-\hat\beta^2}$.

The Critical Window Around $\hat{\beta}_c = 1$

Diffusively rescale space-time and define the random measure

$$\mathcal{Z}_{N;s,t}^{\beta_N}(\mathrm{d}x, \mathrm{d}y), \qquad 0 \le s < t,\ x, y \in \mathbb{R}^2$$

such that for $\varphi \in C_c(\mathbb{R}^2)$ and $\psi \in C_b(\mathbb{R}^2)$,

$$\mathcal{Z}_{N;s,t}^{\beta_N}(\varphi, \psi) := \frac{1}{N} \sum_{x,y \in \mathbb{Z}^2} \varphi\Big(\frac{x}{\sqrt{N}}\Big) Z_{Ns,Nt}^{\beta_N}(x,y) \psi\Big(\frac{y}{\sqrt{N}}\Big).$$

Q. Does $\mathcal{Z}_{N;s,t}^{\beta_N}(\mathrm{d}x, \mathrm{d}y)$ *converge to a unique limit as* $N \to \infty$*?* If yes, the limit can be interpreted as the solution of the critical 2d SHE.

Moment Asymptotics In the Critical Window

Theorem 1 [BC'98, CSZ'19a, CSZ'19b, GQT'21] Let $\hat{\beta} = \hat{\beta}(\theta)$ be in the critical window, $\varphi \in C_c(\mathbb{R}^2)$, and $\psi \in C_b(\mathbb{R}^2)$. Then as $N \to \infty$,

(1) $\mathbb{E}\big[\mathcal{Z}_{N;s,t}^{\beta_N}(\varphi,\psi)\big] \to \int \varphi(x)\, g_{\frac{t-s}{2}}(y-x)\,\psi(y)\mathrm{d}x\mathrm{d}y$, where g is the heat kernel;

(2) $\mathbb{V}\mathrm{ar}(\mathcal{Z}_{N;s,t}^{\beta_N}(\varphi,\psi)) \to \int \cdots \int \varphi(x)\varphi(x') K_{t-s}^{\theta}(x,x';y,y')\psi(y)\psi(y')$, where the correlation kernel $K_{t-s}^{\theta}(x,x';y,y') \sim C \log \frac{1}{|x-x'|}$ as $|x-x'| \to 0$ (similarly for $|y-y'| \to 0$).

(3) $\lim_{N\to\infty} \mathbb{E}\big[\mathcal{Z}_{N;s,t}^{\beta_N}(\varphi,\psi)^3\big]$ exists and is finite.

(4) $\lim_{N\to\infty} \mathbb{E}\big[\mathcal{Z}_{N;s,t}^{\beta_N}(\varphi,\psi)^k\big]$ exists and is finite if $\varphi, \psi \in L^2(\mathbb{R}^2)$.

- The limiting k-th moments for $k \geq 3$ are all expressed as infinite series. All subsequential weak limits of $\mathcal{Z}_{N;s,t}^{\beta_N}(\varphi,\psi)$ will have the same moments. But they grow too fast to uniquely determine the limiting law of $\mathcal{Z}_{N;s,t}^{\beta_N}(\varphi,\psi)$.

Main Result: Critical 2d Stochastic Heat Flow

Main Result [CSZ'23] Let $\beta_N := \hat{\beta}/\sqrt{R_N}$ with $\hat{\beta} = \hat{\beta}(\theta) := 1 + \frac{\theta}{\log N}$.
Then as random measures on $\mathbb{R}^2 \times \mathbb{R}^2$ (with vague topology),

$$(\mathcal{Z}^{\beta_N}_{N;s,t}(\mathrm{d}x, \mathrm{d}y))_{0 \le s \le t} \underset{N\to\infty}{\Longrightarrow} \mathscr{Z}^{\theta} := (\mathscr{Z}^{\theta}_{s,t}(\mathrm{d}x, \mathrm{d}y))_{0 \le s \le t}$$

in finite-dimensional distributions.

Furthermore, $\mathscr{Z}^{\theta}$ satisfies the scaling relation: for all $a > 0$,

$$(\mathscr{Z}^{\theta}_{as,at}(\mathrm{d}(\sqrt{a}x), \mathrm{d}(\sqrt{a}y)))_{0 \le s \le t} \overset{\mathrm{dist}}{=} (a\, \mathscr{Z}^{\theta+\log a}_{s,t}(\mathrm{d}x, \mathrm{d}y))_{0 \le s \le t}.$$

We call $\mathscr{Z}^{\theta}$ the *critical 2d stochastic heat flow*, which can be regarded as *the solution of the critical 2d SHE*.

- [Clark'21, 22] proved similar results for the directed polymer on the diamond hierarchical lattice.

Theorem 2 [CSZ'23+] $\mathscr{Z}^{\theta}_{0,t}(\mathbb{1}, \mathrm{d}y)$ is almost surely singular w.r.t. Lebesgue measure; and $\mathscr{Z}^{\theta}_{0,t}(\mathbb{1}, \mathrm{d}y) \Rightarrow 0$ as $t \uparrow \infty$.

Proof Strategy

Key Proof Ingredients

In the absence of a characterization of the limit $\mathscr{Z}^{\theta}$, we will show that $(\mathcal{Z}^{\beta_N}_{N;s,t}(\varphi,\psi))_{N\in\mathbb{N}}$ forms a Cauchy sequence.

A. *Coarse-Graining*: leading to a family of *coarse-grained models* $\mathscr{Z}^{(\mathrm{cg})}_{\epsilon;s,t}(\,\cdot\,|\Theta_{N,\epsilon})$ that approximate $\mathcal{Z}^{\beta_N}_{N;s,t}$ in L^2. A *mutilinear polynomial* of the coarse-grained disorder variables $\Theta_{N,\epsilon}$.

B. *Time-Space Renewal Structure*: renewal process interpretation of second moment calculations, which leads in the continuum limit to the *Dickman subordinator*;

C. *Lindeberg Principle for multilinear polynomials of dependent random variables*: controls the effect of changing $\Theta_{N,\epsilon}$ to $\Theta_{M,\epsilon}$ in the coarse-grained model $\mathscr{Z}^{(\mathrm{cg})}_{\epsilon}(\,\cdot\,|\Theta)$;

D. *Functional Inequalities for Green's Functions of multiple random walks on $\mathbb{Z}^2$*: yields higher moment bounds for the coarse-grained model, as input for the *Lindeberg principle*.

Can also be implemented for the solution of the mollified SHE using the same coarse-grained models, which will give the same limit.

Coarse Graining of Chaos Expansion

Consider the averaged partition function $\mathcal{Z}_N(\varphi,\psi) := \mathcal{Z}^{\beta_N}_{N;0,1}(\varphi,\psi)$

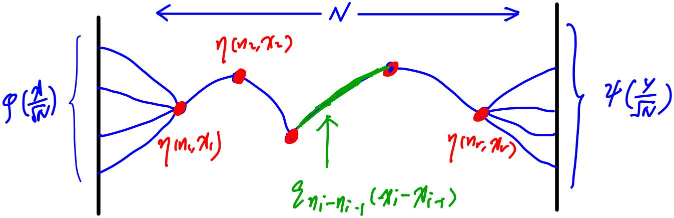
$$\mathcal{Z}_N(\varphi,\psi) = \frac{1}{N}\sum_{x,y\in\mathbb{Z}^2} \overbrace{\varphi_N(x)}^{\varphi(\frac{x}{\sqrt{N}})} q_N(y-x) \overbrace{\psi_N(y)}^{\psi(\frac{y}{\sqrt{N}})} \tag{1}$$

$$+\frac{1}{N}\sum_{r=1}^{\infty} \overbrace{q_{n_1}(\varphi_N,x_1)}^{\sum_x \varphi_N(x) q_{n_1}(x,x_1)} \sum_{\substack{x_1,\dots,x_r\in\mathbb{Z}^2\\ 1\le n_1<\dots<n_r<N}} \Big(\prod_{i=2}^{r} q_{n_i-n_{i-1}}(x_i-x_{i-1})\eta(n_i,x_i)\Big) q_{N-n_r}(x_r,\psi_N),$$

where

$$\eta(n,x) := e^{\beta_N\omega(n,x)-\lambda(\beta_N)} - 1 =: \beta_N X(n,x).$$

Can interpret $(n_1,x_1),\dots,(n_r,x_r)$ as a *time-space renewal sequence*.

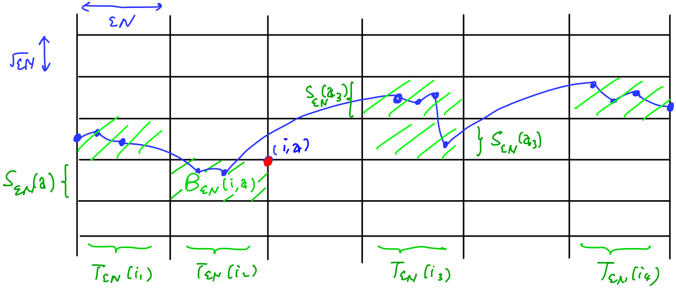


Coarse Graining of Chaos Expansion

Fix $\epsilon \in (0,1)$. Let $N \in \mathbb{N}$. Partition $\mathbb{N} \times \mathbb{Z}^2$ into *mesoscopic time-space boxes* on time-space scale of $(\epsilon N, \sqrt{\epsilon N})$:

$$\mathcal{B}_{\epsilon N}(\mathsf{i}, \mathsf{a}) := \underbrace{((\mathsf{i}-1)\epsilon N, \mathsf{i}\epsilon N]}_{\mathcal{T}_{\epsilon N}(\mathsf{i})} \times \underbrace{((\mathsf{a}-(1,1))\sqrt{\epsilon N}, \mathsf{a}\sqrt{\epsilon N}]}_{\mathcal{S}_{\epsilon N}(\mathsf{a})}. \qquad (2)$$

The *renewal sequence* $(n_1, x_1), \ldots, (n_r, x_r)$ visits a sequence of *mesoscopic time intervals* $\mathcal{T}_{\epsilon N}(\mathsf{i}_1), \ldots, \mathcal{T}_{\epsilon N}(\mathsf{i}_k)$, $1 \leq \mathsf{i}_1 \cdots \leq \mathsf{i}_k \leq \frac{1}{\epsilon}$. For each $\mathcal{T}_{\epsilon N}(\mathsf{i}_j)$, identify the mesoscopic spatial boxes of entry and exit $\mathcal{S}_{\epsilon N}(\mathsf{a}_j)$ and $\mathcal{S}_{\epsilon N}(\mathsf{a}'_j)$, $\mathsf{a}_j, \mathsf{a}'_j \in \mathbb{Z}^2$.



Kernel Replacement and Coarse Grained Variable

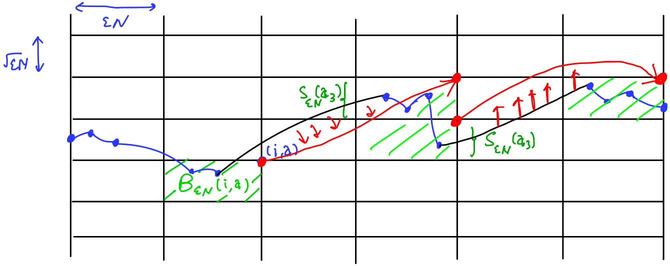
Consecutively visited time-space boxes $\mathcal{B}_{\epsilon N}(\mathsf{i}_j, \mathsf{a}'_j)$ and $\mathcal{B}_{\epsilon N}(\mathsf{i}_{j+1}, \mathsf{a}_{j+1})$ are connected by random walk transition kernel

$$q_{n,m}(x,y), \qquad (n,x) \in \mathcal{B}_{\epsilon N}(\mathsf{i}_j, \mathsf{a}'_j),\ (m,y) \in \mathcal{B}_{\epsilon N}(\mathsf{i}_{j+1}, \mathsf{a}_{j+1}).$$

We *hope* to use local central limit theorem to replace

$$q_{n,m}(x,y) \rightsquigarrow \frac{1}{\epsilon N} g_{\frac{1}{2}(\mathsf{i}_{j+1}-\mathsf{i}_j)}(\mathsf{a}_{j+1} - \mathsf{a}'_j), \tag{3}$$

which *decouples* contributions from different visited time intervals $\mathcal{T}_{\epsilon N}(\mathsf{i})$ (to the chaos expansion of $\mathcal{Z}_N(\varphi, \psi)$).

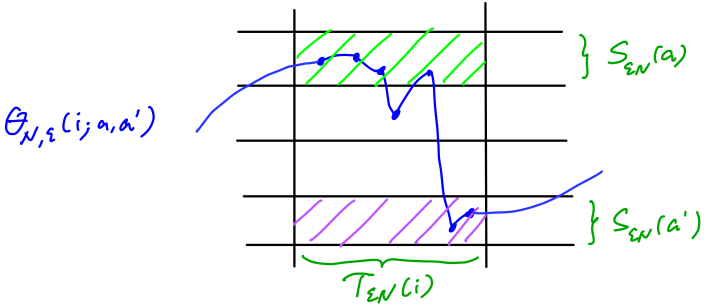


Kernel Replacement and Coarse Grained Variable

If $\mathcal{S}_{\epsilon N}(\mathsf{a})$ and $\mathcal{S}_{\epsilon N}(\mathsf{a}')$ are the spatial boxes of entry and exit in the time interval $\mathcal{T}_{\epsilon N}(\mathsf{i})$, then the contribution is

$$\Theta_{N,\epsilon}(\mathsf{i};\mathsf{a},\mathsf{a}') := \frac{1}{\epsilon N}\sum_{r=1}^{\infty}\sum_{\substack{(n_1,x_1),\dots,(n_r,x_r)\\ x_1\in\mathcal{S}_{\epsilon N}(\mathsf{a}),\, x_r\in\mathcal{S}_{\epsilon N}(\mathsf{a}')\\ n_1<\cdots<n_r,\ n_i\in\mathcal{T}_{\epsilon N}(\mathsf{i})}} \eta(n_1,x_1)\prod_{j=2}^{r} q_{n_{j-1},n_j}(x_{j-1},x_j)\eta(n_j,x_j)\,,$$

which we call a *coarse-grained (CG) disorder variable*.



Kernel Replacement and Coarse Grained Variable

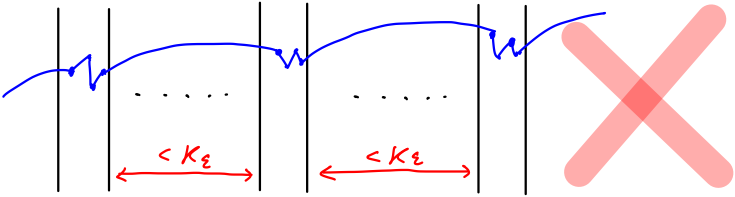
Complication: The kernel replacement

$$q_{n,m}(x,y) \rightsquigarrow \frac{1}{\epsilon N} g_{\frac{1}{2}(\mathsf{i}_{j+1}-\mathsf{i}_j)}(\mathsf{a}_{j+1} - \mathsf{a}'_j),$$

induces a small error only for $|\mathsf{i}_{j+1} - \mathsf{i}_j|$ large (say $\geq K_\epsilon = (\log \frac{1}{\epsilon})^6$) and $|\mathsf{a}_{j+1} - \mathsf{a}'_j|$ not too large (say $\leq M_\epsilon \sqrt{|\mathsf{i}_{j+1} - \mathsf{i}_j|}$, $M_\epsilon = \log\log \frac{1}{\epsilon}$).

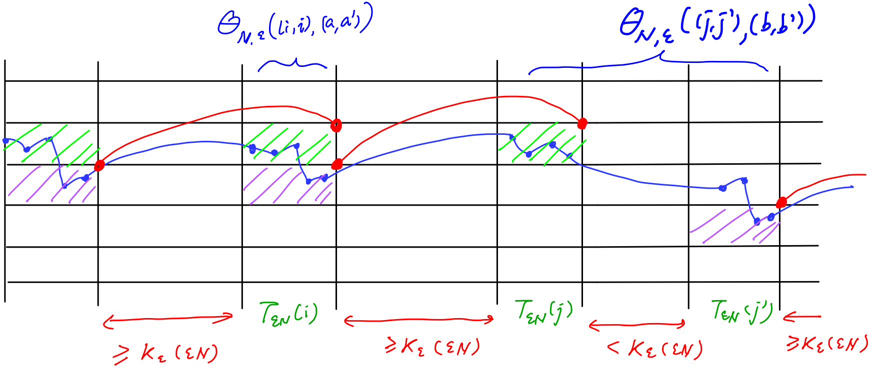
Lemma (No Triple Visited Intervals) Let $\mathcal{Z}_N^{\mathrm{NT}}(\varphi,\psi)$ be the chaos expansion of $\mathcal{Z}_N(\varphi,\psi)$, where $(n_1,x_1),\cdots,(n_r,x_r)$ does not visit any $\mathcal{T}_{\epsilon N}(\mathsf{i}_j)$, $\mathcal{T}_{\epsilon N}(\mathsf{i}_{j+1})$, $\mathcal{T}_{\epsilon N}(\mathsf{i}_{j+2})$ with $\mathsf{i}_{j+1}-\mathsf{i}_j, \mathsf{i}_{j+2}-\mathsf{i}_{j+1} < K_\epsilon$. Then

$$\lim_{\epsilon \downarrow 0} \limsup_{N\to\infty} \|\mathcal{Z}_N^{\mathrm{NT}}(\varphi,\psi) - \mathcal{Z}_N(\varphi,\psi)\|_2 = 0.$$



Kernel Replacement and Coarse Grained Variable

We can replace $q_{n,m}(x, y)$ connecting $\mathcal{T}_{\epsilon N}(\mathsf{i}_j)$ and $\mathcal{T}_{\epsilon N}(\mathsf{i}_{j+1})$ with $\mathsf{i}_{j+1} - \mathsf{i}_j \geq K_\epsilon$. The result decoupling leads to a second type of coarse-grained disorder variable $\Theta_{N,\epsilon}(\vec{\mathsf{i}}, \vec{\mathsf{a}})$, with $\vec{\mathsf{i}} = (\mathsf{i}, \mathsf{i}')$, $\vec{\mathsf{a}} = (\mathsf{a}, \mathsf{a}')$.Its chaos expansion visits $\mathcal{T}_{\epsilon N}(\mathsf{i})$ and $\mathcal{T}_{\epsilon N}(\mathsf{i}')$, with $\mathcal{S}_{\epsilon N}(\mathsf{a})$ the spatial box of entry and $\mathcal{S}_{\epsilon N}(\mathsf{a}')$ the spatial box of exit.



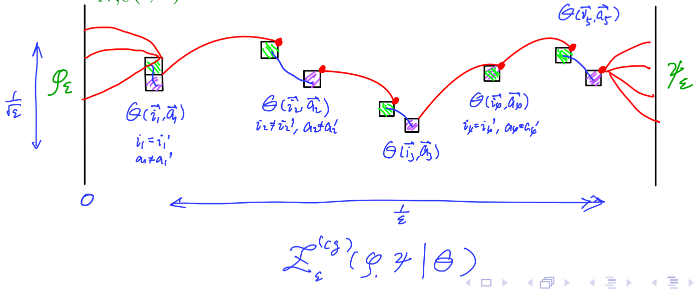
Coarse-Grained Model

This leads to the *Coarse-Grained Model*

$$\mathscr{Z}_\epsilon^{(\mathrm{cg})}(\varphi,\psi|\Theta) := g_{\frac{1}{2}}(\varphi,\psi) + \epsilon \sum_{r=1}^{\infty} \sum_{\substack{(\vec{\mathsf{i}}_1,\dots,\vec{\mathsf{i}}_r) \\ (\vec{\mathsf{a}}_1,\dots,\vec{\mathsf{a}}_r)}}$$

$$g_{\frac{1}{2}\mathsf{i}_1}(\overbrace{\varphi_\epsilon}^{\varphi(\sqrt{\epsilon}\cdot)}, \mathsf{a}_1)\Theta(\vec{\mathsf{i}}_1,\vec{\mathsf{a}}_1)\left\{\prod_{j=2}^{r} g_{\frac{1}{2}(\mathsf{i}_j-\mathsf{i}'_{j-1})}(\mathsf{a}_j-\mathsf{a}'_{j-1})\Theta(\vec{\mathsf{i}}_j,\vec{\mathsf{a}}_j)\right\} g_{\frac{1}{2}(\frac{1}{\epsilon}-\mathsf{i}'_r)}(\mathsf{a}'_r, \overbrace{\psi_\epsilon}^{\psi(\sqrt{\epsilon}\cdot)}),$$

where $\Theta := \Theta_{N,\epsilon}(\vec{\mathsf{i}},\vec{\mathsf{a}})$ varies with N and ϵ.



Coarse-Grained Model

Note that $\mathscr{Z}_\epsilon^{(\mathrm{cg})}(\varphi, \psi|\Theta)$ has the same structure (self-similarity) as

$$\mathcal{Z}_N(\varphi,\psi) = \frac{1}{N}\sum_{x,y\in\mathbb{Z}^2} \varphi_N(x) q_N(y-x)\psi_N(y)$$

$$+ \frac{1}{N}\sum_{r=1}^{\infty} q_{n_1}(\varphi_N, x_1) \sum_{\substack{x_1,\dots,x_r\in\mathbb{Z}^2 \\ 1\le n_1<\dots<n_r<N}} \Big(\prod_{i=2}^{r} q_{n_i-n_{i-1}}(x_i - x_{i-1})\eta(n_i,x_i)\Big) q_{N-n_r}(x_r,\psi_N).$$

Lemma (Coarse-Grain Approximation) Let $\Theta_{N,\epsilon}$ be the coarse grained disorder variables on time-space scale $(\epsilon N, \sqrt{\epsilon N})$. Then

$$\lim_{\epsilon\downarrow 0}\limsup_{N\to\infty} \|\mathscr{Z}_\epsilon^{(\mathrm{cg})}(\varphi,\psi|\Theta_{N,\epsilon}) - \mathcal{Z}_N(\varphi,\psi)\| = 0. \qquad (4)$$

Lindeberg Principle

Since $\mathcal{Z}_N(\varphi,\psi) \approx \mathscr{Z}_\epsilon^{(\mathrm{cg})}(\varphi,\psi|\Theta_{N,\epsilon})$ for ϵ small and N large, only remains to show

$$\mathscr{Z}_\epsilon^{(\mathrm{cg})}(\varphi,\psi|\Theta_{N,\epsilon}) \approx \mathscr{Z}_\epsilon^{(\mathrm{cg})}(\varphi,\psi|\Theta_{M,\epsilon})$$

with an error that is uniform in large $M \geq N$ and tends to 0 as $\epsilon \downarrow 0$.

Prove a *Lindeberg Principle* for multilinear polynomials of dependent random variables $[\Phi_\epsilon(\Theta) := \mathscr{Z}_\epsilon^{(\mathrm{cg})}(\varphi,\psi|\Theta)]$ to compare $\Phi_\epsilon(\Theta_{N,\epsilon})$ and $\Phi_\epsilon(\Theta_{M,\epsilon})$. Need the following:

- Convergence of mean and covariance of $\Theta_{N,\epsilon}(\vec{\mathsf{i}},\vec{\mathsf{a}})$ as $N \to \infty$.
- Control of the influence of each $\Theta_{N,\epsilon}(\vec{\mathsf{i}},\vec{\mathsf{a}})$, more precisely, show

$$\sum_{(\vec{\mathsf{i}},\vec{\mathsf{a}})} \mathbb{E}\Big[\big|\partial_{\Theta(\vec{\mathsf{i}},\vec{\mathsf{a}})}\Phi_\epsilon(\Theta_{N,\epsilon})\big|^3\Big]\mathbb{E}\Big[|\Theta_{N,\epsilon}(\vec{\mathsf{i}},\vec{\mathsf{a}})|^3\Big] \ll \mathbb{E}\Big[\Phi_\epsilon(\Theta_{N,\epsilon})^2\Big],$$

for N large and ϵ small. Note that

$$\sum_{(\vec{\mathsf{i}},\vec{\mathsf{a}})} \mathbb{E}\Big[\big|\partial_{\Theta(\vec{\mathsf{i}},\vec{\mathsf{a}})}\Phi_\epsilon(\Theta_{N,\epsilon})\big|^2\Big]\mathbb{E}\Big[|\Theta_{N,\epsilon}(\vec{\mathsf{i}},\vec{\mathsf{a}})|^2\Big] \approx \mathrm{Deg}(\Phi_\epsilon)\mathbb{E}\Big[\Phi_\epsilon(\Theta_{N,\epsilon})^2\Big].$$

In our case, can show $\mathrm{Deg}(\Phi_\epsilon) \approx \log\frac{1}{\epsilon}$ uniformly in N large.

Moment Bounds for the Coarse-Grained Model

To apply Lindeberg, we need to bound

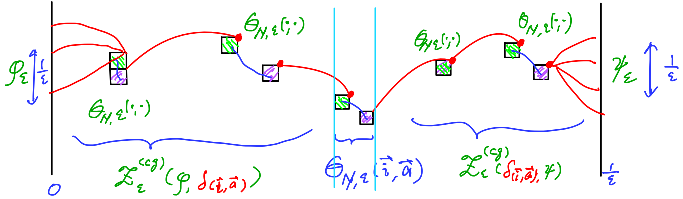
$$\mathbb{E}\Big[\big|\Theta_{N,\epsilon}(\vec{\mathsf{i}},\vec{\mathsf{a}})\big|^3\Big] \quad \text{and} \quad \mathbb{E}\Big[\Big|\partial_{\Theta(\vec{\mathsf{i}},\vec{\mathsf{a}})}\Phi_\epsilon(\Theta_{N,\epsilon})\Big|^3\Big]$$

and gain a factor $\ll \log\frac{1}{\epsilon}$ compared to $\mathbb{E}\Big[\Big|\partial_{\Theta(\vec{\mathsf{i}},\vec{\mathsf{a}})}\Phi_\epsilon(\Theta_{N,\epsilon})\Big|^2\Big]$.

- $\Theta_{N,\epsilon}(\vec{\mathsf{i}},\vec{\mathsf{a}})$ is an averaged partition function comparable to $\mathcal{Z}_N(\varphi,\psi)$. Hence need to bound $\mathbb{E}\big[\mathcal{Z}_N(\varphi,\psi)^4\big]$.

- $$\partial_{\Theta(\vec{\mathsf{i}},\vec{\mathsf{a}})}\Phi_\epsilon(\Theta_{N,\epsilon}) = \partial_{\Theta(\vec{\mathsf{i}},\vec{\mathsf{a}})}\mathscr{Z}_\epsilon^{(\mathrm{cg})}(\varphi,\psi\,|\,\Theta_{N,\epsilon})$$
$$= \mathscr{Z}_\epsilon^{(\mathrm{cg})}(\varphi,\delta_{(\vec{\mathsf{i}},\vec{\mathsf{a}})}\,|\,\Theta_{N,\epsilon})\cdot\Theta_{N,\epsilon}(\vec{\mathsf{i}},\vec{\mathsf{a}})\cdot\mathscr{Z}_\epsilon^{(\mathrm{cg})}(\delta_{(\vec{\mathsf{i}},\vec{\mathsf{a}})},\psi\,|\,\Theta_{N,\epsilon})$$

Hence need to bound $\mathbb{E}\big[\mathscr{Z}_\epsilon^{(\mathrm{cg})}(\varphi,\psi\,|\,\Theta_{N,\epsilon})^4\big]$.



Lindeberg Principle

for

Multilinear Polynomials

of

Independent Random Variables

Lindeberg Principle for Multilinear Polynomials

Theorem A [MOO'10] Let $\mathbb{T}$ be an index set, and let $(\eta_i)_{i\in\mathbb{T}}$, $(\zeta_i)_{i\in\mathbb{T}}$ be i.i.d. random variables with $\mathbb{E}[\zeta_i] = \mathbb{E}[\eta_i] = 0$, $\mathbb{E}[\zeta_i^2] = \mathbb{E}[\zeta_i^2] = 1$, and $M := \max\{\mathbb{E}[|\zeta_i|^3], \mathbb{E}[|\eta_i|^3]\} < \infty$.

Given a kernel $\psi : I \subset \mathbb{T} \to \mathbb{R}$ of degree l, i.e., $\psi(I) = 0$ for $|I| > l$, let

$$\Psi(\eta) := \sum_{I\subset\mathbb{T}} \psi(I) \prod_{i\in I} \eta_i =: \sum_{I\subset\mathbb{T}} \psi(I)\eta_I.$$

Then for any $f \in C_b^3(\mathbb{R})$, there exists $C > 1$ depending only on the law of η and ζ s.t.

$$\Big|\mathbb{E}[f(\Psi(\zeta)) - f(\Psi(\eta))]\Big| \leq \|f'''\|_\infty M C^l \big(\max_{i\in\mathbb{T}} \mathrm{Inf}_i(\Psi)\big)^{\frac{1}{2}} \mathbb{E}[\Psi(\eta)^2], \quad (5)$$

where $\mathrm{Inf}_i(\Psi) := \mathbb{E}[(\partial_i \Psi(\eta))^2] = \sum_{I\ni i} \psi(I)^2$.

Remark The more classic Lindeberg principle assumes that $\Psi(\eta)$ has bounded derivatives w.r.t. each η_i, which does not hold here. But our $\Psi(\eta)$ here has a special multilinear structure of bounded degree.

Proof of Lindeberg for Multilinear Polynomials

Proof. Assume $|\mathbb{T}| = N$. Let $\zeta^{(0)} = \eta, \zeta^{(1)}, \ldots, \zeta^{(N)} = \zeta$ be the successive replacements of each η_i by ζ_i. Then

$$\mathbb{E}[f(\Psi(\zeta)) - f(\Psi(\eta))] = \sum_{k=1}^{N} \mathbb{E}\Big[f(\Psi(\zeta^{(k)})) - f(\Psi(\zeta^{(k-1)}))\Big],$$

$$\text{with} \quad f(\Psi(\zeta^{(k)})) - f(\Psi(\zeta^{(k-1)})) =: g_{k;\eta,\zeta}(\eta_k) - g_{k;\eta,\zeta}(\zeta_k),$$

$$\text{and} \quad g_{k;\eta,\zeta}(\eta_k) - g_{k;\eta,\zeta}(\zeta_k) = g'_{k;\eta,\zeta}(0)(\eta_k - \zeta_k) + \frac{g''_{k;\eta,\zeta}(0)}{2}(\eta_k^2 - \zeta_k^2) + R_{\eta,\zeta}(\eta_k) - R_{\eta,\zeta}(\zeta_k),$$

with remainders

$$|R_{\eta,\zeta}(\eta_k)| \le \|g'''_{\eta,\zeta}\|_\infty |\eta_k|^3 \le \|f'''\|_\infty |\partial_k \Psi(\zeta^{(k)})|^3 |\eta_k|^3,$$
$$|R_{\eta,\zeta}(\zeta_k)| \le \|f'''\|_\infty |\partial_k \Psi(\zeta^{(k-1)})|^3 |\zeta_k|^3.$$

Proof of Lindeberg for Multilinear Polynomials

Since $g_{k;\eta,\zeta}(x)$ and $\partial_k \Psi(\zeta^{(k)}) = \partial_k \Psi(\zeta^{(k-1)})$ are independent of η_k, ζ_k, while $\mathbb{E}[\eta_k - \zeta_k] = \mathbb{E}[\eta_k^2 - \zeta_k^2] = 0$, we have

$$\Big| \sum_{k=1}^{N} \mathbb{E}[g_{\eta,\zeta}(\eta_k) - g_{\eta,\zeta}(\zeta_k)] \Big| \le 2M \|f'''\|_\infty \sum_{k=1}^{N} \mathbb{E}[|\partial_k \Psi(\zeta^{(k)})|^3]. \qquad (6)$$

Note that $\partial_k \Psi(\zeta^{(k)})$ is a multilinear polynomial in $\zeta^{(k)}$ and

$$\partial_k \Psi(\zeta^{(k)}) = \sum_{I \ni k} \psi(I) \prod_{i \in I \setminus \{k\}} \zeta_i^{(k)},$$

$$\mathbb{E}[(\partial_k \Psi(\zeta^{(k)}))^2] = \sum_{I \ni k} \psi(I)^2 = \mathrm{Inf}_k(\Psi).$$

Using *hypercontractivity* for polynomial chaos expansions, we have

$$\mathbb{E}[|\partial_k \Psi(\zeta^{(k)})|^3] \le \Big(\sum_{I \ni k} c^{2|I|} \psi(I)^2 \Big)^{\frac{3}{2}} \le c^{3l}\, \mathrm{Inf}_k(\Psi)^{\frac{3}{2}}.$$

Since $\psi(I) = 0$ for $|I| > l$, the RHS of (6) can be bounded by

$$C_f M c^{3l} (\max_k \mathrm{Inf}_k(\Psi))^{\frac{1}{2}} \sum_k \sum_{I \ni k} \psi(I)^2 \le C_f M c^{3l} (\max_k \mathrm{Inf}_k(\Psi))^{\frac{1}{2}} \cdot l \sum_{I \ne \emptyset} \psi(I)^2.$$

Applying Lindeberg to Directed Polymer

For the averaged point-to-plane directed polymer partition function, with $\varphi \in C_c(\mathbb{R}^2)$,

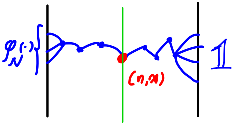
$$\Psi(X) := \frac{1}{N} \sum_{x \in \mathbb{Z}^2} \underbrace{\varphi_N(x)}_{\varphi(\frac{x}{\sqrt{N}})} (Z_N^{\beta_N}(x) - 1)$$

$$= \frac{1}{N} \sum_{l=1}^{\infty} q_{n_1}(\varphi_N, x_1) \sum_{\substack{x_1,\dots,x_l \in \mathbb{Z}^2 \\ 1 \le n_1 < \cdots < n_l < N}} \prod_{i=2}^{l} q_{n_i - n_{i-1}}(x_i - x_{i-1}) \beta_N X(n_i, x_i).$$

By L^2 approximation, we can restrict to degree $l = O(\log N)$ when β_N is in the critical window. Theorem A and the hypercontractivity argument in the proof is too crude, which gives $C^l = C^{O(\log N)}$.

Instead, we work with the bound (6), which becomes

$$\sum_{x \in \mathbb{Z}^2, 1 \le n \le N} \mathbb{E}\Big[\big|\partial_{(n,x)} \Psi(X)\big|^3\Big],$$



where

$$\partial_{(n,x)} \Psi(X) = \frac{\beta_N}{N} Z_{N;0,n}^{\beta_N}(\varphi_N, x) Z_{N;n,N}^{\beta_N}(x, \mathbb{1}).$$

Applying Lindeberg to Directed Polymer

Therefore

$$\sum_{\substack{x\in\mathbb{Z}^2\\1\le n\le N}} \mathbb{E}\big[|\partial_{(n,x)}\Psi(X)|^3\big] = \sum_{\substack{x\in\mathbb{Z}^2\\1\le n\le N}} \frac{\beta_N^3}{N^3}\mathbb{E}\big[(Z^{\beta_N}_{N;0,n}(\varphi_N,x))^3\big]\mathbb{E}\big[(Z^{\beta_N}_{N;n,N}(x,\mathbb{1}))^3\big].$$

It can be shown (see Part 3) that

$$\mathbb{E}\big[(Z^{\beta_N}_{N;0,n}(\varphi_N,x))^3\big],\ \mathbb{E}\big[(Z^{\beta_N}_{N;n,N}(x,\mathbb{1}))^3\big] = o(N^\delta) \quad \text{for any } \delta>0.$$

Because $\varphi \in C_c(\mathbb{R}^2)$, the dominant contribution in $\sum_{(n,x)}$ comes from $x = O(\sqrt{N})$, and there are $O(N^2)$ such (n,x), $\sum_{(n,x)} \to 0$ as $N\to\infty$.

Conclusion: Applying Lindeberg for multilinear polynomials of i.i.d. random variables directly without coarse-graining can show that the law of the averaged partition functions does not depend on the law of the disorder ω as $N\to\infty$. But coarse-graining is necessary to show convergence to a uniquene limit.

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