Towards Abstract Wiener Model Spaces

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Motivation and Recollection

Reminder: Classical Definition

Gives rigorous meaning to

$$\mathrm{d}\mu(\phi) = \frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2}\|\phi\|_{\mathscr{H}}^2\right) \mathcal{D}\phi. \tag{1}$$

Definition
An **abstract Wiener space** is a quadruple consisting of

- a separable (real) Banach space E,
- a separable (real) Hilbert space $\mathscr{H}$ "Cameron-Martin space",
- a continuous, linear injection $j : \mathscr{H} \hookrightarrow E$, and
- a (centred Gaussian) probability measure μ on $(E, \mathfrak{B}_E)$ s.t.

$$\int_E \exp\left(\mathrm{i}\,\ell(x)\right) \mathrm{d}\mu(x) = \exp\left(-\frac{1}{2}\|\mathfrak{C}_\mu \ell\|_{\mathscr{H}}^2\right), \quad \ell \in E^*. \tag{2}$$

where $\mathfrak{C}_\mu : E^* \to E \subseteq E^{**}$ is the covariance operator of μ on E.

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Examples include

Gaussian Measure μ	**Hilbert Space** $\mathscr{H}$	**Banach Space** E
Brownian Motion	$\left(W_0^{1,2}([0,1]), \int_0^1 x'(s)y'(s)\,\mathrm{d}s\right)$	$C[0,1], \mathcal{C}^{\frac{1}{2}-\kappa}[0,1], \ldots$
Space-time White Noise	$\left(L^2(\mathbb{R}^d), \langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^d)}\right)$	$\mathcal{C}_{\mathfrak{s}}^{0,-\frac{d+2}{2}-\kappa}(\mathbb{R}^d), \ldots$
Dirichlet Gaussian Free Field	$\left(\dot{H}_0^1(U), \int_U \langle\nabla\phi,\nabla\psi\rangle\,\mathrm{d}\lambda^d\right)$	$\dot{H}^{-\frac{d-2}{2}-\kappa}(U), \ldots$
Mult. Gaussian $\mathcal{N}(0,\Sigma)$	$(\mathbb{R}^d, \langle\cdot,\Sigma^{-1}\cdot\rangle_{\mathbb{R}^d})$	$\left(\mathbb{R}^d, \langle\cdot,\cdot\rangle_{\mathbb{R}^d}\right), \ldots$
β-fractional Brownian Motion	$\left(\dot{H}^{\beta+\frac{1}{2}}, \left\langle\cdot,(-\Delta)^{\beta+\frac{1}{2}}\cdot\right\rangle_{L^2}\right)$	$\mathcal{C}^{0,\beta-\kappa}, \ldots$

What is it good for?

- **Large Deviation Principle**: Rate function of $(\mu_\varepsilon)_{\varepsilon>0}$ is $\frac{1}{2}\|\cdot\|^2_{\mathscr{H}}$
- **Cameron-Martin Theorem**: $\mu(\cdot) \sim \mu(\cdot + h) \Leftrightarrow h \in \mathscr{H}$
- **Malliavin Calculus**: $\mathscr{H}$-derivative is non-degenerate $\Rightarrow$ Wiener functional has density
- **Support Theorems**: $\operatorname{supp} \mu = \overline{j(\mathscr{H})}$
- **Structure theorem of Gaussian measures**: every Gaussian measure arises through an AWS and is characterized by $\mathscr{H}$

Brave New World:

Gaussian Stochastic Analysis $\rightarrow$ Rough Paths and Regularity Structures:

Theorem/Theory	**Classical**	**RP & Reg. Structures**
Large Deviation Principles	e.g. [5, Sec. 3.4]	[8], [11], [6], [10]
Cameron-Martin Theorem	e.g. [1, Sec. 4.2]	[8, Sec. 15.8]
Malliavin Calculus	e.g. [12]	[3], [13], [2]
Support Theorem	e.g. [7, Sec. 9.3]	[4], [9], [8, Sec. 15.8]
Underlying Structure	Abstract Wiener Spaces	???

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Underlying Structure	Abstract Wiener Spaces	**???**

Abstract Wiener Model Spaces

General Setup

- an **ambient space**: a separable Banach space $\mathbf{E} := \bigoplus_{\tau\in\mathcal{W}} E_\tau$, graded over a finite set $\mathcal{W}$ with a function $[\cdot] : \mathcal{W} \to \mathbb{N}_{\geq 1}$ (state space of enhanced noise).
  Comes with projection maps

$$\pi_\tau : \mathbf{E} \to E_\tau \quad \text{and} \quad \pi^{(k)} : \mathbf{E} \to \bigoplus_{\substack{\tau\in\mathcal{W} \\ [\tau]=k}} E_\tau \tag{3}$$

  and scaling and homogeneous norm

$$\delta^\varepsilon(\mathbf{x}) = \sum_{\tau\in\mathcal{W}} \varepsilon^{[\tau]}\pi_\tau(\mathbf{x}) \quad \text{and} \quad |||\mathbf{x}|||_{\mathbf{E}} = \sum_{\tau\in\mathcal{W}} \|\pi_\tau \mathbf{x}\|_\tau^{\frac{1}{[\tau]}}, \quad \mathbf{x}\in\mathbf{E}\,. \tag{4}$$

- an abstract Wiener space $(\pi^{(1)}(\mathbf{E}), \mathscr{H}, j, \mu)$ (state space of noise)
- a μ-a.s. defined measurable lift $\hat{\mathfrak{L}} : E \to \mathbf{E}$ (enhancement)

Two Philosophies

Naively one would like to define $\mathbf{H} := \hat{\mathfrak{L}}|_{\mathscr{H}}(\mathscr{H})$, but $\mu(\mathscr{H}) = 0$ if $\dim(\mathscr{H}) = \infty$.

$\Rightarrow \hat{\mathfrak{L}}|_{\mathscr{H}}$ depends on the μ-a.s. version of $\hat{\mathfrak{L}}$.

Two options:

Bottom-Up:

Define $\mathfrak{L}$ on $\mathscr{H}$ and extend to E

Top-Down:

Define $\hat{\mathfrak{L}}$ on E and proxy-restrict to $\mathscr{H}$

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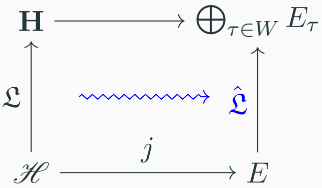

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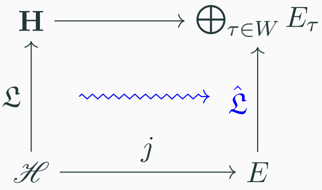

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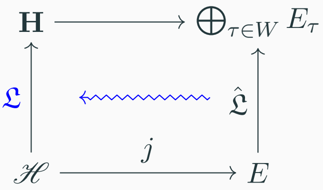

Definition of Abstract Wiener Model Spaces

Definition
An **abstract Wiener Model Space** $(\mathbf{E}, \mathbf{H}, \iota, \boldsymbol{\mu}, \mathfrak{L}, \hat{\mathfrak{L}})$ consists of

1. an **ambient space** $(\mathcal{W}, \mathbf{E}, [\cdot])$
2. a separable Hilbert space $\mathbf{H}$ with a continuous (non-linear) injection $\iota : \mathbf{H} \hookrightarrow \mathbf{E}$, called **enhanced Cameron-Martin space**.
3. a Borel probability measure $\boldsymbol{\mu}$ on $\mathbf{E}$ such that $\mu := \pi^{(1)}_* \boldsymbol{\mu}$ is centred Gaussian on E such that $\mathscr{H} := \pi^{(1)}(\mathbf{H})$ is the Cameron-Martin space associated to μ
4. a continuous two-sided inverse $\mathfrak{L} : \mathscr{H} \to \iota(\mathbf{H})$ of $\pi^{(1)}|_{\iota(\mathbf{H})}$, called **skeleton lift**
5. a measurable μ-almost surely right-inverse $\hat{\mathfrak{L}}$ of $\pi^{(1)}$, called **full lift** such that $\hat{\mathfrak{L}}_* \mu = \boldsymbol{\mu}$ and for every $\tau \in \mathcal{W}$ the measurable map $\pi_\tau \circ \hat{\mathfrak{L}} : E \to E_\tau$ lies in $\mathcal{C}^{(\leq[\tau])}(E, \mu; E_\tau)$.

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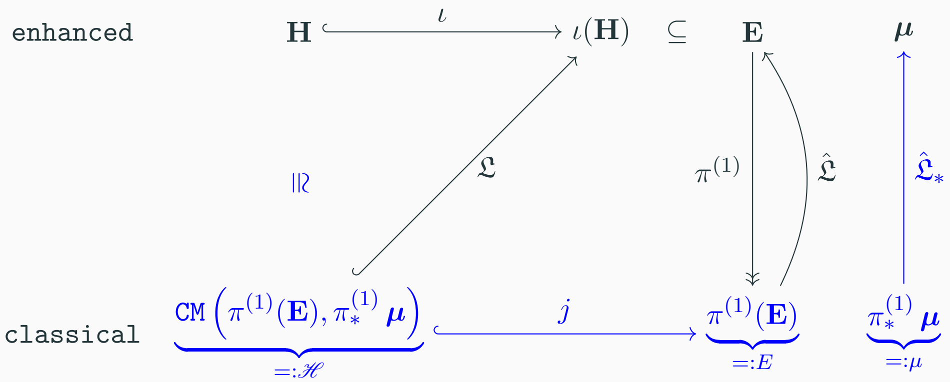
Definition of Abstract Wiener Model Spaces - Diagram



$\Rightarrow$ How to construct AMWS?

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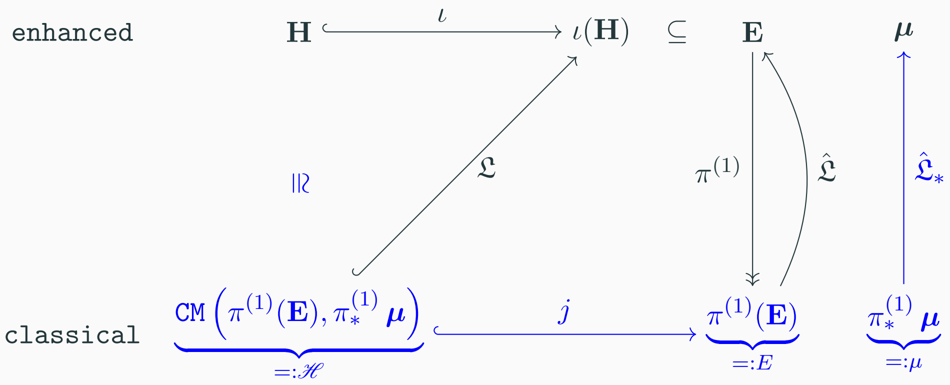


$\Rightarrow$ How to construct AMWS?

Bottom-Up Construction

Theorem
Let $(\mathcal{W}, \mathbf{E}, [\cdot])$ *be an ambient space, with* $E = E^{(1)}$ *s.t.* $(E, \mathscr{H}, \mu)$ *is an abstract Wiener space. let* $\mathfrak{L} : \mathscr{H} \to \mathbf{E}$ *be a skeleton lift and let* $(\Phi_m)_{m\in\mathbb{N}} : E \to \mathscr{H}$ *be an admissible approximation s.t.*

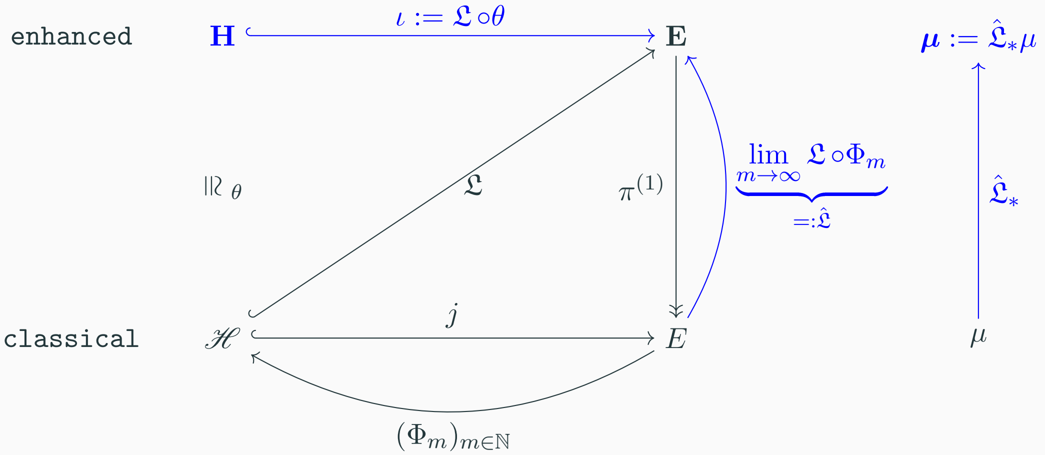
$$\pi_\tau \circ \mathfrak{L} \circ \Phi_m \in \mathcal{C}^{(\leq[\tau])}(E, \mu; E_\tau), \quad m \in \mathbb{N}, \tau \in \mathcal{W} \tag{5}$$

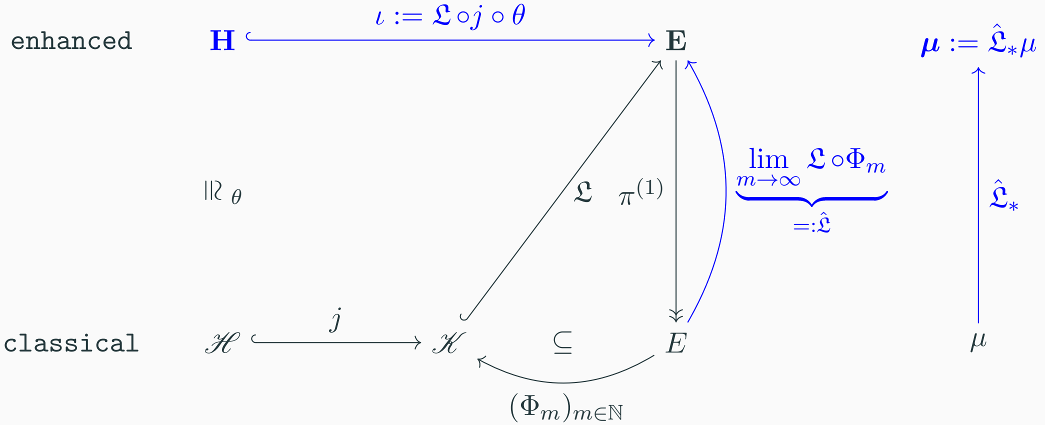
where $\mathcal{C}^{(\leq[\tau])}(E, \mu; E_\tau)$ *denotes the inhomogeneous* E_τ*-valued Wiener-Ito-Chaos of order* $[\tau]$*.*

Then the following data constitutes an AWMS:

1. *the ambient space* $\mathbf{E}$
2. *the skeleton lift* $\mathfrak{L} : \mathscr{H} \to \mathbf{E}$
3. *a separable Hilbert space* $\mathbf{H} \cong \mathscr{H}$ *together with* $\iota = \mathfrak{L}$
4. $\boldsymbol{\mu} := \hat{\mathfrak{L}}_* \mu$*, where*
5. $\hat{\mathfrak{L}} := \lim_{m\to\infty} \mathfrak{L} \circ \Phi_m$.

Bottom-Up Construction: Diagram




enhanced
H
ι := 𝔏∘j∘θ
E
μ := 𝔏̂*μ
≀ θ
𝔏
π(1)
lim m→∞ 𝔏∘Φm
=:𝔏̂
𝔏̂*
classical
ℋ
j
𝒦
⊆
E
μ
(Φm)m∈ℕ

Theorem (LDP)
Let $(\mathbf{E}, \mathbf{H}, \iota, \boldsymbol{\mu}, \mathfrak{L}, \hat{\mathfrak{L}})$ *be the AWMS obtained by the Bottom-Up construction via the data* $\mathbf{E}, (E, \mathscr{H}, \mu), \mathfrak{L}, (\Phi_m)_{m\in\mathbb{N}}$ *and recall the natural scaling*

$$\delta^{\varepsilon}\,\mathbf{x} = \sum_{\tau\in\mathcal{C}} \varepsilon^{[\tau]}\pi_{\tau}\,\mathbf{x}, \quad \varepsilon > 0, \tau \in \mathcal{W}. \tag{6}$$

Then the family $(\delta^{\varepsilon}_{*}\,\boldsymbol{\mu})_{\varepsilon>0}$ *satisfies an LDP on* $\mathbf{E}$ *with good rate function*

$$\mathscr{J}(\mathbf{x}) = \begin{cases} \frac{1}{2}\|\pi^{(1)}(\mathbf{x})\|^2_{\mathscr{H}} & , \quad \mathbf{x} \in \mathfrak{L}(\mathscr{H}) \\ +\infty & , \quad \textit{else.} \end{cases} \tag{7}$$

Theorem (Exp. Integrability)
Let $(\mathbf{E}, \mathbf{H}, \iota, \boldsymbol{\mu}, \mathfrak{L}, \hat{\mathfrak{L}})$ *be the AWMS obtained by the Bottom-Up construction via the data* $\mathbf{E}, (E, \mathscr{H}, \mu), \mathfrak{L}, (\Phi_m)_{m\in\mathbb{N}}$*. Then the measurable function* $\left|\!\left|\!\left| \hat{\mathfrak{L}} \right|\!\right|\!\right|_{\mathbf{E}}$ *is exponentially integrable, in the sense that there exists an* $\eta_0 > 0$ *s.t.*

$$\mathbb{E}\left[\exp\left(\eta \left|\!\left|\!\left| \hat{\mathfrak{L}} \right|\!\right|\!\right|_{\mathbf{E}}^2\right)\right] = \int_{\mathbf{E}} e^{\eta |||\mathbf{x}|||_{\mathbf{E}}^2} d\,\boldsymbol{\mu}(\mathbf{x}) < \infty, \quad \forall \eta < \eta_0. \tag{8}$$

Furthermore,

$$\eta_0 := \inf\left\{\frac{1}{2}\|\pi^{(1)}\mathbf{h}\|_{\mathscr{H}}^2 : \mathbf{h} \in \mathbf{H}, |||\mathbf{h}|||_{\mathbf{E}} > 1\right\}. \tag{9}$$

Theorem (CM Theorem)
Let $(\mathbf{E}, \mathbf{H}, \iota, \boldsymbol{\mu}, \mathfrak{L}, \hat{\mathfrak{L}})$ *be the AWMS obtained by the Bottom-Up construction via the data* $\mathbf{E}, (E, \mathscr{H}, \mu), \mathfrak{L}, (\Phi_m)_{m\in\mathbb{N}}$. *Then for every* $\mathbf{h} \in \mathbf{H}$

$$\underbrace{(\mathbf{T_h})_* \boldsymbol{\mu}}_{=:\boldsymbol{\mu}_\mathbf{h}} \approx \boldsymbol{\mu}\,. \tag{10}$$

where $\mathbf{T} : \hat{\mathfrak{L}}(E) \to \hat{\mathfrak{L}}(E)$ *is an appropriate shift operator. Furthermore, the density has the form*

$$\frac{\mathrm{d}\boldsymbol{\mu}_\mathbf{h}}{\mathrm{d}\boldsymbol{\mu}}(\mathbf{x}) = \exp\left(\mathfrak{C}^{-1}\left(\pi^{(1)}\mathbf{h}\right)\left(\pi^{(1)}\,\mathbf{x}\right) - \frac{1}{2}\|\pi^{(1)}\mathbf{h}\|^2_{\mathscr{H}}\right), \quad \mathbf{x} \in \mathbf{E}, \tag{11}$$

where $\mathfrak{C}$ *is the covariance operator associated to* μ.

Top-Down Construction

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Let $(\mathcal{W}, \mathbf{E}, [\cdot])$ be an ambient space, $\boldsymbol{\mu}$ a Borel measure on $\mathbf{E}$ s.t. $\mu := \pi^{(1)}_* \boldsymbol{\mu}$ is a centred Gaussian measure on E, a measurable μ-a.e. right-inverse $\hat{\mathfrak{L}} : E \to \mathbf{E}$ of $\pi^{(1)}$ s.t. $\hat{\mathfrak{L}}_* \mu = \boldsymbol{\mu}$ and

$$\pi_\tau \circ \hat{\mathfrak{L}} : E \to E_\tau \in \mathcal{C}^{(\leq [\tau])}(E, \mu; E_\tau), \quad \tau \in \mathcal{W}. \tag{12}$$

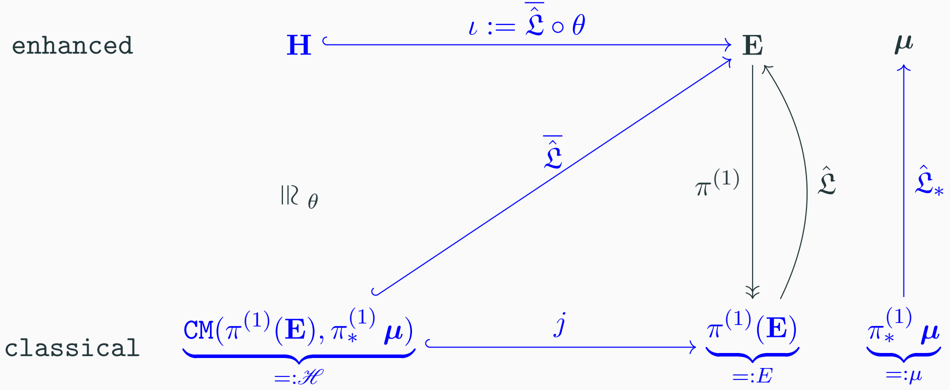
Then the following data constitutes an AWMS:

1. the ambient space $\mathbf{E}$
2. a separable Hilbert space $\mathtt{CM}(\pi^{(1)}(\mathbf{E}), \pi^{(1)}_* \boldsymbol{\mu})$ together with $\iota = \overline{\hat{\mathfrak{L}}}$
3. the measure $\boldsymbol{\mu}$
4. the skeleton lift $\mathfrak{L} := \overline{\hat{\mathfrak{L}}}$
5. the full lift $\hat{\mathfrak{L}}$.

where $\mathtt{CM}(\pi^{(1)}(\mathbf{E}), \pi^{(1)}_* \boldsymbol{\mu})$ denotes the Cameron-Martin space associated to the Gaussian measure $\pi^{(1)}_* \boldsymbol{\mu}$ on E.

Top-Down Construction: Diagram

Produce restriction of $\hat{\mathfrak{L}}$ by perturbing on $\mathscr{H}$.



Top-Down Construction: Proxy-Restriction

Recall that $\pi_\tau \circ \hat{\mathfrak{L}} : E \to E_\tau$ lies in $\mathcal{C}^{(\leq[\tau])}(E,\mu;E_\tau)$. Denote by $\Pi_{[\tau]} : \mathcal{C}^{(\leq[\tau])}(E,\mu;E_\tau) \to \mathcal{C}^{([\tau])}(E,\mu;E_\tau)$ the projection onto the top-chaos.

Define proxy-restriction $\hat{\mathfrak{L}} \mapsto \overline{\hat{\mathfrak{L}}}$ as

$$\overline{\hat{\mathfrak{L}}}(h) := \mathbb{E}\left[\left(\Pi_{[\tau]}\hat{\mathfrak{L}}\right)(\cdot + h)\right], \quad h \in \mathscr{H} .$$

Assume $\hat{\mathfrak{L}}(\cdot) = H_\alpha(\cdot) = \prod_{\alpha_i \in \alpha} \mathsf{h}_{\alpha_i}(\langle e_i, \cdot \rangle)$ some Hermite polynomial. Then for any $h \in \mathscr{H}$

$$\mathsf{h}_{\alpha_i}(\langle e_i, x + h\rangle) = \sum_{k=0}^{n} \binom{n}{k} \underbrace{\mathsf{h}_{\alpha_i}(\langle e_i, x\rangle)}_{\substack{\mathbb{E}[\dots]=0 \\ \text{unless k = 0}}} \langle e_i, h\rangle^{n-k} \quad \Rightarrow \quad \mathbb{E}\left[\hat{\mathfrak{L}}(\cdot + h)\right] = \underbrace{\prod_{\alpha_i \in \alpha} \langle e_i, h\rangle^{\alpha_i}}_{\text{"leading part" of } \hat{\mathfrak{L}}}$$

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Consistency

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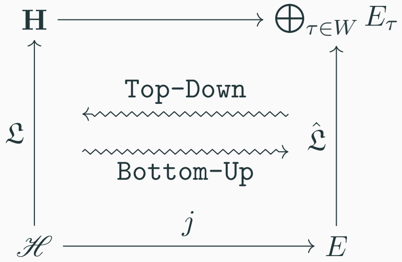
Recall that classically:

$$\mathscr{H} \rightsquigarrow (E, j, \mu) \quad \texttt{L. Gross} \tag{13}$$

$$(E, \mu) \rightsquigarrow (\mathscr{H}, j) \quad \texttt{X. Fernique/H. Satô} \tag{14}$$

In the enhanced setting:



Theorem (Consistency)

Let $(\mathbf{E}, \mathbf{H}, \boldsymbol{\mu}, \mathfrak{L}, \hat{\mathfrak{L}})$ *denote the AWMS obtained by the Bottom-Up construction via* $(\mathcal{W}, \mathbf{E}, [\cdot]), (E, \mathscr{H}, \mu), \mathfrak{L}, (\Phi_m)_{m\in\mathbb{N}}$.

Let $\underline{\mathbf{H}}$ *and* $\overline{\hat{\mathfrak{L}}}$ *be the enhanced Cameron-Martin space and the skeleton lift produced by applying the Top-Down construction to the AWMS obtained through the Bottom-Up construction.*

Then

$$\overline{\hat{\mathfrak{L}}} = \mathfrak{L} \quad \textit{and} \quad \underline{\mathbf{H}} = \mathbf{H}\,. \tag{15}$$

Applications

Application of Bottom-Up: Gaussian Rough Paths

Let $X = (X_1, \dots, X_d)$ be a centred, cont. Gaussian process with independent components of finite ρ-variation, $p > 2\rho$. Define

$$E = \bigoplus_{i=1}^{d} C^{0,p-\text{var}}([0,T];\mathbb{R}), \quad \mathscr{H} := \bigoplus_{i=1}^{d} \mathscr{H}^{X_i}, \quad \mu(\cdot) = \mathbb{P}(X \in \cdot)$$

$$W^{(1)} = \{1,\dots,d\}, \quad W^{(2)} = \{ij : 1 \le i,j \le d\}, \quad W^{(3)} = \{ijk : 1 \le i,j,k \le d\}$$

$$E_\tau = C^{0,\frac{p}{[\tau]}-\text{var}}([0,T]^2,\mathbb{R}), \quad \Phi_m = \sum_{i=0}^{m} \langle e_i, \cdot \rangle e_i \text{ ... Karhunen–Loève approx.}$$

(Using the shuffle relations) define for every $1 \le i,j,k \le d$, $0 \le s,t \le T$

$\left[\mathfrak{L}_i^{\text{GRP}}(h)\right](t) = h^i(t)$	$\left[\mathfrak{L}_{iij}^{\text{GRP}}(h)\right](s,t) = \int_s^t (h^i(s,r))^2 dh^j(r)$
$\left[\mathfrak{L}_{ij}^{\text{GRP}}(h)\right](s,t) = \int_s^t h^i(s,r) dh^j(r)$	$\left[\mathfrak{L}_{iji}^{\text{GRP}}(h)\right](s,t) = [\mathfrak{L}_{ij}(h)](s,t) \cdot [\mathfrak{L}_i(h)](s,t) - 2[\mathfrak{L}_{iij}(h)](s,t)$
$\left[\mathfrak{L}_{ii}^{\text{GRP}}(h)\right](s,t) = \frac{1}{2}(h^i(s,t))^2$	$\left[\mathfrak{L}_{jii}^{\text{GRP}}(h)\right](s,t) = [\mathfrak{L}_{ii}(h)](s,t) \cdot [\mathfrak{L}_j(h)](s,t) - [\mathfrak{L}_{iji}(h)](s,t) - [\mathfrak{L}_{iij}(h)](s,t)$
$\left[\mathfrak{L}_{ijk}^{\text{GRP}}(h)\right](s,t) = \int_s^t \int_s^r h^i(s,u) dh^j(s,u) dh^k(r)$	$\left[\mathfrak{L}_{iii}^{\text{GRP}}(h)\right](s,t) = \frac{1}{6}(h^i(s,t))^3$

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$$\left[\mathfrak{L}_{ij}^{\mathsf{GRP}}(h)\right](s,t) = \int_s^t h^i(s,r)dh^j(r)$$

$$\left[\mathfrak{L}_{ii}^{\mathsf{GRP}}(h)\right](s,t) = \tfrac{1}{2}(h^i(s,t))^2$$

$$\left[\mathfrak{L}_{ijk}^{\mathsf{GRP}}(h)\right](s,t) = \int_s^t \int_s^r h^i(s,u)dh^j(s,u)dh^k(r)$$

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$$\left[\mathfrak{L}_{jii}^{\mathsf{GRP}}(h)\right](s,t) = [\mathfrak{L}_{ii}(h)](s,t)\cdot[\mathfrak{L}_j(h)](s,t) - [\mathfrak{L}_{iji}(h)](s,t) - [\mathfrak{L}_{iij}(h)](s,t)$$

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Application of Top-Down: Rough Volatility with β-fractional Brownian motion

Let $\beta \in (0,1)$ and $K^\beta(t) := \sqrt{2\beta}\, t^{\beta-\frac{1}{2}} 1_{\{t>0\}}$ be the Volterra kernel and $\widehat{W} = K^\beta * \xi$. Define

$$E = \mathcal{C}^{0,-\frac{1}{2}}([0,T];\mathbb{R}), \quad \mathscr{H} = L^2([0,T]), \quad \mu = \mathbb{P}(\xi \in \cdot)$$

$$W^{(1)} = \{\Xi\}, \quad W^{(i)} = \{\Xi\mathcal{I}(\Xi)^{i-1}, \mathcal{I}(\Xi)^i\}, \quad i = 2,\dots,M_\beta$$

$$E_\tau = \overline{\{f : [0,T]^2 \to \mathbb{R} \text{ smooth}\}}^{\|\cdot\|_{E_\tau}}, \quad \|f\|_{E_\tau} := \sup_{\lambda,\varphi,s} \lambda^{-[\tau]} \left|\langle f_s, \varphi_s^\lambda\rangle\right|$$

The proxy restriction is given by

$$\left[\mathfrak{L}^{\mathsf{RV}}_{\Xi}(h)\right]_s(\cdot) = h.$$

$$\left[\mathfrak{L}^{\mathsf{RV}}_{\mathcal{I}(\Xi)^m}(h)\right]_s(\cdot) = ((K^\beta * h)_{s,\cdot})^m$$

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Thank you!

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