Self-repelling Brownian polymer $(X_t)_{t\geq 0}$ in the critical dimension, $d = 2$

and the environment profile $(\eta_t)_{t\geq 0}$

Harry Giles
joint work with Giuseppe Cannizzaro

CIME, Cetraro

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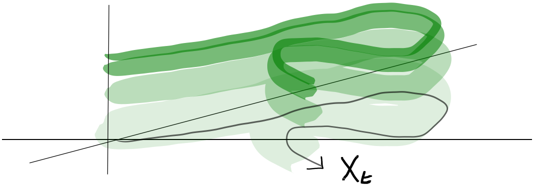
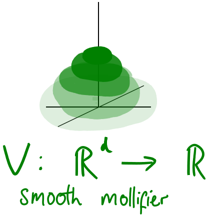
Defining SRBP. General dimensions $d \geq 2$

Consider $(X_t)_{t\geq 0}$ a continuous stochastic process on $\mathbb{R}^d$, mollifier $V : \mathbb{R}^d \to \mathbb{R}$, and mollified occupation field $x \mapsto \int_0^t V(x - X_s)ds$.





Let $(B_t)_{t\geq 0}$ a Brownian motion in $\mathbb{R}^d$. Define Self-repelling Brownian Polymer (SRBP) as solution $(X_t)_{t\geq 0}$ to, $X_0 = 0$

$$\mathrm{d}X_t = \mathrm{d}B_t - \Big(\int_0^t \nabla V(X_t - X_s)\mathrm{d}s\Big)\mathrm{d}t$$

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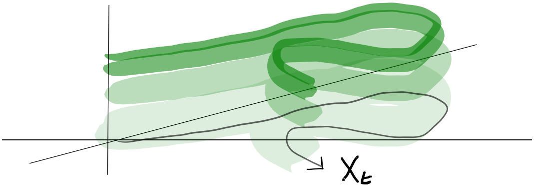

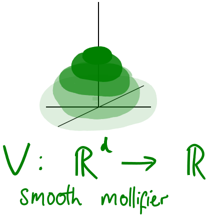

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Scaling limit

Let $\varepsilon > 0$. Consider diffusively rescaled $(X^\varepsilon{}_t)_{t\geq 0}$, where $X^\varepsilon{}_t = \varepsilon X_{t/\varepsilon^2}$.

$$\mathrm{d}X^\varepsilon{}_t = \mathrm{d}B^\varepsilon{}_t - \varepsilon^{d-2}\Big(\int_0^t \nabla V^\varepsilon(X^\varepsilon - X^\varepsilon{}_s)ds\Big)\mathrm{d}t$$

The mollifier $V^\varepsilon(x) := \varepsilon^{-d}V(\varepsilon x)$ is *sharper*. And $B^\varepsilon{}_t := \varepsilon B_{t/\varepsilon^2}$.

Conjecture. What happens as $\varepsilon \to 0$

$\mathbb{E}[\lvert X^\varepsilon{}_{t=1}\rvert^2] \sim 1$	$d \geq 3$
$\mathbb{E}[\lvert X^\varepsilon{}_{t=1}\rvert^2] \sim \sqrt{\lvert\log\varepsilon\rvert}$	$d = 2$
$\mathbb{E}[\lvert X^\varepsilon{}_{t=1}\rvert^2] \sim (1/\varepsilon)^{2/3}$	$d = 1$

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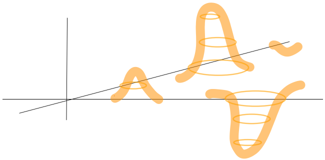
$$\mathbb{E}[|X^\varepsilon{}_{t=1}|^2] \sim \sqrt{|\log \varepsilon|} \qquad d = 2$$

$$\mathbb{E}[|X^\varepsilon{}_{t=1}|^2] \sim (1/\varepsilon)^{2/3} \qquad d = 1$$

Introduce potential ω, and α-weak coupling

Consider a smooth potential

$$\omega : \mathbb{R}^d \to \mathbb{R}$$



chosen independently at random from a *nice* space of potentials, $\Omega \subset C^\infty(\mathbb{R}^d, \mathbb{R})$.

From now on, critical dimension $d = 2$

For $\alpha > 0$, redefine the SRBP $(X_t)_{t \geq 0}$ as solution $X_0 = 0$

$$\mathrm{d}X_t = \mathrm{d}B_t - \left(\frac{\alpha}{\sqrt{|\log \varepsilon|}}\right)\left(\nabla\omega(X_t)\right)\mathrm{d}t + \left(\frac{\alpha}{\sqrt{|\log \varepsilon|}}\right)^2\left(\int_0^t \nabla V(X_t - X_s)ds\right)\mathrm{d}t$$

What happens to the diffusively scaled $({X^\varepsilon}_t)_{t \geq 0}$ under weak-coupling? For which $\alpha / \sqrt{|\log \varepsilon|} \to 0$ as $\varepsilon \to 0$

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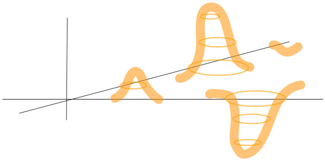

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Main theorem

Definition. Smoothed out Gaussian Free Field ($\sqrt{V} * \mathrm{GFF}$)

Let $(\Omega, \mathcal{F}, \pi)$ be the canonical probability space such that $(\omega(x) : x \in \mathbb{R}^2)$ is a Gaussian process

$$\int \omega(x)\omega(y)\pi(\mathrm{d}\omega) = V * g(x-y), \qquad g(x) := -\log|x|$$

Theorem (WIP).

Under the annealed law, $\pi \otimes \mathbb{P}$, and under α-weak coupling, we have as $\varepsilon \to 0$

$$({X^\varepsilon}_t)_{t \geq 0} \xrightarrow{d} BM(\sigma^2(\alpha))$$

with the explicit formula

$$\sigma^2(\alpha) = \sqrt{4\pi\alpha^2 + 1}$$

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Comparisons.

Strong coupling

- Strong coupling refers to absence of factors $\alpha/\sqrt{|\log \varepsilon|}$
- (Tóth, Valkó 2011) Under which SRBP is superdiffusive:

$$\log|\log\varepsilon| \lesssim \mathbb{E}[|X^{\varepsilon}{}_{t=1}|^2] \lesssim |\log\varepsilon|$$

- Constructing a limit for (X_t) under a *superdiffusive scaling*?
- For us, strong coupling corresponds to choice $\alpha = \sqrt{|\log\varepsilon|}$. This gives $\sigma^2(\alpha) \sim \sqrt{|\log\varepsilon|}$, non-rigourly suggesting the conjectured super-diffusivity.

High dimensions $d \geq 3$

(Horváth, Tóth, Vető, Bálint 2012) Under strong coupling, Gaussian limit obtained using "Kipnis-Varadhan" technique. Fails in $d = 2$

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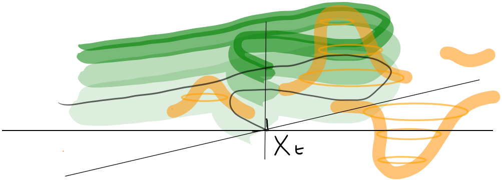
Standard techniques. Environment seen by the particle

Let $(\eta_t)_{t\geq 0}$ be Ω-valued process, for $x \in \mathbb{R}^2$



$$\eta_t(x) = \omega(x + X_t) + \frac{\alpha}{|\log \varepsilon|}\left(\int_0^t V(x + X_t - X_s)ds\right)$$

- Representation $X_t = B_t - \frac{\alpha}{|\log \varepsilon|}\int_0^t \nabla \eta_s(0)\mathrm{d}s$
- The process $(\eta_t)_{t\geq 0}$ is a stationary Markov process: for all $t \geq 0$, $\mathrm{Law}(\eta_t) = \sqrt{V} * \mathrm{GFF}$
- Drift term is **additive functional of a stationary Markov process**

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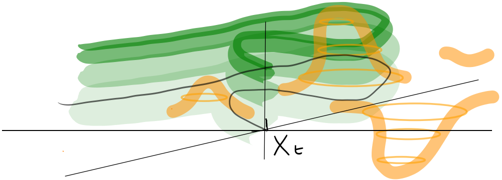

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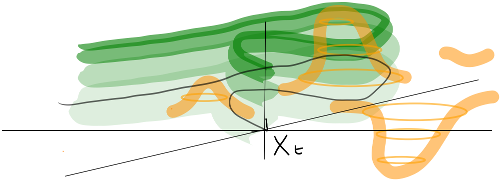

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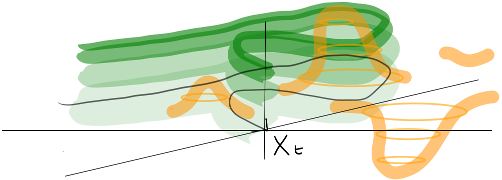

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Idea for studying $\int_0^t f(\eta_s) ds$

1. Solve generator equation $-\mathcal{L}u = f$
2. For any u, Dynkin martingale $M_t = u(\eta_t) - u(\eta_0) - \int_0^t \mathcal{L}u(\eta_s) ds$
3. Obtain CLT for $\varepsilon M_{t/\varepsilon^2} \approx \varepsilon \int_0^{t/\varepsilon^2} f(\eta_s) ds$

We call this the **Kipnis-Varadhan** program for obtaining CLT.

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Critical $d = 2$ techniques. Replacement Lemma.

- Want to solve $-\mathcal{L}u = f$, with $u \in L^2(\pi)$
- Extra structure: chaos decomposition $L^2(\pi) = \oplus_{n=0}^{\infty} \mathcal{H}_n$
- Generator respects the structure

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{A}_- + \mathcal{A}_+, \qquad \mathcal{L}_0 : \mathcal{H}_n \to \mathcal{H}_n, \qquad \mathcal{A}_\pm : \mathcal{H}_n \to \mathcal{H}_{n\pm 1}$$

- (Cannizzaro, Gubinelli, Toninelli 2023) We can find $\tilde{\mathcal{L}} \approx \mathcal{L}$

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- Solving "replacement equation" much easier. For us $f \in \mathcal{H}_1$

$$u_1 = -(\mathcal{L}_0 + \tilde{\mathcal{L}}_0)^{-1} f, \qquad u_n = -(\mathcal{L}_0 + \tilde{\mathcal{L}}_0)^{-1} \mathcal{A}_+ u_{n-1}, \qquad u_n \in \mathcal{H}_n$$

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Critical $d = 2$ techniques. Finding $\tilde{\mathcal{L}}_0$.

Benefits of choosing u solving $\tilde{\mathcal{L}}u = f$ (replacement equation)

- Using explicit form of the solution, we can close the estimates needed for Kipnis-Varadhan to succeed.
- The limiting diffusivity can be found explicitly!

Additional difficulty. We split $\mathcal{L}$ into two *regions*, $\mathcal{L} = \mathcal{L}^{\text{good}} + \mathcal{L}^{\text{bad}}$

$$\mathcal{L}^{\text{good}} = \mathcal{L}_0 + \mathcal{A}_-^{\text{good}} + \mathcal{A}_+^{\text{good}}, \qquad \mathcal{L}^{\text{bad}} = \mathcal{A}_-^{\text{bad}} + \mathcal{A}_+^{\text{bad}}$$

- We have $\mathcal{L}^{\text{good}} \approx \mathcal{L}_0 + \tilde{\mathcal{L}}_0 + \mathcal{A}_+^{\text{good}}$ satisfies a *replacement lemma*

$$\tilde{\mathcal{L}}_0 = \mathcal{L}_0\left(\sqrt{4\pi\alpha^2 \frac{|\log(\varepsilon^2 + \mathcal{L}_0)|}{|\log \varepsilon|} + 1} - 1\right)$$

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- For $\mathcal{L}^{\text{bad}}$, entirely different argument

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Summary

- The SRBP $(X_t)_{t\geq 0}$ is a non-Markovian process avoiding regions of space it previously occupied
- For $d = 2$ it is super-diffusive. To see something in the diffusive limit, we tune down the self-interaction as we zoom out (α-weak coupling).
- We obtain a central limit theorem using Kipnis-Varadhan
- Big changes needed to work in critical dimension
  - Splitting $\mathcal{L} = \mathcal{L}^{\text{good}} + \mathcal{L}^{\text{bad}}$
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