Scaling-supercritical SDEs with nearly divergence-free distributional drift

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Content

1. Introduction

- Consider a singular SDE $X\colon [0,T] \to \mathbb{R}^d$

$$dX_t = b(X_t)dt + dB_t,$$

where B is d-dimensional Brownian motion and $b \in \mathcal{S}'(\mathbb{R}^d)$ is a fixed **distribution**.

- We are interested in the case where b makes the equation **scaling-supercritical**, as for S(P)DEs in general this regime is not well-understood and the methods from Paracontrolled distrubutions [Gubinelli, Imkeller, Perkowski, '15], Regularity structures [Hairer '14] etc. do not apply

- Assume $\mathrm{div} \cdot b \approx 0$. Then, formally, Leb is an (quasi-) invariant measure and the infinitesimal generator has the form

$$\mathcal{L} = \Delta + b \cdot \nabla,$$

where Δ is symmetric, $b \cdot \nabla$ is (almost) anti-symmetric

- This structure allows results in the **supercritical regime** under certain conditions

- We consider

$$dX_t = b(X_t)dt + dB_t, \tag{1}$$

on $\mathbb{T}^d$ (for this talk), where B is a d-dimensional Brownian motion

- Helmholtz decomposition: For $b \in \mathcal{S}'(\mathbb{T}^d; \mathbb{R}^d)$ we can write

$$b_i = (b_{\text{div}-\text{free}})_i + (b_{\text{grad}})_i = \text{div}\,(A_i^b) + \partial_i V^b + \text{const.},$$

where $A_{ij}^b = \Delta^{-1}(\partial_j b^i - \partial_i b^j)$ is a (distributional) *antisymmetric* matrix field, $V^b = \Delta^{-1}\text{div}\cdot b \in \mathcal{S}'(\mathbb{T}^d)$

- The Kolmogorov backward equation for equation (1) formally is

$$\partial_t u = \Delta u + b\cdot\nabla u, \tag{2}$$

- Assume $\text{div}\, b = 0$ for simplicity. There are two ways of making sense of this equation:
  - i $\partial_t u = \Delta u + \text{div}\,(bu)$,
  - ii $\partial_t u = \Delta u + \text{div}\,(A\nabla u)$.

We assume

i.

$$b \in B^{-\gamma}_{p,\infty}\, \gamma < 1, p > \frac{2}{1-\gamma}, \tag{3}$$

OR

ii. $A \in L^p$ for some $p > 2$ and there exists a compact set K such that for $\varepsilon > 0$ with $B^\varepsilon = \{x\colon d(x, K) \leq \varepsilon\}$

$$\sup_{\varepsilon>0} \varepsilon^{-2} V(B^\varepsilon), \sup_{\varepsilon>0} \varepsilon^{-2} \int_{B^\varepsilon} |A|^2 < \infty, \tag{4}$$

and for all $\varepsilon > 0$,

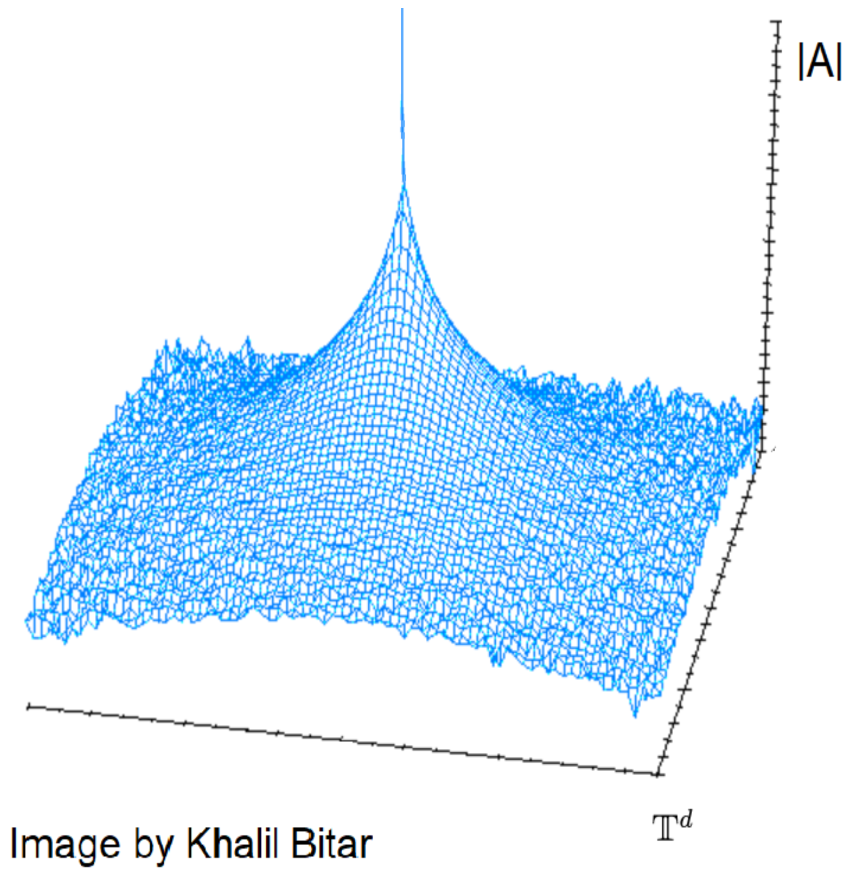
$$A 1_{B^c_\varepsilon} \in L^\infty. \tag{5}$$

- The treshhold for criticality is

$$p > \frac{d}{1-\gamma},$$

  thus, under assumption i, equation (1) and the singular PDE (2) can be posed in the **scaling-supercritical** regime for $d \geq 3$
- Assumption ii produces "proper" elements $A \in L^p, p > 2$ ($A \in L^\infty$ would be critical)

Assumptions on the drift



- Consider the classically well-posed approximation of (1)

$$dX_t^m = b^m(X_t^m)dt + dB_t, \tag{6}$$

where $b^m = \sum_{|k|\leq m} \hat{b}(k)e^{2\pi i\langle k,\cdot\rangle}$.

- In particular **no renormalization**, the limit is not expected to be Gaussian

Theorem 1. *Let $b \in \mathcal{S}'$ satisfy assumption i or ii and assume $\mathrm{div}\cdot b \in \mathcal{C}^{-1+}(\mathbb{T}^d)$. Then, for any $\eta \in L^2(\mathbb{T}^d)$ the law on $C([0,T],\mathbb{T}^d)$ of the solutions to equation (6) with $X_0^m \sim \eta d\lambda_{\mathbb{T}^d}$ converges to a unique limit, which is a Markov process. Its law is described by the solutions to the singular PDE (2).*

3. Proof: Existence

- Existence follows from a forward backward martingale argument (Itô trick):
- For $f \in C^2(\mathbb{T}^d)$, $\mathrm{div}\cdot b = 0$ (for simplicity) and $X_0^m \sim \mathrm{Leb}$,

$$f(X_t^m) - f(X_0^m) - \int_0^t b\cdot\nabla f(X_s^m) - \frac{1}{2}\Delta f(X_s^m) ds = \int_0^t \nabla f(X_s^m)\cdot dB_s,$$

and with $\hat{X}_t = X_{T-t}$

$$f(\hat{X}_t^m) - f(\hat{X}_0^m) + \int_0^t b\cdot\nabla f(X_s^m) - \frac{1}{2}\Delta f(X_s^m) ds = \int_0^t \nabla f(\hat{X}_s^m)\cdot d\hat{B}_s,$$

- Adding both equations, applying BDG and replacing $f \to \Delta^{-1} f$ yields

$$\mathbb{E}\left[\left|\sup_{0\le t\le T}\int_0^t f(X_s^m) ds\right|^p\right] \lesssim T^{p/2}\|(-\Delta)^{-1/2} f\|_{L^p}^p .$$

- Kolmogorov continuity criterion yields

Proof: Existence

Proposition 2. *Let* $b \in \mathcal{S}'$ *such that* $\nabla \cdot b \in \mathcal{C}^{-1+}$ *and* i *or* ii *hold. Then, the sequence* $(\mathrm{law}\,(X^m))_{m\in\mathbb{N}}$ *is tight on* $C([0,T];\mathbb{R}^d)$ *and any accumulation point satisfies*

$$\mathbb{E}\left[\left|\sup_{0\le t\le T}\int_0^t f(X_s)ds\right|^p\right] \lesssim T^{p/2}\|(-\Delta)^{-1/2}f\|_{L^p}^p\,. \tag{7}$$

Furthermore,

$$f(X_t) - f(X_0) - \lim_m \int_0^t (\mathcal{L}^m f)(X_s)\,ds\,,$$

is a martingale in the filtration generated by X *for each* $f \in \mathcal{S}$.

Proof: Uniqueness

- Assume again $\operatorname{div} b = 0$ for simplicity
- Consider the generators $\mathcal{L}^m = \Delta + b^m \cdot \nabla =: \mathcal{L}_0 + \mathcal{G}_m$ of the semigroups $S_t^m u(x_0) = \mathbb{E}_{x_0}[u(X_t^m)]$ with domains $\mathcal{D}(\mathcal{L}^m)$, where $\mathcal{L}_0$ is symmetric, $\mathcal{G}_m$ is antisymmetric w.r.t. L^2.
- The latter implies that $\|(1-\mathcal{L}^m)^{-1} u^\sharp\|_{H^1} \lesssim \|u\|_{H^{-1}}$, uniformly in m
- Using the Banach-Alaoglu theorem and a diagonal sequence argument on a countable dense subset of H^{-1} one can show that

$$(1-\mathcal{L}^{m_k})^{-1} \to \mathcal{K} \in L(H^{-1}, H^1),$$

  pointwise - weakly.
- Furthermore

$$\tilde{\mathcal{D}}(\mathcal{L}) := \mathcal{K} L^2 \subset H^1,$$

  is a domain for $\mathcal{L} = \Delta + \operatorname{div}(b\cdot)$, respectively $\mathcal{L} = \Delta + \operatorname{div}(A\nabla\cdot)$, on which it generates a contraction semigroup on L^2, i.e. solves the KBE

Proof: Uniqueness

- Under assumption i:
  - Show that *each* accumlation point X solves the martingale problem for *each* generator such $(\tilde{\mathcal{D}}(\mathcal{L}), \mathcal{L})$ which implies uniqueness of X by uniqueness of the martingale problem
  - key ingredients are the uniform bound $\|\mathcal{K}u^\sharp\|_{L^\infty} \leq \|u^\sharp\|_{L^\infty}$, which follows from the Markov property of the approximations, the Itô trick 7 and the interpolation estimate

$$\|b\cdot\nabla\|_{H^{-1}} \lesssim \|u\|_{B^{2/q}_{q,\infty}} \lesssim \|u\|_{L^\infty \cap H^1},$$

for a certain $2/q > \gamma$.

Proof: Uniqueness

- Under assumption ii:
  - Picking further subsequences one can show that each accumulation point X is a Markov process with a generator $(\tilde{\mathcal{D}}(\mathcal{L}), \mathcal{L})$ obtained by the diagonal sequence argument
  - Thus, uniqueness follows if we can show that

$$\tilde{\mathcal{D}}(\mathcal{L}) = \mathcal{D}_{\max} := \{u \in H^1 : \mathcal{L}u \in L^2\}$$

  - This can be reduced to the showing

$$u \in H^1, \mathcal{L}u = 0 \Rightarrow u = 0 \quad ,$$

    which turns out to be true

- $\mathbb{R}^d$
- time-dependent b, probably $b \in L_T^\infty B_{p,\infty}^{-\gamma}$, $p > \frac{2}{1-\gamma}$.
- general $b \in H^{-1}$? ($\exists$ counterexamples for a related, non-singular problem [Székelyhidi, Modena '18])
- infinite dimensions??
  - There are SPDEs with a similar structure (nice invariant measure, singularity contained in the anti-symmetric part) like the *Fractional Stochastic Burgers equation* $u: \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}$

$$\partial_t u = (-\Delta)^\theta + \partial_x u^2 + (-\Delta)^{\theta/2} \xi \,, \tag{8}$$

or *Fractional Stochastic Quasi-geostrophic equation*

$$\partial_t u = (-\Delta)^\theta + \nabla^\perp (-\Delta)^{-1/2} u \cdot \nabla u + (-\Delta)^{\theta/2} \xi \,,$$

for which we can show existence of unique non-Gaussian solutions in the critical regime for θ, but existence is known partially in the super-critical regime ([Gubinelli, Jara, 2013], [Tots, 2020]).

5. References

1. Gubinelli, M., Imkeller, P., & Perkowski, N. (2015). Paracontrolled distributions and singular PDEs. *Forum of Mathematics, Pi, 3*, E6.
2. Hairer, M. A theory of regularity structures. *Invent. math.* **198**, 269–504 (2014).
3. Modena, S., Székelyhidi, L. Non-uniqueness for the Transport Equation with Sobolev Vector Fields. *Ann. PDE* **4**, 18 (2018).
4. Perkowski, N., Graefner, G. (2023). MPI Summer School "PDEs and Randomness", lecture notes. https://www.mis.mpg.de/fileadmin/pdf/slides_randompde_5751.pdf