A priori bounds for 2-d gPAM

Guilherme de Lima Feltes
University of Münster

CIME Summer School, Cetraro – September 11-15, 2023

(ongoing) work with Ajay Chandra (London) & Hendrik Weber (Münster)

The general equation:

$$\begin{cases} (\partial_t - \Delta) u = \sigma(u)\xi & \text{in } (0,\infty)\times\mathbb{T}^d \\ u = u_0 & \text{on } \{t=0\}\times\mathbb{T}^d \end{cases} \qquad (1)$$

$\sigma : \mathbb{R} \to \mathbb{R}$ smooth, ξ random distribution over $\mathbb{T}^d$ or $\mathbb{R}_+ \times \mathbb{T}^d$

of negative regularity

Goal: establish global (in time) solutions to (1)

The ~~generalised~~ Parabolic Anderson Model (gPAM)

$(\partial_t - \Delta) u = \sigma(u)\, \xi$, ξ spatial white noise on $\mathbb{T}^2$

- If $\sigma(u) = u$, Continuous Parabolic Anderson Model (PAM)

random motions on random potential, directed polymers, trapping of random paths, branching processes in random medium, Anderson localisation, etc..

[Hairer'14], [Gubinelli-Imkeller-Perkowski'15]

$\rightsquigarrow$ Global in time solutions u on $\mathbb{T}^2$, $u_0 \in C^\alpha$

[Hairer-Labbé'18] —"— on $\mathbb{R}^3$, $u_0 = \delta_x$

The generalised Parabolic Anderson Model (gPAM)

$$(\partial_t - \Delta) u = \sigma(u)\xi \, , \quad \xi \text{ spatial white noise on } \mathbb{T}^2$$

- $\sigma : \mathbb{R} \to \mathbb{R}$ smooth   Nonlinear generalisation of PAM

generic description of (nonlinear) evolution of particles in a random stationary medium

[Hairer'14], [Gubinelli-Imkeller-Perkowski'15]

$\rightsquigarrow$ only provide short time (local) solutions u.

Parab. Stoch. quantisation of Sine-Gordon EQFT (S-G)

$$\begin{cases} (\partial_t - \Delta)\Phi = \sin(\beta\Phi) + \xi & \text{in } (0,\infty)\times\mathbb{T}^2 \\ \Phi = \Phi_0 & \text{on } \{t=0\}\times\mathbb{T}^2 \end{cases} \quad (2)$$

ξ space-time white noise on $\mathbb{R}_+\times\mathbb{T}^2$, $\beta > 0$ (Temp^{-1})

2-d Sine-Gordon measure

$V(\Phi) = \cos(\beta\Phi)$ in $\nu(d\varphi) \propto e^{-\int V(\varphi)}\mu(d\varphi)$ ← 2-d GFF

⇝ Dynamics provides information on the measure

Parab. Stoch. quantisation of Sine-Gordon EQFT (S-G)

$$\boxed{(\partial_t - \Delta)\Phi = \sin(\beta\Phi) + \zeta} \rightsquigarrow \boxed{(\partial_t - \Delta) u = \sigma(u)\xi \quad (1)}$$

set $\Phi = \varphi + u$, $(\partial_t - \Delta)\varphi = \zeta \Rightarrow u = \Phi - \varphi$ solves (1)

$$\sigma(u)\xi = \frac{-i}{2}\Big(\underbrace{e^{i\beta u}}_{\sigma_+(u)}\underbrace{e^{i\beta\varphi}}_{\xi_+} - \underbrace{e^{-i\beta u}}_{\sigma_-(u)}\underbrace{e^{-i\beta\varphi}}_{\xi_-}\Big), \quad \xi_\pm = e^{\pm i\beta\varphi}$$

[Chandra-Hairer-Shen '18] $\rightsquigarrow$ local sol. for $\beta^2 \in (0, 8\pi)$ $\left[\xi_\pm \in C^{-\frac{\beta^2}{4\pi}}\right]$

[Hairer-Shen '16] $\rightsquigarrow$ "Da Prato-Debussche regime" $\beta^2 \in (0, 4\pi)$

Some definitions

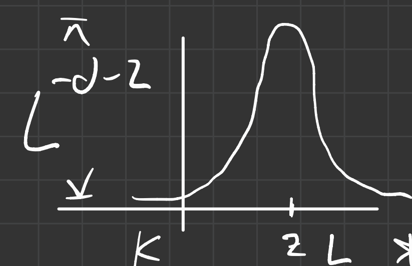
$$D := [0,1]\times\mathbb{T}^d \,; \quad D_a^b := [a,b]\times\mathbb{T}^d \,; \quad D \ni z = (t,x)$$

$$|z-\bar z| = \max\{|t-\bar t|^{1/2}, |x-\bar x|\}$$ "parabolic distance"

$$B(z,L) := \{\bar z\in D \,/\, |z-\bar z|<L,\ \bar t<t\}$$ "only past"

$\xi\in C^{-\alpha}$, $\alpha>0$, if $$[\xi]_{-\alpha} := \sup_{z\in D}\sup_{L<1} |\langle\xi,\varphi_z^L\rangle|\, L^\alpha$$

$$\varphi_z^L(\bar z) := L^{-d-2}\,\varphi\left(\frac{\bar t - t}{L^2}, \frac{\bar x - x}{L}\right),$$



Main Result: $\exists\, 0 < \bar{\kappa} << 1/3$ (independent of everything here), such that for any $0 < \kappa \leq \bar{\kappa}$, if $\xi \in C^{-1-\kappa}$ can be lifted to a model in the sense of [Hairer'14], $\sigma \in C_b^2(\mathbb{R})$ and $u_0 \in L^\infty(\mathbb{T}^d)$, then

$$\boxed{\|u\|_{D_0^1} \leq C(d, \kappa, C_\sigma, C_\xi) \cdot \|u_0\| \wedge 1}$$

Here $C_\sigma = \max\{\|\sigma\|, \|\sigma'\|, \|\sigma''\|\}$, $C_\xi = \max\{[\bullet]_{-1-\kappa}, [\,{}^{\bullet}_{|}\,]_{1-\kappa}, \dots\}$

$\rightsquigarrow$ Include 2-d gPAM ($\kappa > 0$) & S-G ($\beta^2 \in (4\pi, (1+\bar{\kappa})4\pi)$)

Applications

- 2-d gPAM $\quad \xi \in C^{-\frac{d}{2}-}$ model by [Hairer '14]

$\Rightarrow$ any $\kappa > 0$ works for $d=2$ and only $\kappa > 1/2$ for $d=3$ $\ddot{\frown}$

- S-G $\beta^2 \in (4\pi, (1+\frac{1}{2})4\pi)$ $\quad \xi_\pm = e^{\pm i\beta\varphi} \in C^{-\frac{\beta^2}{4\pi}}$,

since $\quad -\frac{\beta^2}{4\pi} = -1-\kappa > -1-\bar{\kappa}$.

model by [Chandra-Hairer-Shen '18]

Why difficult?

$$\sigma(\overset{1-\kappa}{u})\,\overset{-1-\kappa}{\xi}$$

$\rightsquigarrow$ need somehow $\quad u \longrightarrow U \in C^\gamma \qquad \gamma > 1+\kappa$

use model $\qquad \sigma(u) \longrightarrow \sigma(U) \in C^\gamma$

Chain rule: $F, f \in C^2(\mathbb{R})$, $F(f)'' = F''(f)\,f'^2 + F'(f)\,f''$

Problem: $[\sigma(U)]_\gamma \leq [U]_\gamma \, \|U\|^{\gamma-1} + \dots$

Strategy:

$$\begin{cases} (\partial_t - \Delta) u = \sigma(u)\xi \\ u|_{t=0} = u_0 \end{cases}$$

For any $C_\star > 0$, $\exists\, T_\star = T_\star(C_\star)$

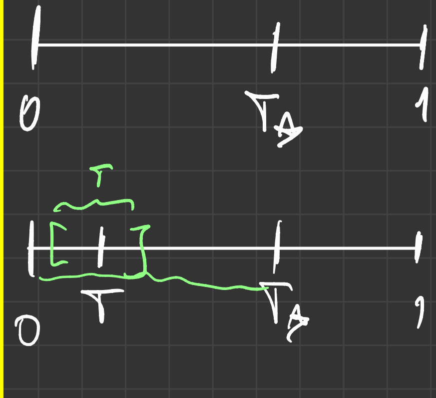
$$T_\star := \inf\{ t \geq 0 \,/\, \|u\|_{D_0^t} \geq C_\star(\|u_0\| \wedge 1) \}$$

$$\Rightarrow \|u\|_{D_0^{T_\star}} \leq C_\star(\|u_0\| \wedge 1) =: \check{C}_\star$$

Goal: Show $T_\star \geq 1$ by taking $C_\star$ big enough

Strategy:

$$\begin{cases} (\partial_t - \Delta) u = \sigma(u)\,\xi \\ u|_{t=0} = u_0 \end{cases}$$



1) Interior estimate with blow up in an interval of size T small enough;

Reconstruction + Schauder with blow up

Strategy:

$$\begin{cases}(\partial_t-\Delta)u=\sigma(u)\xi\\ u|_{t=0}=u_0\end{cases}$$

1) Interior estimate with blow up in an interval of size T small enough;

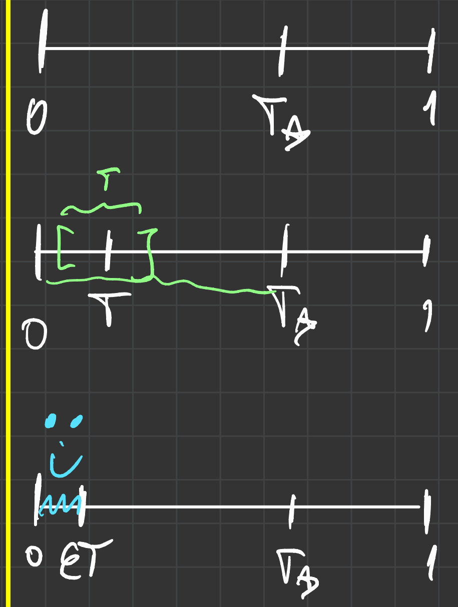
2) Post process to get $\|u\|_{D_0^{\varepsilon T}} \le 2\|u_0\| \wedge 1$ for ε small enough;

$\boxed{\text{indeped. of } C^{\vee}_{\star}}$

$$\begin{cases}(\partial_t-\Delta)u_1=0\\ u_1|_{t=0}=u_0\end{cases}$$

$$\begin{cases}(\partial_t-\Delta)u_2=\sigma(u_1+u_2)\xi\\ u_2|_{t=0}=0\end{cases}$$

extend u_2 to $[-\infty,1]\times\mathbb{T}^d$ and use Schauder without blow up



Strategy:

$$\begin{cases}(\partial_t-\Delta)u=\sigma(u)\xi\\ u|_{t=0}=u_0\end{cases}$$

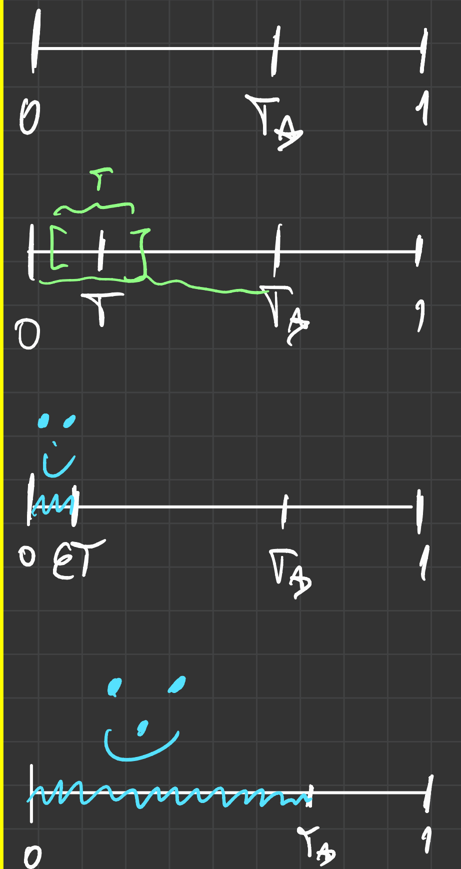
1) Interior estimate with blow up in an interval of size T small enough;

2) Post process to get $\|u\|_{D_0^{\varepsilon T}} \le 2\|u_0\|_{\infty}$ for ε small enough; $\boxed{\text{independ. of } C_*^\vee}$

3) Away from $t=0$, show a "maximum principle" for $z \in D_{\varepsilon T,1}^{T_*}$, which gives

$$\|u\|_{D_0^{T_*}} \le \|u\|_{D_0^{\varepsilon T}} + g(C_\sigma, C_\xi)\,(C_*^\vee)^\beta \qquad \boxed{\beta<1}$$

"the juice of the argument" $\beta=\beta(\bar{K})$



Close the deal

$$C_\star^{\vee} = C_\star \cdot (\|u_0\| \wedge 1)$$

$$\|u\|_{D_0^{T_\star}} \leq \partial(\|u_0\| \wedge 1) + g(C_\sigma, C_\xi)\left(C_\star \|u_0\| \wedge 1\right)^{\beta}$$

$$\leq C_\star(\|u_0\| \wedge 1)$$

$$\Longleftrightarrow \quad \frac{\partial}{C_\star} + \frac{g(C_\sigma, C_\xi)}{C_\star^{1-\beta}} \cdot \underbrace{(\|u_0\| \wedge 1)}_{\leq 1}^{\beta - 1} \leq 1$$

$\beta < 1$ implies result for $C_\star = C_\star(\partial, g(C_\sigma, C_\xi))$ big enough!

Thank You!