The Uniform Even Subgraph and the Ising Model

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(Joint work with Boris Kjær and Frederik Ravn Klausen)
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September 14th, 2023

Plan

- Introduce/recall main characters.
- Segue into our main tool.
- Warm-up exercise.
- Main Theorem.

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  (And first theorem.)
- Main Theorem.

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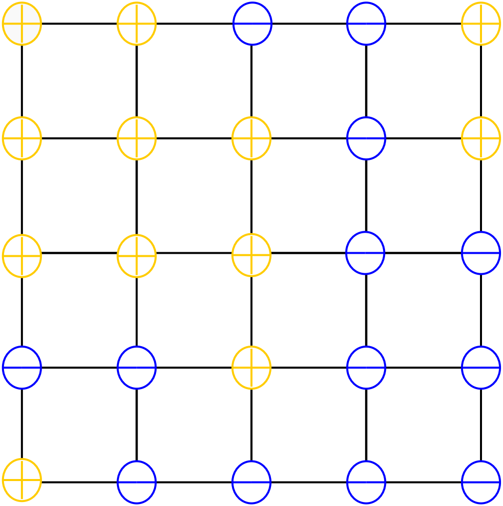
  (The Uniform Even Subgraph and Haar measures.)
- Warm-up exercise.
  (And first theorem.)
- Main Theorem.
  (And a sketch of proof).

Our main characters



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The Ising model at inverse temperature $\beta \geq 0$ is the measure on $\{-1, +1\}^V$ given by

$$\mathbf{I}_{\beta,G}[\sigma] = \frac{1}{Z_{\beta,G}} \exp\left(\beta \sum_{(v,w)\in E} \sigma_v \sigma_w\right)$$

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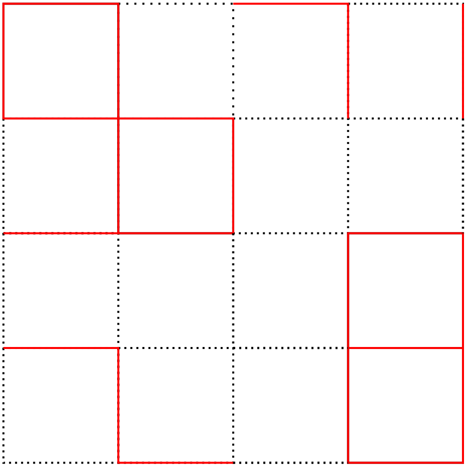

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$$\mathbb{P}_{\tilde{p},G} \preceq \phi_{p,G} \preceq \mathbb{P}_{p,G},$$

where $\tilde{p} = \frac{p}{2-p}$.

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Theorem (Peierls '36, Aizenman, Barsky, Fernandez '87)

For $d \geq 2$, there exists $\beta_c > 0$ such that

$$\langle \sigma_v, \sigma_w \rangle_{\beta,\mathbb{Z}^d} \begin{cases} \leq \exp\left(-C\|v-w\|\right) & \beta < \beta_c \\ \geq C & \beta > \beta_c \end{cases}$$

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For $d \geq 2$, there exists $p_c = p_c(\phi, d) \in (0,1)$ such that

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Aside

We also have a $p_c(\mathbb{P}, d)$.

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$p_c(\mathbb{P}, d) < \frac{1}{2}$ for $d \geq 3$.

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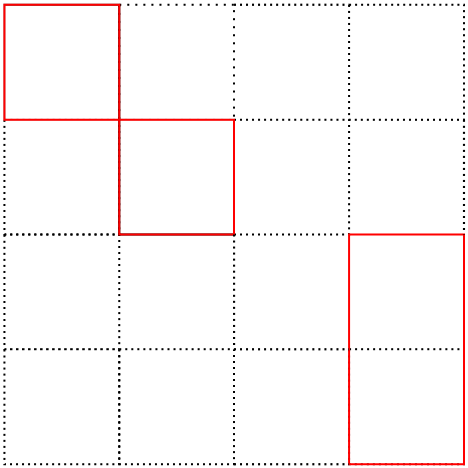

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A graph is even if the degree of every vertex is even.
No FKG!

The two-dimensional case

Theorem (Garet, Marchand, Marcovici '18)

On $\mathbb{Z}^2$, *there is an* x_c, *and it is equal to* $\tanh(\beta_c)$.

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Proof:

$$\phi_{p,G}[\mathtt{UEG}_\omega[\eta]] \propto \sum_{\omega \supseteq \eta} \frac{2^{\kappa(\omega)}}{|\{\text{Even subgraphs of } \omega\}|} \left(\frac{p}{1-p}\right)^{o(\omega)}$$

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$$= 2^{|V|} \sum_{\omega' \in \{0,1\}^{E \setminus \eta}} \left(\frac{p}{2(1-p)}\right)^{o(\omega)} \left(\frac{p}{2(1-p)}\right)^{o(\eta)}$$

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$$\Omega_\emptyset(\omega) := \{\text{Even subgraphs of } \omega\}.$$

$\mathtt{UEG}_\omega$ is the Haar measure on $\Omega_\emptyset(\omega)$.

Edge separation

For a graph $G = (V, E)$, and $E' \subset E$, say that $\Omega_\emptyset(G)$ **separates** E' if for $e \in E'$, there exists $\eta \in \Omega_\emptyset(G)$ with $\eta \cap E' = \{e\}$.

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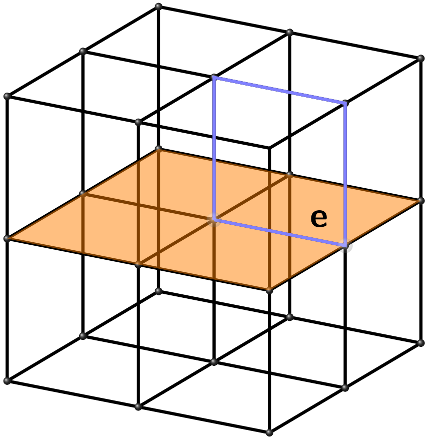
Proof: The restriction map $\Omega_\emptyset(G) \to \{0,1\}^{E'}$ is a homomorphism.

Percolation

Theorem (H., Kjær, Klausen, '23)

For $d \geq 3$, $\texttt{UEG}_{\mathbb{Z}^d}$ percolates.

Percolation



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For the proof, use that an FK-edge is open with probability at least $\frac{p}{2-p}$.

Main result

Theorem (H., Kjær, Klausen, '23)

For $d \geq 3$ and $x \in (\frac{p_c}{2-p_c}, 1]$, there exists $c > 0$ such that

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In particular, expected cluster sizes are infinite.

Input from the torus

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Works for $G = \omega \sim \phi_{p,\mathbb{T}_n^d}$ when $p > p_c$.

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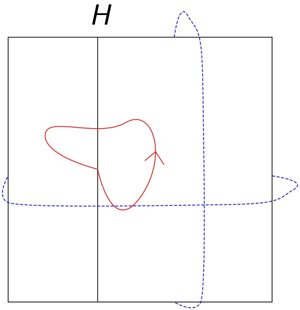

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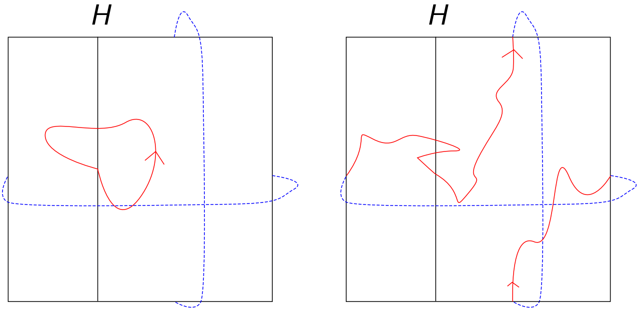

To finish proof: If $\eta \sim \mathrm{UEG}_G$, then $\eta \overset{d}{=} \eta\Delta\gamma$.

A miracle lemma

Lemma

If $G = G_1 \cup G_2 \cup G_3$ *and* G_2 *contains a connected separating surface between* G_1 *and* G_3*, the marginals of* UEG_G *on* G_1 *and* G_3 *are independent.*

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Again, applies to the supercritical random-cluster model!

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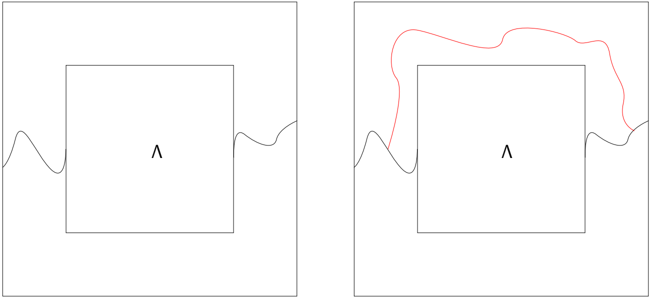

Other results in the paper:

- Uniqueness of infinite volume measures for the loop $\mathrm{O}(1)$ model in the supercritical regime.
- Exponential mixing of the infinite volume loop $\mathrm{O}(1)$ model in the supercritical regime.
- All results carry over to the random current.

Thank you for the attention.