$\sqrt{\log t}$ - superdiffusivity for a Brownian particle in the curl of the 2d GFF

CIME Summer School Statistical Mechanics and Stochastic PDEs

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The Model

Consider the following stochastic differential equation.

$$\mathrm{d}X(t) = \omega(X(t))\mathrm{d}t + \mathrm{d}B(t), \quad X(0) = 0\,,$$

where B_t is 2d-Brownian motion and

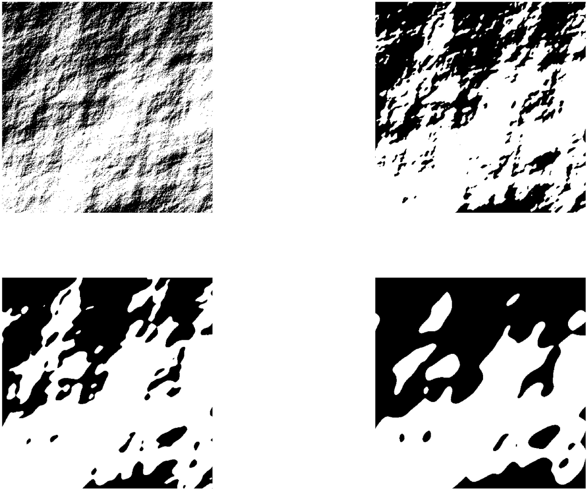
$$x \mapsto \omega(x) = (\omega_1(x), \omega_2(x))$$

is the curl of a smoothened Gaussian Free Field. That is

$$(\omega_1(x), \omega_2(x)) = (\partial_{x_2} U * \xi(x), -\partial_{x_1} U * \xi(x))\,,$$

where U is a radially symmetric smooth bump function and ξ is a $2d$ GFF.

Level lines



Superdiffusivity

Theorem ([CHST22])

As $t \to \infty$

$$E[X_t^2] \approx t\sqrt{\log t}$$

where $\approx$ *is in a Tauberian sense.*

This was conjectured by Toth and Valko in [TV12].

Markov process

Lemma

The environment as seen from the particle

$$\eta_t(x) = \omega(X_t + x)$$

is a stationary Markov process. Its generator can be written as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{A}_+ + \mathcal{A}_-.$$

Wiener Chaos Decomposition

Let

$$L^2(\omega) = \bigoplus_{k\geq 0} H_k$$

be the Wiener Chaos decomposition. We identify random variables in $\mathcal{H}_k$ with their symmetric Fourier kernels in $L^2_{\text{sym}}\left((\mathbb{R}^2)^n\right)$.
The generator $\mathcal{L} = \mathcal{L}_0 + \mathcal{A}_+ + \mathcal{A}_-$ satisfies

$$\begin{aligned} \mathcal{L}_0 &: \ H_n \to H_n\,, \\ \mathcal{A}_+ &: \ H_n \to H_{n+1}\,, \text{ and} \\ \mathcal{A}_- &: \ H_n \to H_{n-1} \end{aligned}$$

and acts on a kernel $\psi_n \in \mathcal{H}_n$ as

$$\mathcal{L}_0\psi_n(p_{1:n}) = -|\sum_{i=1}^{n} p_i|^2 \psi_n(p_{1:n})$$

$$\mathcal{A}_+\psi_n(p_{1:n+1}) = i\frac{1}{n+1}\sum_{i=1}^{n+1}(p_i \times \sum_{j=1}^{n+1} p_j)\psi(p_{1:n+1\setminus i})$$

Laplace transform

Let $\phi(\omega) = \omega_1(0) \in \mathcal{H}_1$. Considering the SDE in integral form

$$X_1(t) = B_1(t) + \int_0^t \phi(\eta_s)\mathrm{d}s\,.$$

Lemma ([CES21])

The Laplace transform of $\mathbb{E}(\int_0^t \phi(\eta_s)\mathrm{d}s\,)^2$ *satisfies*

$$\int_0^\infty e^{-\lambda t}\mathbb{E}[\int_0^t \phi(\eta_s)\mathrm{d}s]\mathrm{d}t = \frac{2}{\lambda^2}\left\langle \phi(\lambda - \mathcal{L})^{-1}\phi \right\rangle\,.$$

Cut Resolvent equation

Lemma (Lemma 2.1 in [LQSY04])

Let Π_n be the projection onto $\bigoplus_{k=0}^{n} H_k$ and $\mathcal{L}_n = \Pi_n \mathcal{L} \Pi_n$. Further let $\psi^n = (\lambda - \mathcal{L}_n)^{-1}\phi$. Then

$$\langle \phi, \psi^{(2n)} \rangle \leq \langle \phi, (\lambda - \mathcal{L})^{-1}\phi \rangle \leq \langle \phi, \psi^{(2n+1)} \rangle.$$

Furthermore $\langle \phi, \psi^{(2n)} \rangle$ and $\langle \phi, \psi^{(2n+1)} \rangle$ are increasing and decreasing respectively, and both converge to $\langle \phi, (\lambda - \mathcal{L})^{-1}\phi \rangle$.

System of Equations

We want to solve $\psi^n = (\lambda - \mathcal{L}_n)^{-1}\phi$, this leads to a system of equations:

$$
\begin{aligned}
(\lambda - \mathcal{L}_0)\psi_n^n - \mathcal{A}_+\psi_{n-1}^n &= 0 \\
(\lambda - \mathcal{L}_0)\psi_k^n - \mathcal{A}_+\psi_{k-1}^n - \mathcal{A}_-\psi_{k+1}^n &= 0 \\
(\lambda - \mathcal{L}_0)\psi_1^n - \mathcal{A}_-\psi_2^n &= \phi
\end{aligned}
$$

Solving this leads to $\psi_1^n = \mathcal{H}_n\phi$ where $\mathcal{H}_n$ is defined by

$$
\begin{aligned}
\mathcal{H}_1 &= 0 \\
\mathcal{H}_{k+1} &= \mathcal{A}_+^*(\lambda - \mathcal{L}_0 + H_k)^{-1}\mathcal{A}_+.
\end{aligned}
$$

Iterative estimates

Lemma

All $\mathcal{H}_k$ are positive self adjoint operators which map H_n into H_n. If S is another positive self adjoint operator then:

$$\mathcal{H}_k \leq S \Rightarrow \mathcal{H}_{k+1} \geq \mathcal{A}_+^*(\lambda - \mathcal{L}_0 + S)^{-1}\mathcal{A}_+$$

The goal is to estimate each operator $\mathcal{H}_k$ from above or below by a simpler operator $\mathcal{S}_k$.

Diagonal Operators and Integrals

A diagonal operator $\mathcal{S}$ with Fourier multiplier $(\sigma_k)_{k\geq 0}$ is an operator that acts on $\psi_n \in H_k$ as:

$$\mathcal{S}\psi_n(p_{1:n}) = \sigma_k(p_{1:n})\psi_n(p_{1:n}).$$

Lemma

Let $\mathcal{S}$ and $\tilde{\mathcal{S}}$ be diagonal operators with Fourier multipliers σ and $\tilde{\sigma}$ respectively. If for every $p_{1:n}$ s.t. $\sum_{i=1}^{n} p_i \neq 0$ it holds that:

$$\int_{\mathbb{R}^2} \hat{V}(q)\sin(\theta)^2\sigma_{n+1}(p_{1:n}, q)dq \leq \tilde{\sigma}_n(p_{1:n})$$

then

$$\langle \psi, \mathcal{A}_+^* \mathcal{S} \mathcal{A}_+ \psi \rangle_{diag} \leq \langle \psi, (-\mathcal{L}_0)\tilde{\mathcal{S}}\psi \rangle$$

for every ψ.

Some functions

Define the following functions

$$\mathrm{L}(x,z) = z + \log(1 + x^{-1})$$

$$\mathrm{LB}_k(x,z) = \sum_{0 \le j \le k} \frac{(\frac{1}{2}\log \mathrm{L}(x,z))^j}{j!} \quad \text{and} \quad \mathrm{UB}_k(x,z) = \frac{\mathrm{L}(x,z)}{LB(x,z)}$$

and further

$$\sigma_k(x,z) = \begin{cases} \mathrm{UB}_{\frac{k-2}{2}}(x,z), & \text{if } k \text{ is even,} \\ \mathrm{LB}_{\frac{k-1}{2}}(x,z), & \text{if } k \text{ is even.} \end{cases}$$

and finally

$$z_k(n) = K_1(n+k)^{2+2\varepsilon} \quad \text{and} \quad f_k(n) = K_2\sqrt{(}z_k(n)) \,.$$

Diagonal operators

Let $\mathcal{S}_k$ be defined as:

$$\mathcal{S}_k = \begin{cases} f_k(n)\sigma_k(\lambda - \mathcal{L}_0, z_k(\mathcal{N})) & \text{if } k \text{ is even and} \\ \frac{1}{f_k(n)}(\sigma_k(\lambda - \mathcal{L}_0, z_k(\mathcal{N})) - f_k(n)) & \text{if } k \text{ is odd,} \end{cases}$$

where $\mathcal{N}$ is the number operator.

Theorem

For any $\varepsilon > 0$, for suitable K_1, K_2, for any $0 < \lambda \leq 1$ and $k \geq 1$ we have

$$\mathcal{H}_{2k-1} \geq c_{2k-1}(-\mathcal{L}_0)\mathcal{S}_{2k-1}$$
$$\mathcal{H}_{2k} \leq c_{2k}(-\mathcal{L}_0)\mathcal{S}_{2k},$$

where c_{2k} and c_{2k+1} are both converging to finite positive values.

Outlook

Development

- Evolving field: [FW22]
- Stochastic homogenization: [CMOW23]
- Brownian Polymer: See Harry's talk

Open Problems

- Weak coupling CLT
- Diffusivity to 0
- Quenched behaviour

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