Exercises: Bernoulli percolation

Observe that p_c may be defined on any vertex-transitive graph in the same way as on $\mathbb{Z}^d$.

Exercise 1.1. Show that $p_c(\mathbb{Z}) = 1$.

Exercise 1.2. Show that for the "ladder" graph $\mathbb{Z} \times \{0, 1\}$, $p_c = 1$.

Exercise 1.3. Let T_d denote the $d+1$-regular tree (with the root having degree d rather than $d+1$). Prove that $p_c(T_d) = \frac{1}{d}$ and observe that the Peierls argument works all the way up to p_c.

Prove that $\mathbb{P}_{p_c}(T_d) = 0$. Show that $\mathbb{P}_{p_c}[0 \text{ is connected to distance } n] = n^{-\alpha_1+o(1)}$ for an exponent $\alpha_1 > 0$ to be determined.

Exercise 1.4. Show that $p_c(\mathbb{Z}^d)$ is decreasing in d. Show that it is strictly decreasing. Show that $p_c(\mathbb{Z}^d) \to 0$ as $d \to \infty$.

Exercise 1.5. For $d = 2$, use the self-duality of percolation to prove that, for all $n \geq 1$,

$$\mathbb{P}_{\frac{1}{2}}\Big[\{0\} \times [0, n] \text{ connected to } \{n+1\} \times [0, n] \text{ inside } [0, n+1] \times [0, n]\Big] = \tfrac{1}{2}.$$

Using the sharpness of the phase transition to conclude that $p_c(\mathbb{Z}^2) \geq \frac{1}{2}$.

Exercise 1.6. Show that $p \mapsto \theta(p)$ is right-continuous.
Indication: $\theta(p)$ is the decreasing limit of the continuous and increasing functions $p \mapsto \mathbb{P}_p[0 \leftrightarrow \partial\Lambda_n]$.