Exercises: Bernoulli and FK-percolation

Exercise 2.1. Consider Bernoulli percolation on $\mathbb{Z}^2$. Assuming that $\theta(p) > 0$, use the FKG inequality to prove that

$$\mathbb{P}_p[\{n\} \times [-n, n] \xleftrightarrow{\Lambda_n^c} \infty] \to 1 \qquad \text{as } n \to \infty.$$

Using the uniqueness of the infinite cluster and the self-duality of $\mathbb{P}_{1/2}$ prove that

$$\theta(1/2) = 0.$$

Combine this with the lower bound on $p_c \geq 1/2$ (Exercise 1.5) to conclude that $p_c = 1/2$ for Bernoulli percolation on $\mathbb{Z}^2$, and that the phase transition is continuous.

Exercise 2.2. Consider Bernoulli percolation on $\mathbb{Z}^d$. The goal of this exercice is to prove continuity of $p \mapsto \theta(p)$ for all $p \neq p_c$. Recall from Exercise 1.6 that this function is right-continuous. Thus, we only need to prove left-continuity for $p \neq p_c$.

Fix p is such that $\theta(p) > \lim_{u \nearrow p} \theta(u)$. In particular $\theta(p) > 0$ and $p \geq p_c$.

(a) Consider the increasing coupling P of Bernoulli percolation using uniforms $(U_e)_{e \in E}$. Argue that

$$P[0 \xleftrightarrow{\omega_p} \infty \text{ but } 0 \mathrel{\overset{\omega_u}{\not\longleftrightarrow}} \infty \text{ for all } u < p] > 0.$$

(b) Argue that, conditionally on ω_p, $(U_e)_{e \in \omega_p}$ are i.i.d. uniforms on $[0, 1-p]$. Conclude that a.s. for any n there exists $u < p$ such that $\omega_u = \omega_p$ on Λ_n.

(c) Call a vertex v *fragile* (for some configuration $(U_e)_e$) if $v \xleftrightarrow{\omega_p} \infty$ but $v \mathrel{\overset{\omega_u}{\not\longleftrightarrow}} \infty$ for all $u < p$. Prove that if 0 is fragile, then a.s. all vertices of the infinite cluster are fragile.

(d) Deduce that, for any $u < p$, $\mathbb{P}_p$[there exists no infinite cluster] > 0. Show that this implies $p \leq p_c$ and conclude.

Exercise 2.3. Consider Bernoulli percolation on $\mathbb{Z}^d$ and $p > p_c(\mathbb{Z}^d)$. Using the supercritical sharpness and the uniqueness of the infinite cluster, prove that

$$\theta(p)^2 \leq \mathbb{P}_p[x \leftrightarrow y] \leq \theta(p)^2 + e^{-c(p)\|x-y\|} \qquad \text{for all } x, y \in V,$$

for some $c(p) > 0$.

Exercise 2.4. Fix $d \geq 2$ and $q \geq 1$. Prove that, for all p,

$$\lim_{u \nearrow p} \phi^0_{u,q} = \lim_{u \nearrow p} \phi^1_{u,q} = \phi^0_{p,q} \qquad \text{and} \qquad \lim_{u \searrow p} \phi^0_{u,q} = \lim_{u \searrow p} \phi^1_{u,q} = \phi^1_{p,q},$$

in the sense that, for all A depending on finitely many edges $\lim_{u \nearrow p} \phi^0_{u,q}[A] = \phi^0_{p,q}[A]$.

Exercise 2.5. Fix $d \geq 2$, $q \geq 1$ and $p \in (0,1)$.

(a) Show that for any sequence of boundary conditions $(\xi_n)_n$, the following limit exists and does not depend on the sequence

$$f(p,q) = \lim_{N\to\infty} f_N^{\xi}(p,q) = \lim_{N} \tfrac{1}{N^2} \log Z_{\Lambda_N,p,q}^{\xi_N}.$$

(b) Show that $p \mapsto f_N^{\xi}(p,q)$ is convex, and conclude that $p \mapsto f(p,q)$ is also a convex function.

Indication: compute the differential of $f_N^{\xi}(p,q)$.

(c) Prove that $\partial_p^- f(p,q) = \phi_{p,q}^0[e \text{ open}]$ and $\partial_p^+ f(p,q) = \phi_{p,q}^1[e \text{ open}]$.

(d) Conclude that, when $p \mapsto f(p,q)$ is differentiable, $\phi_{p,q}^0[e \text{ open}] = \phi_{p,q}^1[e \text{ open}]$. Using $\phi_{p,q}^0 \leq_{\text{st}} \phi_{p,q}^1$, deduce that $\phi_{p,q}^0 = \phi_{p,q}^1$.

Exercise 2.6. Consider FK-percolation on $\mathbb{Z}^d$ with $q > 1$ and some p. Prove that for any $n < N$ and A an increasing event depending only on the edges in Λ_n,

$$0 \leq \phi_{\Lambda_N}^1[A] - \phi_{\Lambda_N}^0[A] \leq \phi_{\Lambda_N}^1[\Lambda_n \leftrightarrow \partial\Lambda_N],$$

Indication: explore the cluster of $\partial\Lambda_N$ under $\phi_{\Lambda_N}^1$. If it does not reach Λ_n, prove that the probability of A is then smaller $\phi_{\Lambda_N}^0[A]$.

Deduce that if p is such that $\phi^1[0 \leftrightarrow \infty] = 0$, then $\phi^0 = \phi^1$.

Exercise 2.7. Consider FK-percolation on $\mathbb{Z}^2$ with $q > 1$. Use duality to show that

$$\phi_{\Lambda_n}^{\xi}[\Lambda_n \text{ crossed from left to right by open path}] = c,$$

for all n, where ξ is the b.c. where the left and right sides are wired, while the top and bottom are free. Why is $c \neq 1/2$?

Assuming the sharpness of the phase transition and the uniqueness of the infinite cluster, proceed as in Exercise 2.1 to prove that $p_c(d) = \frac{\sqrt{q}}{1+\sqrt{q}}$.

Why can't we conclude that the phase transition is continuous?

Exercise 2.8. Consider FK-percolation on $\mathbb{Z}^2$ with some $q \geq 1$ and $p \in [0,1]$. Assume that we are in the case (b) of Theorem 3.1, that is the RSW regime. Show that there exists $c > 0$ such that

$$c\,n^{-1} \leq \phi_{\Lambda_n}^{\xi}[0 \leftrightarrow \partial\Lambda_n] \leq n^{-c} \qquad \text{for all } n \geq 1.$$

Deduce that $\phi[|\mathcal{C}_0|] = \infty$, where $\mathcal{C}_0$ is the cluster of 0, and ϕ is the unique infinite volume measure with this set of parameters (here it is used as an expectation).

Exercise 2.9. Consider FK-percolation on $\mathbb{Z}^2$ with some $q \geq 1$ and $p \in [0,1]$. Assume that we are in the case (c) of Theorem 3.1, that is the discontinuous phase transition regime, where $\phi^0 \neq \phi^1$.

(a) Using $\phi^0 = (\phi^1)^*$, show that, for any fixed edge e, $\phi^0[e \text{ open}] < 1/2$.

(b) Write $\phi^{\text{per}}_{\Lambda_n}$ for the FK-percolation measure on the square torus of side-length $2n$. Prove that, for any fixed edge e, $\phi^{\text{per}}_{\Lambda_n}[e \text{ open}] = 1/2$.

(c) Assume that the only Gibbs measures[2] of FK-percolation on $\mathbb{Z}^2$ which are invariant under translations and rotations by $\pi/2$ are the linear combinations of ϕ^0 and ϕ^1 (this may be proved using relatively soft tools). Prove that

$$\phi^{\text{per}}_{\Lambda_n} \longrightarrow \tfrac{1}{2}\phi^0 + \tfrac{1}{2}\phi^1 \qquad \text{as } n \to \infty,$$

in the sense that the probabilities of any event depending on finitely many edges converges.

(d) Is $\lim_n \phi^{\text{per}}_{\Lambda_n}$ ergodic?

Exercise 2.10. Fix $q \geq 1$ and $p \in [0,1]$. Let ϕ be an **ergodic** Gibbs measure for FK-percolation on $\mathbb{Z}^2$ which is invariant under translations and rotations by $\pi/2$. Use the same construction as in Exercise 2.1 to prove that a.s. there exists no infinite primal cluster or a.s. there exists no infinite dual cluster.

Deduce that the only Gibbs measures of FK-percolation on $\mathbb{Z}^2$ which are invariant under translations and rotations by $\pi/2$ are the linear combinations of ϕ^0 and ϕ^1.

Exercise 2.11. Fix $q \geq 1$ and $p \in [0,1]$. Let ϕ be a Gibbs measure for FK-percolation on $\mathbb{Z}^2$ which is invariant under translations and rotations by $\pi/2$. Assume that there exists $c > 0$ such that

$$\begin{aligned}
&\phi\big[\omega \text{ contains a horizontal crossing of } [0,2n]\times[0,n]\big] \geq c \qquad \text{and}\\
&\phi\big[\omega^* \text{ contains a horizontal crossing of } [0,2n]\times[0,n]\big] \geq c \qquad \text{for all } n \geq 1.
\end{aligned}$$

Prove (without using the quadrichotomy theorem) that

$$\begin{aligned}
&\phi\big[\omega \text{ contains a horizontal crossing of } [0,kn]\times[0,n]\big] \geq c^{2k} \qquad \text{and}\\
&\delta < \phi[H_n] < 1-\delta \qquad \text{for all } n \geq 1,
\end{aligned}$$

and the same for the dual, for some $\delta > 0$.

Does this imply that $\phi\big[0 \leftrightarrow \partial\Lambda_n\big] \to 0$?

[2] For the purpose of this exercise, Gibbs measures should be understood as the potential limits of finite volume mesures