Rough walks

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Plan

Rough paths and homogenization

Regenerative processes

Kipnis-Varadhan

Application: random conductance model

Quenched result

Reminder: Rough paths, integrals, differential equations

Definition by Lyons '98: Let $p \in (2,3)$. A p-variation rough path is a pair $(Z, \mathbb{Z})$ defined by $(Z_{s,t}, \mathbb{Z}_{s,t}) \in \mathbb{R}^d \times \mathbb{R}^{d \otimes d}$ for $0 \le s < t \le T$ so that

(i) $\|Z\|_{p-\mathrm{var},[0,T]} + \|\mathbb{Z}\|_{p/2-\mathrm{var},[0,T]} < \infty$

(ii) Algebraic relations $0 \le s < u < t \le t$

1st level: $Z_{s,t} = Z_{s,u} + Z_{u,t}$ increments of a path
($Z_t := Z_{o,t}$ implies $Z_{s,t} = Z_t - Z_s$)

2st level: $\mathbb{Z}_{s,t} - \mathbb{Z}_{s,u} - \mathbb{Z}_{u,t} = Z_{s,u} \otimes Z_{u,t}$ "Chen's relation".
(Example: $\mathbb{Z}_{s,t} = \int_{(s,t]} \int_{(s,r_1]} dZ_{r_2} \otimes dZ_{r_1}$ for fixed notion of integration.)

Definition is more general, if $p \ge 3$ the rough path has $\lfloor p \rfloor$ levels.

Reminder: Rough paths, integrals, differential equations

[Lyons '98, Gubinelli '04]: $\mathcal{Z} := (Z, \mathbb{Z})$. Construction of rough integral s.t. $(Y, \mathcal{Z}) \mapsto \int_0^{\cdot} Y_s d\mathcal{Z}_s$ is continuous (if Y is "controlled by $\mathcal{Z}$");
Existence of solutions of rough differential equations in space of paths "controlled by $\mathcal{Z}$";
Itô-Lyons map $\mathcal{Z} \mapsto Y_t = Y_0 + \int_0^t f(Y_s) d\mathcal{Z}_s$ is continuous.

In particular, if $(Z^n, \mathbb{Z}^n) =: \mathcal{Z}^n \to \mathcal{Z}$ in rough path topology then (under conditions)

$$Y_t^n = Y_0^n + \int_0^t f(Y_s^n) d\mathcal{Z}_s^n \to Y_t = Y_0 + \int_0^t f(Y_s) d\mathcal{Z}_s.$$

Random processes as rough paths

Cadlag/discrete-time process Z on $\mathbb{R}^d$ is fixed.
Interested to lift $(Z, \mathbb{Z})$ to rough path space by setting

$$\mathbb{Z}_{s,t} := \int_{(s,t]} Z_{s,r} \otimes dZ_r,$$

where the notion of integration is fixed to be Stratonovich / Riemann–Stieltjes (interpolate linearly for discrete time).

Example. Stratonovich Brownian rough path $(B, \mathbb{B})$ is the Stratonovich lift of Brownian motion, i.e. $\mathbb{B}_{s,t} = \int_s^t \int_s^{r_1} dB_{r_2} \otimes \circ dB_{r_1}$, the iterated Stratonovich integral of the Brownian motion B (with a certain covariance matrix Σ).

Effect of second level limit on SDE approximations

(Σ, Γ) - Stratonovich Brownian rough path is a pair $(B, \mathbb{B})$ so that B is a Brownian motion with covariance Σ and $\mathbb{B} = \mathbb{B}^{\mathrm{Str}} + \Gamma\cdot$, that is

$$\mathbb{B}_{s,t} = \int_s^t \int_s^{r_1} dB_{r_2} \otimes \circ dB_{r_1} + (t-s)\Gamma.$$

Kelly '16: Assume Stratonovich lift $(Z^n, \mathbb{Z}^n)$ of semimartingales satisfies Functional CLT in the p-variation rough path topology, for some $p > 2$, where the limit $(B, \mathbb{B})$ is a (Σ, Γ) - Stratonovich Brownian.

Fix $f \in C^1(\mathbb{R}, \mathbb{R}^d)$. Solutions to $Y_t^n = Y_0^n + \int_0^t f(Y_s^n) \circ dZ_s^n$ converge weakly to the solution to

$$Y_t = Y_0 + \int_0^t f(Y_s) \circ dB_s + \int_0^t \underbrace{\Gamma f(Y_s) \cdot f'(Y_s)}_{=\sum_{i,j=1}^d \Gamma^{i,j} f_i'(Y_s) f_j(Y_s)} \, ds.$$

Related: Chevyrev, Friz, Korepanov, Melbourne, Zhang, Hairer, Li,...
Djurdjevac-Kremp-Perkowski, Kifer,...

Functional CLT in rough path topology

Some possible questions and challenges:

- Stronger convergence; may require stronger tools.
- Interesting examples for nontrivial (Σ, Γ) - FCLT.
- Interpretation of Γ.

Natural candidates: various RWRE which satisfy classical FCLT.

Regenerative processes

Definition

A process $X = (X_k)_{k \geq 0}$ on $\mathbb{R}^d$ is called *regenerative* if its increments form a delayed renewal process: there are almost surely (random) times $0 =: \tau_0 < \tau_1 < \tau_2 < ... < \infty$ so that

$$\left(\{X_{\tau_k, \tau_k + m}\}_{0 \leq m \leq \tau_{k+1} - \tau_k}, \ \tau_{k+1} - \tau_k \right)_{k \geq 1} \text{ are i.i.d}$$

and are independent of $(\{X_m\}_{0 \leq m \leq \tau_1}, \tau_1)$.

Simple examples

- Random walk $X_n = \sum_{k=1}^n \xi_k$, by taking $\tau_k = k$.
- Additive functionals $X_n = \sum_{k=1}^n f(Y_k)$, Y recurrent irred. Markov; τ_k is k-th hitting time.

Functional CLT in the rough path topology

- X - regenerative, $\mathbb{E}[X_{\tau_k,\tau_{k+1}}] = 0$ (centered).
- Diffusive rescaling + Stratonovich lift $(X^n_{s,t}, \mathbb{X}^n_{s,t})_{0\le s<t\le T}$.

$(X^n_t := \frac{X_{\lfloor n^2 t\rfloor}}{n} + \frac{n^2 t - \lfloor n^2 t\rfloor}{n} X_{\lfloor n^2 t\rfloor, \lfloor n^2 t\rfloor+1}.)$

Theorem (Lopusanschi - O '21, O '21)

Assume $\mathbb{E}\Big[(\tau_{k+1}-\tau_k)\sup_{\tau_k<m\le\tau_{k+1}} \|X_{\tau_k,m}\|^2\Big] < \infty,\ k \ge 0.$
Then, $(X^n, \mathbb{X}^n) \Rightarrow (B, \mathbb{B}^{Str} + \Gamma\cdot)$, *a Stratonovich rough Brownian, in the p-variation rough path topology,* $p \in (2,3)$. *Moreover*

$$\Gamma = \frac{\mathbb{E}[\mathrm{Antisym}(\mathbb{X}^1_{\tau_1,\tau_2})]}{\mathbb{E}[\tau_2-\tau_1]},\ \textit{the area anomaly.}$$

Remarks

- Optimal moment condition (without delay).
  Counter-example: second moment condition for jumps of random walks:
  $$(\tau_{k+1}-\tau_k)\sup_{\tau_k<m\leq\tau_{k+1}}\|X_{\tau_k,m}\|^2 = 1\cdot\|\xi_{k+1}\|^2$$
  for all $k\geq 0$.
- In fact, enough any moment for the delaying epoch ($k=0$).
- Stationary regenerative: Green-Kubo type formula (in writing with M. Engel and P. Friz). To be mentioned again later.
- Area anomaly: Γ is the expected signed stochastic area in a regeneration interval, normalized by its expected length.

From the proof: How to see the area correction? $X_{\tau_k}=\sum_{j=1}^{k-1}X_{\tau_{j-1},\tau_j}$ is a centred random walk with second moments jumps. Also,

$$S_{0,\tau_k}(X_\cdot)=S_{0,k}(X_{\tau_\cdot})+\sum_{i=1}^{k}A_{\tau_{i-1},\tau_i}(X_\cdot),$$

where S is the **middle-point** iterated sum, Q the QV (sum of product of increments) and A is the antisymmetric part of S.

Applications and a question

- RW in deterministic box-periodic environment (or on periodic graphs): straight-forward construction of Markovian examples with non-vanishing area anomaly Γ.
- Ballistic random walks in random environments (that is i.i.d, uniformly elliptic, Sznitman T'), $d \geq 2$, annealed.

  Sznitman-Zerner '99: delayed regenerative.

  Sznitman '00: all moments are finite. Showed LLN and classical CLT.
- RW in Dirichlet environments. Annealed, $d \geq 2$, trap parameter $\kappa > 3$ + extra condition.

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Q. Is $\Gamma \neq 0$ for (non-degenerate) ballistic RWRE?

Kipnis-Varadhan in rough path topology

Theorem (Deuschel - O - Perkowski '21)

X Markov, $X_0 \sim \mu$ stationary and ergodic for $\mathcal{L}$,$\mathcal{L}^*$.
$L^2(\mu) \ni F : E \to \mathbb{R}^d$ s.t. $\int F d\mu = 0$ and assume $\mathcal{H}^{-1}$ condition.
$\begin{cases} \lambda \int |\Phi_\lambda|^2 d\mu \to 0 \\ \int (\Phi_\lambda - \Phi_{\lambda'}) \otimes (-\mathcal{L})(\Phi_\lambda - \Phi_{\lambda'}) d\mu \to 0 \end{cases}$ if $(\lambda - \mathcal{L})\Phi_\lambda = F$. *Set*
$Z^n_t = n^{-1} \int_0^{n^2 t} F(X_s) ds$. *Then for the Stratonovich lift*

$$(Z^n, \mathbb{Z}^n) \to \left(B, \mathbb{B}^{Str} + \cdot \Gamma\right)$$

in p-variation rough path topology, $p \in (2,3)$, where B is a Brownian motion with covariance

$$\Sigma = 2 \lim_{\lambda \to 0} \mathbb{E}[\Phi_\lambda \otimes (-\mathcal{L}_S)\Phi_\lambda]$$

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Remarks. (i) In particular, correction vanishes if $\mathcal{L} = \mathcal{L}^*$.

(ii) Can be applied to regenerative in stationary Markov setting.

(iii) [Engel-Friz-O '23+]: Beyond Markov (stationary processes): the formulae expressed in terms of time correlations (Green-Kubo type).

Random conductances and additive functionals

- Environments $\{\omega(x,y) = \omega(y,x) : x, y \in \mathbb{Z}^d, x \sim y\}$.
- For fixed ω consider P_o^ω the law of continuous time random walk X^ω with jumps rates $\omega(x,y)$ from x to y stating at o.
- Idea: the environment seen from the walker $(\omega_t := \tau_{X_t}\omega)_{t\geq 0}$ is Markov.
- $\mathbb{P}$ initial law $\{\omega(x,y)\}_{x\sim y}$ shift-invariant ergodic implies the process $(\omega_t)_{t\geq 0}$ is ergodic reversible (e.g. **[Kozlov '85]**).
- Can decompose

$$X_t = X_t^\omega = M_t + Z_t, \qquad M \text{ martingale}$$

and $Z_t = \int_0^t F(\omega_s)ds$ additive functional with

$$F(\omega) = \sum_{e\sim 0} e\,\omega(0,e) \text{ the empirical drift.}$$

Application to RW in random conductances

Assume conductances on $\mathbb{Z}^d$ independent but stationary (e.g., i.i.d) and uniformly elliptic (bdd away from 0 and ∞, uniformly) under P.
Let $\mathbb{P}_o$ be the annealed law (averaging P_o^ω with respect to P)

Theorem (Deuschel - O - Perkowski '21)

For the Stratonovich lift $(X^n, \mathbb{X}^n)$ under the annealed law $\mathbb{P}_o$

$$(X^n, \mathbb{X}^n) \to (B, \mathbb{B}^{Str})$$

weakly in p-variation rough path topology, $p \in (2, 3)$,

Note convergence to Stratonovich (without anomaly!).

Quenched FCLT in rough path topology

With Johaness Bäumler, Noam Berger and Martin Slowik.

Settings. $\mathbb{Z}^d$, $d \geq 3$, i.i.d nearest neighbor conductances in $\{0\} \cup [a, b]$.

The assertion remains as in the annealed case except that the weak convergence takes place with respect to $\mathbb{P}_0^\omega$ for P-a.e.-ω.

On the proof

In the proof we use two crucial pieces of information:

- **Gloria-Neukamm-Otto '14, Dario '18** moments of the corrector $\mathbb{E}|\chi(\omega, x)|^p < C_p$ for all $p > 0$.
- Quenched Heat kernel bounds: **Mathieu-Remy 04, Barlow 04**.

Key Lemma. Corrector χ not only approximated by gradients, but is a gradient: $\chi(\omega, x) = D\varphi(\omega, x) = \varphi(\tau_x\omega) - \varphi(\omega)$ for some φ. Moreover,

$$\sup_{t\geq 0} E_0^\omega[\|\varphi(\tau_{X_t}\omega)\|^q] < c_q(\omega) \text{ a.s., for all } q > 0.$$

- Tightness. Corrector: "uniform" ellipticity guaranties jumps proportional to time, then deducing by heat kernel. Martingale: bounded jumps enables transferring estimates to Martingale.
- Identification of limit: **Kurtz-Protter '91** classical result on convergence of stochastic integrals + ergodic theorem for deterministic limit terms (plus Slutsky's theorem).

Thank you for your attention!