Large-scale behavior of (super)-critical SPDEs

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CIME Summer school - Cetraro 2023

based on joint works with: G. Cannizzaro (Warwick), D. Erhard (Bahia)
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Plan :

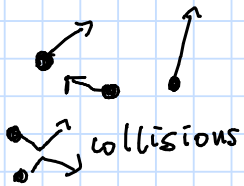
Part I

- Motivations : driven diffusive systems & interface growth
- The SPDEs : Stochastic Burgers & KPZ ($d \geq 2$)
- questions, heuristics (scaling argument) & some theorems : Gaussian limits, $d \geq 2$

Part II $d = 2$: Weak coupling limit

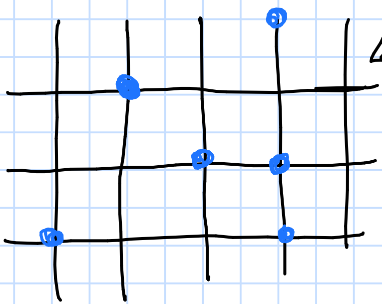
Part III $d \geq 3$: Gaussian scaling limit of Stoch. Burgers eq.

Motivations I: Driven diffusive systems



$\xrightarrow{\text{External field}}$ $\rightsquigarrow$ macroscopic particle flow

Lattice discretization: Asymmetric (Simple) Exclusion Process (ASEP)



$\mathbb{Z}^d$ $\quad d \geq 1$

exclusion: at most 1 particle per vertex

dynamics: continuous-t Markov chain

$x \to y$ $\quad$ jump rate from x to y $= \begin{cases} 0 & \text{if } y \text{ occupied} \\ p(y-x) & y \text{ empty} \end{cases}$

Assume that p has the form

$$p(x) = \underbrace{\frac{1}{2d}}_{p_0(x)} + \lambda \cdot \begin{cases} 1 & x = e_1 \\ -1 & x = -e_1 \\ 0 & \text{else} \end{cases}$$

$p_0(x)$ symmetric part

$\lambda \in \left[-\frac{1}{2d}, \frac{1}{2d}\right]$

i.e. jump kernel has "drift" $w = 2\lambda e_1$

"asymmetry"

$\boxed{\text{Fact}}$: $\mu_\rho = \bigotimes_{x \in \mathbb{Z}^d} \mathrm{Bern}(\rho)$ $\quad \rho \in [0,1]$ is stationary measure

Exercise!

Asymmetry $\Longrightarrow$ Irreversible process

Question : Large-scale behavior of fluctuations

$\eta(x,t) = \{ \exists \text{ particle at } x \text{ at time } t \}$

$$C(x,t) = \langle \eta(\underbrace{x,t}_{\to\, x - v(\rho)t})\, \eta(0,0) \rangle_\rho - \overbrace{\langle \eta(x,t) \rangle_\rho}^{\rho} \overbrace{\langle \eta(0,0) \rangle_\rho}^{\rho}$$

↳ average w.r.t. μ_ρ

$\langle \quad \rangle_\rho$ = stationary average with density ρ

Symmetric exclusion : $w = 0 \Longrightarrow$ normal diffusion

$$C(x,t) \underset{\substack{t\to\infty \\ |x|\to\infty}}{\simeq} \frac{e^{-\frac{|x|^2}{2Dt}}}{(2\pi D t)^{d/2}}$$

D = constant → Diffusion coefficient

Dynamic correl. length $\ell(t) \approx \sqrt{t}$

Asymmetry $\overset{?}{\Longrightarrow}$ normal diffusion or superdiffusion ? $\left(D = D(t) \underset{\to}{\overset{t\to\infty}{}} \infty \right)$

Recap: Main features of ASEP

- symmetric part p_s of p induces diffusion
- asymmetric " of p induces drift in direction ω
- exclusion rule $\to$ non-linearity

  e.g. rate of jump $x \to y$ $= \eta_x (1-\eta_y) \cdot p(y-x)$
- particle number is conserved locally
- randomness (jumps occur at random space-time points)

"Continuum limit" of ASEP (van Beijeren – Kutner – Spohn '85)
(MESOSCOPIC)

$\eta(x,t) \in \{0,1\}$, $x \in \mathbb{Z}^d$ $\longrightarrow$ $\phi(x,t) \in \mathbb{R}$, $x \in \mathbb{R}^d$, (fluctuation of) density profile

jump Markov process $\longrightarrow$ stochastic PDE

if $w = 0$: linear equation, Gaussian process

$$\partial_t \phi = \tfrac{1}{2}\Delta\phi + w\cdot\nabla(\phi^2) + \operatorname{div}\xi$$

$\boxed{\text{Stochastic Burgers Equation (SBE)}}$

diffusion

non-linearity $\leftrightsquigarrow$ asymmetry

($w \neq 0$: irreversibility)

noise $\xi = (\xi_1, \dots, \xi_d)$

$\xi_i =$ Gaussian space-time white noise

divergence $\longleftrightarrow$ particle conservation

N.B. $\operatorname{div}\xi \overset{d}{=} (-\Delta)^{\frac{1}{2}}\xi_1$

"Particle conservation"

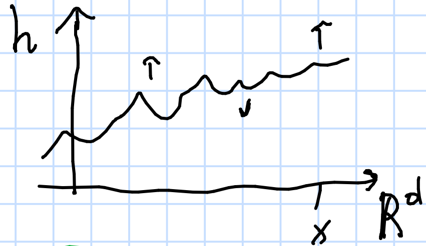
Note: all terms in r.h.s. are of the form $\nabla \cdot$ something

$$\Delta \phi = \nabla \cdot \nabla \phi, \quad \omega \cdot \nabla \phi^2 = \nabla \cdot (\omega \phi^2), \quad \operatorname{div} \xi = \nabla \cdot \xi$$

$$\partial_t \int_V \phi(x,t)\, dv = \int_V \nabla \cdot F \, dx = \int_{\partial V} F \cdot \hat{n} \, dS \qquad \text{(Gauss)}$$

i.e. "particle #" changes because of flow through boundary, <u>not</u> creation/destruction

Motivation II: stochastic interface growth



d-dimensional, generalised KPZ equation

$h = h(x,t) \in \mathbb{R}$, $x \in \mathbb{R}^d$

$$\partial_t h = \frac{1}{2}\Delta h + \lambda\,(\nabla h, Q\nabla h) + \xi$$

λ: coupling strength

Q: $d \times d$ matrix

if $Q = \mathbb{I}$
$\Rightarrow (\ ,\ ) = |\nabla h|^2$

$\frac{1}{2}\Delta h$: smoothing mechanism

$(\nabla h, Q\nabla h)$: non-linearity "lateral growth effect"

ξ: space-time noise (not conservative)

$\lambda = 0 \Rightarrow$ Edwards-Wilkinson equation (= Stoch Heat Equation)
Linear, Gaussian

Some Features of SBE and KPZ

- $w=0 \Rightarrow$ linear equation

$$\partial_t \phi = \frac{1}{2}\Delta \phi + (-\Delta)^{\frac{1}{2}} \xi$$

$\lambda = 0 \Rightarrow$ linear equation

$$\partial_t h = \frac{1}{2}\Delta h + \xi$$

Fourier: $\hat{\phi}(k,t) = \int \phi(x,t)\, e^{ikx}\, dx$

$$d\hat{\phi}(k,t) = -\frac{|k|^2}{2}\hat{\phi}(k,t)\,dt + |k|\,dB_k(t)$$

$$d\hat{h}(k,t) = -\frac{|k|^2}{2}\hat{h}(k,t)\,dt + dB_k(t)$$

$B_k(t)$: iid (complex) Brownian Motions

$\Rightarrow$ Fourier modes are independent Ornstein-Uhlenbeck processes

$$dX_t = -\theta X_t\, dt + \sigma\, dB_t$$

Some Features of SBE and KPZ

- $w \neq 0$ or $\lambda \neq 0$:

SBE and KPZ are singular SPDEs

(ill-posed in every dimension
ϕ, h distribution
$\phi^2, |\nabla h|^2$??
$\longrightarrow$ Regularisation)

serious problem

- Classical analytical Tools & M. Hairer's "Regularity Structures", "paracontrolled calculus" (Gubinelli-Imkeller-Perkowski) do not apply if $d \geq 2$

- $d=1$: $h = \int^x \phi$ "solves" the Kardar-Parisi-Zhang equation (KPZ) or

$$\partial_t h = \frac{1}{2}\Delta h + \omega (\nabla h)^2 + \xi$$

$\hookrightarrow$ space-time white noise

Stationary distributions

- SBE: space white noise is stationary $\forall d \geq 1$

distribution

η white noise : $\eta = (\eta(x))_{x \in \mathbb{R}^d}$ average-zero Gaussian "function"

with covariance $\mathbb{E}(\eta(x)\,\eta(y)) = \delta(x-y)$

in Fourier: $\hat{\eta}(k)$ are iid $N(0,1)$ $\mathbb{E}\,\hat{\eta}(k)\,\hat{\eta}(e) = 1\!\!1_{e+k=0}$

- White noise is stationary also for the non-linear eq!! ($w \neq 0$)

- Analogy : ASEP $\longleftrightarrow$ Burgers

$\bigotimes_x$ Bernoulli white noise

Stationary distributions for KPZ

- $d=1$: white noise is stationary for $\partial_x h$
- $d=2$ if $Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\left[\partial_t h = \frac{1}{2}\Delta h + \lambda\left((\partial_x h)^2 - (\partial_y h)^2\right) + \xi \right]$

ANISOTROPIC KPZ, AKPZ

then 2-d GFF is stationary$^{(*)}$

$$\mathbb{E}\, \hat{\eta}(k)\, \hat{\eta}(\ell) = \frac{\mathbb{1}_{\ell+k=0}}{|k|^2}$$

(*) (except the "zero mode")

- else, no Gaussian stationary distribution (or any other known stationary distribution)

Scaling argument and predicted large-scale behavior

First remark: Linear equation is (diffusively) scale invariant

SBE $\partial_t \phi = \frac{1}{2}\Delta\phi + (-\Delta)^{\frac{1}{2}}\xi$

$$\boxed{\phi^{(\varepsilon)}(t,x) := \varepsilon^{-\frac{d}{2}}\,\phi\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right)}$$

$\varepsilon > 0$

$$\Rightarrow \partial_t \phi^{(\varepsilon)} = \frac{1}{2}\Delta\phi^{(\varepsilon)} + (-\Delta)^{\frac{1}{2}}\tilde{\xi}$$

$\tilde{\xi}$: is still a space-time white noise

Scaling argument: Pretend that non-linear eq. has solution. Then

$$\partial_t \phi^{(\varepsilon)} = \frac{1}{2}\Delta\phi^{(\varepsilon)} + \varepsilon^{\frac{d-2}{2}}\, w\cdot\nabla\left(\phi^{(\varepsilon)}\right)^2 + (-\Delta)^{\frac{1}{2}}\tilde{\xi}$$

Scaling argument and predicted large-scale behavior

Scaling argument: Pretend that non-linear eq. has solution. Then

$$\partial_t \phi^{(\varepsilon)} = \frac{1}{2}\Delta \phi^{(\varepsilon)} + \varepsilon^{\frac{d-2}{2}} w \cdot \nabla \left(\phi^{(\varepsilon)}\right)^2 + \operatorname{div} \tilde{\xi}$$

Suggests:

- $d > 2$ irrelevance of non-linearity, asymptotic diffusive scaling
- $d < 2$ relevance, superdiffusion
- $d = 2$ marginal dimension, finer analysis needed

Comments

- purely formal argument (but usually gives right prediction)

- $d \geq 3$ Too naif to expect that non-linearity has no effect on $\varepsilon \to 0$ limit

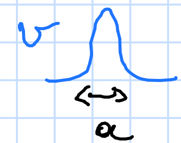
- same scenario conjectured for discrete models (universality)

- Marginal dimension $d=2$: finer structure (symmetries) of non-linearity can determine the large-scale behavior
  SBE $\neq$ KPZ

Regularisation

at least two options

- smoothing the noise
  $\mathbb{E}\,\xi(x,t)\xi(y,s) = \delta(t-s)\,\delta(x-y) \longrightarrow \delta(t-s)\cdot v(x-y)$



- Fourier cut-off on non-linearity

physically: microscopic cutoff $\approx$ lattice spacing $(=1)$

$\phi^2 \to \Pi_{\le 1}\left(\Pi_{\le 1}\phi\right)^2$ $\qquad$ $\Pi_{\le a}$ projects on modes $|k| \le a$

Remark Large-scale analysis $\Longleftrightarrow$ Removal of cut-off $a \to 0$

That is, $\phi^{(\varepsilon)} = \varepsilon^{-\frac{d}{2}}\,\phi\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right)$ has cutoff $\frac{1}{\varepsilon}$

i.e. it solves

$$\partial_t \phi^{(\varepsilon)} = \frac{1}{2}\Delta\phi^{(\varepsilon)} + \varepsilon^{\frac{d}{2}-1}\, w\cdot\nabla\left[\Pi_{\le\frac{1}{\varepsilon}}\left(\Pi_{\le\frac{1}{\varepsilon}}\phi^{(\varepsilon)}\right)^2\right] + (-\Delta)^{\frac{1}{2}}\xi$$

Some known results

- $d=1$ KPZ scaling $D(t) \sim t^{\frac{1}{3}}$ superdiffusion

that is, $\mathrm{Cov}(\phi(x,t), \phi(0,0)) \approx 0 \iff \frac{|x|^2}{t\,D(t)} \gg 1$ $\quad |x| \gg t^{\frac{2}{3}}$

Superdiffusion

(many exactly solvable / integrable models)

- $d=2$ for SBE, $D(t) \sim (\log t)^{\frac{2}{3}}$ (non-rigorous arguments)

van Beijeren - Kutner - Spohn 85

Yau '04 : rigorous proof for 2-dimensional ASEP

logarithmic corrections to diffusivity

Some known results

- $d \geq 3$: ASEP has Gaussian large-scale limit, $D(t) \approx 1$

  (Landim, Olla, Yau, Varadhan, ...)
  ≥ 1997 ...

  (precise statement: see next page for stochastic Burgers equation)
  $d \geq 3$

- $d \geq 3$: KPZ with $Q = I_d$ has Gaussian limit

  [Magnen-Unterberger, Gu-Zeitouni, Comets-Cosco-Mukherjee, Lygkonis-Zygouras, ...]
  ≥ 2018

New results I : Scaling limit, $d \geq 3$

Theorem 1 [G. Cannizzaro, M. Gubinelli, F.T., '23] $d \geq 3$ SBE

$$\partial_t \phi^{(\varepsilon)} = \frac{\Delta \phi^{(\varepsilon)}}{2} + \varepsilon^{\frac{d}{2}-1} w \cdot \nabla (\phi^{(\varepsilon)})^2 + (-\Delta)^{\frac{1}{2}} \xi$$

↳ with regularisation, i.e. $\Pi_{\leq \frac{1}{\varepsilon}} (\Pi_{\leq \frac{1}{\varepsilon}} \phi^{(\varepsilon)})^2$

As $\varepsilon \to 0$, $\phi^{(\varepsilon)} \Rightarrow$ solution of linear equation

$$\partial_t \phi = \frac{\Delta \phi}{2} + \underbrace{c(w)(w \cdot \nabla)^2 \phi} + \left(-\Delta - c(w)(w \cdot \nabla)^2\right)^{\frac{1}{2}} \xi$$

→ new Gaussian noise

$c(w) > 0$ Non-Trivial effect of non-linearity

↳ not explicit

Remark : initial $t=0$ distribution stationary (space white noise)

New results II. 2D AKPZ: logarithmic divergence of diffusivity

remember: diffusion coeff $D(t)$ is such that correlations small

$\iff$ distance $\gtrsim \sqrt{t\, D(t)}$

Theorem 2 [Cannizzaro - Erhard - T. '20]

(Extremely informal version) For 2d AKPZ, whenever $\lambda \neq 0$

$$D(t) = (\log t)^{\frac{1}{2}+o(1)} \qquad \text{as} \quad t \to \infty$$

- Analogous result for "diffusion in curl of the GFF": Cannizzaro - Haunschmid - T '21

Scaling limits for $d=2$? (critical dimension)

In view of the $D(t) \sim (\log t)^{\gamma}$ superdiffusivity, $\gamma = \begin{cases} \frac{1}{2} & \text{AKPZ} \\ \frac{2}{3} & \text{SBE} \end{cases}$

we cannot expect analog of Theorem 1 to hold for $d=2$

Non-linearity "divergent" on large scales

$\Rightarrow$ must be "Tamed" somehow

New results III: Gaussian limit in weak coupling regime, $d=2$

Theorem 3 [G. Cannizzaro, D. Erhard, F.T. '21] AKPZ
[G. Cannizzaro, M. Gubinelli, F.T., '23] SBE
} $d=2$, stationary

$$\partial_t \phi^{(\varepsilon)} = \frac{\Delta}{2}\phi^{(\varepsilon)} + \cancel{\varepsilon^{\frac{d}{2}-1}} \cdot \frac{1}{\sqrt{|\log \varepsilon|}} \, w \cdot \nabla (\phi^{(\varepsilon)})^2 + (-\Delta)^{\frac{1}{2}} \xi$$

↳ with regularization, i.e. $\Pi_{\leq \frac{1}{\varepsilon}} \left(\Pi_{\leq \frac{1}{\varepsilon}} \phi^{(\varepsilon)} \right)^2$

↳ Weak coupling limit (see lectures by CSZ)

As $\varepsilon \to 0$, $\phi^{(\varepsilon)} \Rightarrow$ solution of linear equation

$$\partial_t \phi = \frac{\Delta}{2}\phi + \underbrace{c(w) (w \cdot \nabla)^2}_{c(w)>0 \text{ explicit}} \phi + \left(-\Delta - c(w)(\nabla \cdot w)^2\right)^{\frac{1}{2}} \xi$$

Non-Trivial effect of non-linearity

"Universality" of weak scaling in $d=2$

- 2D KPZ $\quad \partial_t h = \frac{\Delta h}{2} + \lambda |\nabla h|^2 + \xi \qquad D(t) \sim t^{\beta}, \ \beta > 0$ → conjectural
- 2D AKPZ $\quad \partial_t h = \frac{\Delta h}{2} + \lambda\left[(\partial_x h)^2 - (\partial_y h)^2\right] + \xi \qquad D(t) \sim \sqrt{\log t}$
- Burgers Eq. $\quad \partial_t \phi = \frac{\Delta \phi}{2} + w \cdot \nabla \phi^2 + (-\Delta)^{\frac{1}{2}} \xi \qquad D(t) \sim (\log t)^{\frac{2}{3}}$ → conjectural

$\Big\}$ very different

However, weak coupling scaling $\lambda_\varepsilon = \frac{1}{\sqrt{|\log \varepsilon|}}$ is the same for all of them

$\boxed{\longrightarrow \text{see part II}}$

Refs: 2D KPZ — Chatterjee-Dunlap, Gu, Caravenna-Sun-Zygouras

2D AKPZ — Cannizzaro, Erhard, F.T. '21

Burgers — Cannizzaro, Gubinelli, F.T. '23

Open problems

- $d=2$: instead of weak coupling limit, we'd like to do:

$$\phi\left(\frac{t}{\varepsilon^2|\log\varepsilon|^{\gamma}}, \frac{x}{\varepsilon}\right) \underset{\varepsilon\to 0}{\Longrightarrow} \text{Scaling limit} \qquad \gamma = \begin{cases} \frac{1}{2} & \text{AKPZ} \\ \frac{2}{3} & \text{SBE} \end{cases}$$

"strong coupling limit"

- prove analog of theorem 3 for 2-dimensional (W)ASEP

End of Part I

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Part II & III

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Reminder: Stochastic Burgers Equation (SBE), $d \geq 2$

$\eta^\varepsilon = \eta^\varepsilon(t,x)$ $\quad t \geq 0, \quad x \in \mathbb{T}^d$ (d-dim Torus, side 2π)

$$\partial_t \eta^\varepsilon = \frac{1}{2}\Delta \eta^\varepsilon + \lambda_\varepsilon w \cdot \Pi_{\frac{1}{\varepsilon}} \nabla \left(\Pi_{\frac{1}{\varepsilon}} \eta^\varepsilon\right)^2 + (-\Delta)^{\frac{1}{2}} \xi$$

$\hookrightarrow$ space-time Gaussian white noise

$$\lambda_\varepsilon = \begin{cases} \frac{1}{\sqrt{|\log \varepsilon^2|}} & d = 2 \\ \varepsilon^{\frac{d}{2}-1} & d \geq 3 \end{cases}$$

$\varepsilon > 0$

$\eta^\varepsilon(0) =$ space white noise with $\int_{\mathbb{T}^d} \eta^\varepsilon(0,x)\,dx = 0$ (0 space average)

$\eta(\varphi) \stackrel{d}{=} \mathcal{N}(0, \|\varphi\|_2^2)$

$\int_{\mathbb{T}^d} \varphi(x)\,dx = 0$

Notation: $\mathbb{P}, \mathbb{E}$: refers to space white noise

$\mathbf{P}, \mathbf{E}$: refers to law of process. Depends on ε

The equation in Fourier space

$\varphi : \mathbb{T}^d \to \mathbb{R}$ $\qquad \hat{\varphi}(k) = \int_{\mathbb{T}^d} \frac{e^{-ikx}}{(2\pi)^{d/2}} \varphi(x)\,dx \qquad k \in \mathbb{Z}^d \setminus \{0\}$

Note: $\widehat{(-\Delta)^{\frac{1}{2}} \varphi}(k) = |k|\, \hat{\varphi}(k)$

The Fourier modes $\hat{\eta}^{\varepsilon}(k)$, $k \in \mathbb{Z}^d \setminus \{0\}$ solve system of coupled SDEs

$$d\hat{\eta}^{\varepsilon}(k) = -\frac{|k|^2}{2} \hat{\eta}^{\varepsilon}(k)\,dt + i\lambda_{\varepsilon}(w \cdot k) \sum_{\substack{\ell, m : \\ \ell + m = k}} \mathcal{J}^{\varepsilon}_{\ell, m}\, \hat{\eta}^{\varepsilon}(\ell)\,\hat{\eta}^{\varepsilon}(m)\,dt + |k|\,dB_t(k)$$

$$\mathcal{J}^{\varepsilon}_{\ell, m} \equiv \mathbb{1}_{0 < |\ell|, |m|, |\ell + m| \le \frac{1}{\varepsilon}} \qquad d\langle B(k), B(\ell) \rangle_t = \mathbb{1}_{k + \ell = 0}$$

NB: Fourier modes $\hat{\eta}^{\varepsilon}(k)$, $|k| > \frac{1}{\varepsilon}$ evolve as indep. Ornstein-Uhlenbeck

$\to$ trivial

Wiener chaos decomposition

$$L^2(\mathbb{P}) = \bigoplus_{n \geq 0} \mathcal{H}_n$$

$\downarrow$

$n=0$: constant functions

$\mathcal{H}_1$: span of $\{ \hat{\eta}(k) \, , \, k \in \mathbb{Z}^d_0 \}$

$\mathcal{H}_n$: span of $\{ :\hat{\eta}(k_1) \ldots \hat{\eta}(k_n): \}$ $\qquad$ $:\ :$ Wick product or Hermite poly.

that is, $\mathcal{H}_n$ = span of monomials $\hat{\eta}(k_1) \ldots \hat{\eta}(k_n)$,

projected orthogonally to $\mathcal{H}_0, \ldots \mathcal{H}_{n-1}$

E.g.: $\quad :\hat{\eta}(k)\hat{\eta}(\ell): \; = \; \hat{\eta}(k)\hat{\eta}(\ell) - \mathbb{E}\,\hat{\eta}(k)\hat{\eta}(\ell) = \hat{\eta}(k)\hat{\eta}(\ell) - \mathbb{1}_{\ell+k=0}$

always interested in $k_i \neq 0$

Given $F \in L^2(\mathbb{P})$, unique decomposition

$$F = \sum_{n \geq 0} F_n \qquad F_n \in \mathcal{H}_n \qquad F_n = \sum_{k_1 \ldots k_n} f_n(k_1, \ldots, k_n) :\hat{\eta}(k_1) \cdots \hat{\eta}(k_n):$$

$\downarrow$ Symmetric Kernel

e.g. $F(\eta) = \hat{\eta}(k)\hat{\eta}(\ell)\hat{\eta}(m) + \hat{\eta}(p)\hat{\eta}(q) \in \bigoplus_{j \leq 3} \mathcal{H}_j$

NB we identify F with its kernels f

Linear operators on $L^2(\mathbb{P})$ $\sim$ Linear operators on the space of kernels (Fock space)

One can check: $\mathbb{E}\, F G = \langle f, g \rangle := \sum_n n! \langle f_n, g_n \rangle_{L^2_m}$

Example $\quad F = G = :\hat\eta(k)\hat\eta(-k): = \hat\eta(k)\hat\eta(-k) - 1 \in \mathcal{H}_2$

$$\mathbb{E} F^2 = \mathbb{E}\,\hat\eta(k)\hat\eta(k)\hat\eta(-k)\hat\eta(-k) + 1 - 2\,\mathbb{E}\,\hat\eta(k)\hat\eta(-k) = 2 + 1 - 2 = 1$$

$$F = \sum_{k_1,k_2} f_2(k_1,k_2) :\hat\eta(k_1)\hat\eta(k_2): \qquad f_2(k_1,k_2) = \begin{cases} \frac{1}{2} & k_1 = k,\ k_2 = -k \text{ or } k_1 = -k,\ k_2 = k \\ 0 & \text{else} \end{cases}$$

$$2\,\|f_2\|_2^2 = 2\cdot\left(\frac{1}{4} + \frac{1}{4}\right) = 1$$

The Generator.

$\mathcal{L} = \mathcal{L}^{\varepsilon}$: generator of the Markov process

Recall Itô's formula for a diffusion: (n-dimensional)

if $dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t$ $\quad \sigma \quad n \times n$

$\hookrightarrow$ n-dimensional

Then $(\mathcal{L}f)(x) = \sum_{i=1}^{n} b_i(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j} \left(\sigma(x)\sigma^T(x)\right)_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$

Applying to our SDEs [exercise. Be careful with complex BM!]

$$\mathcal{L}^{\varepsilon} = \mathcal{L}_0 + A^{\varepsilon}$$

$$(\mathcal{L}_0 F)(\eta) = \frac{1}{2} \sum_k |k|^2 \left(-\hat{\eta}(-k) D_k + D_{-k} D_k \right) F(\eta)$$

with $D_k = \dfrac{\partial}{\partial \hat{\eta}(-k)}$ (can be defined more elegantly via Malliavin derivative)

$$(A^{\varepsilon} F)(\eta) = i \lambda_{\varepsilon} \sum_{m,\ell} J^{\varepsilon}_{m,\ell} \, w \cdot (m+\ell) \, \hat{\eta}(m) \, \hat{\eta}(\ell) \, D_{-m-\ell} F(\eta)$$

Remark: $\mathcal{L}_0$ generator of linear eq. Independent of ε

Properties of derivative D_k

- $\mathbb{E}(G D_k F) = \mathbb{E}(-F D_k G + F G \eta(k))$ integration by parts

- if $F \in \mathcal{H}_n$ with kernel f_n then $D_p F \in \mathcal{H}_{n-1}$ with kernel $f_{n-1}(k_1 \dots k_{n-1}) = n f_n(k_1 \dots k_{n-1}, p)$

Notation $k_{1:n} \equiv (k_1, \dots, k_n)$

Stationarity of white noise

Claim: • for every (cylinder) function F,

$\mathbb{E}\,\mathcal{L}^{\varepsilon} F = 0$ Actually, $\mathbb{E}\,\mathcal{L}_0 F$, $\mathbb{E}\,A_\varepsilon F$ are separately 0.

• A_ε is skew-symmetric: $\mathbb{E}\, G A_\varepsilon F = -\mathbb{E}\, F A_\varepsilon G$

Proof: exercise, using integration by parts formula

$\Rightarrow$ $\mathbb{P}$ is stationary

Action of $\mathcal{L}_0$ and A_ε on kernels

Proposition

- $\mathcal{L}_0 : \mathcal{H}_n \longrightarrow \mathcal{H}_n$ Laplacian

$$(\mathcal{L}_0 f)(k_{1:n}) = -\frac{1}{2}|k_{1:n}|^2 f(k_{1:n}) \qquad |k_{1:n}|^2 = \sum_{i=1}^{n} |k_i|^2$$

Diagonal in $\underline{\underline{n}}$ and in $\underline{\underline{k}}$

- $A_\varepsilon = A_\varepsilon^+ + A_\varepsilon^- \qquad (A_\varepsilon^+)^* = -A_\varepsilon^-$

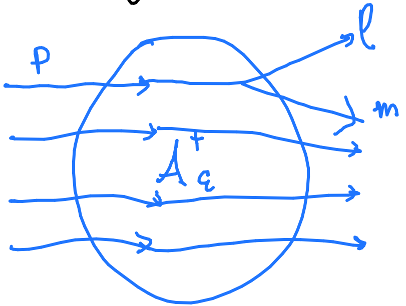
$A_\varepsilon^+ : \mathcal{H}_n \to \mathcal{H}_{n+1}$ "creation"

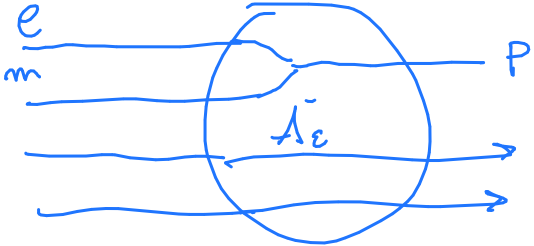
$A_\varepsilon^- : \mathcal{H}_n \to \mathcal{H}_{n-1}$ "annihilation"

$$(A_\varepsilon^+ f)(k_{1:n+1}) = - i \frac{\lambda_\varepsilon}{n+1} \sum_{1 \le i < j \le n+1} \omega \cdot (k_i + k_j)\, J^\varepsilon_{k_i,k_j} f(k_i+k_j, k_{1:n+1} \setminus \{i,j\})$$

$$(A_\varepsilon^- f)(k_{1:n-1}) = i\, n\, \lambda_\varepsilon \sum_{j=1}^{n-1} \omega \cdot k_j \sum_{\ell + m = k_j} J^\varepsilon_{\ell,m} f(\ell, m, k_{1:n-1} \setminus \{j\})$$

Proof: "standard" but lengthy. Uses properties of Wick products.





$\ell + m = p$

More concretely (examples)

1) let $F(\eta) = \eta(\varphi) = \int_{\mathbb{T}^d} dx\, \varphi(x)\, \eta(x)$

$$F(\eta) = \sum_k \hat{\eta}(k)\, \hat{\varphi}(-k)$$

$$(\mathcal{L}_0 F)(\eta) = \sum_k \hat{\varphi}(-k)\, \mathcal{L}_0\, \hat{\eta}(k)$$

$$= \sum_k -\frac{|k|^2}{2}\, \hat{\varphi}(-k)\, \hat{\eta}(k) =$$

$$= \eta\left(\frac{\Delta}{2}\varphi\right)$$

2) similarly if $F(\eta) = \eta(\varphi)\,\eta(\psi)$ then

$$(\mathcal{L}_0 F)(\eta) = \eta\left(\frac{\Delta}{2}\varphi\right)\eta(\psi) + \eta(\varphi)\,\eta\left(\frac{\Delta}{2}\psi\right)$$

3) $F(\eta) = \eta(\varphi) = \sum_k \hat{\varphi}(-k)\hat{\eta}(k)$

$$(A_+^\varepsilon F)(\eta) = \sum_k \hat{\varphi}(-k)\, \underbrace{A_+^\varepsilon \hat{\eta}(k)}_{\text{Kernel} = \mathbb{1}_{\cdot = k}} = -i \frac{\lambda_\varepsilon}{2} \sum_k \hat{\varphi}(-k) \sum_{\substack{\ell, m: \\ \ell+m=k}} (\omega \cdot \overset{\overset{\ell+m}{\uparrow}}{k}) \cdot \hat{\eta}(\ell)\hat{\eta}(m) \cdot \mathbb{1}_{\substack{0<|k|\le\frac{1}{\varepsilon} \\ 0<|\ell|\le\frac{1}{\varepsilon} \\ 0<|m|\le\frac{1}{\varepsilon}}}$$

$$= -\frac{\lambda_\varepsilon}{2} \sum_k \hat{\varphi}(-k)\, i(\omega\cdot k)\, \mathbb{1}_{0<|k|\le\frac{1}{\varepsilon}} \underbrace{\sum_\ell \hat{\eta}(\ell)\hat{\eta}(k-\ell)\, \mathbb{1}_{0<|\ell|\le\frac{1}{\varepsilon}} \mathbb{1}_{0<|\ell-k|\le\frac{1}{\varepsilon}}}_{(\Pi_{\frac{1}{\varepsilon}}\eta)^2(k)}$$

$$= -\frac{\lambda_\varepsilon}{2}\sum_k \hat{\varphi}(-k)\, i(\omega\cdot k)\, \mathbb{1}_{0<|k|\le\frac{1}{\varepsilon}} \underbrace{\sum_{\ell} \hat{\eta}(\ell)\hat{\eta}(k-\ell)\, \mathbb{1}_{0<|\ell|\le\frac{1}{\varepsilon}}\, \mathbb{1}_{0<|\ell-k|\le\frac{1}{\varepsilon}}}_{\left(\Pi_{\frac{1}{\varepsilon}}\eta\right)^2(k)}$$

$$= -\frac{\lambda_\varepsilon}{2}\sum_k \hat{\varphi}(-k)\, \underbrace{i(\omega\cdot k)\, \Pi_{\frac{1}{\varepsilon}}\left(\Pi_{\frac{1}{\varepsilon}}\eta\right)^2(k)}_{\Pi_\varepsilon(\nabla\cdot\omega)\left(\Pi_{\frac{1}{\varepsilon}}\eta\right)^2(k)} = -\frac{\lambda_\varepsilon}{2}\int \varphi(x)(\omega\cdot\nabla)\,\Pi_{\frac{1}{\varepsilon}}\left(\Pi_{\frac{1}{\varepsilon}}\eta\right)^2(x)$$

$$\mathcal{A}_+^\varepsilon \eta(\varphi) = -\frac{\lambda_\varepsilon}{2}\int \varphi(x)(\omega\cdot\nabla)\,\Pi_{\frac{1}{\varepsilon}}\left(\Pi_{\frac{1}{\varepsilon}}\eta^2(x)\right)$$

Reminder: the theorem for $d=2$

Th. Let η^ε be the solution of

$$\begin{cases} \partial_t \eta^\varepsilon = \frac{1}{2}\Delta \eta^\varepsilon + \lambda_\varepsilon\, w\cdot \Pi_{\frac{1}{\varepsilon}} \nabla \left(\Pi_{\frac{1}{\varepsilon}} \eta^\varepsilon\right)^2 + (-\Delta)^{\frac{1}{2}} \xi \\ \eta^\varepsilon(0) = \text{white noise} \end{cases}$$

One has $\eta^\varepsilon \Rightarrow \eta$ (in $C([0,T], \mathcal{S}'(\mathbb{T}^2))$, $\forall\, T>0$)

where η solves

$$\partial_t \eta = \mathcal{L}^{\text{eff}} \eta + \left(-2\mathcal{L}^{\text{eff}}\right)^{\frac{1}{2}} \xi \qquad \text{(S.H.E.)}$$

$$\mathcal{L}^{\text{eff}} = \underbrace{\frac{\Delta}{2}}_{=\mathcal{L}_0} + \overbrace{c(w)\,\frac{(w\cdot\nabla)^2}{2}}^{\equiv \mathcal{L}_0^w}$$

$$c(w) = \frac{1}{|w|^2}\left[\left(\frac{3|w|^2}{2\pi} + 1\right)^{\frac{2}{3}} - 1\right]$$

General scheme.

Not surprisingly, 2 steps

- tightness in $C([0,T], S'(\mathbb{T}^2))$ (easy)
- every limit point solves S.H.E.

Characterisation of S.H.E. (Classical)

η solves S.H.E. $\iff$ $\forall\ \varphi, \psi \in S(\mathbb{T}^2)$ test functions and

$F(\eta) = \eta(\varphi) \in \mathcal{H}_1$ or $F(\eta) = \eta(\varphi)\eta(\psi) - \langle \varphi, \psi \rangle = :\eta(\varphi)\eta(\psi): \quad \in \mathcal{H}_2$,

$F(\eta_t) - F(\eta_0) - \int_0^t \mathcal{L}^{\text{eff}} F(\eta_s)\, ds$ is a martingale

N.B:

- $\mathcal{L}^{\text{eff}} \eta(\varphi) = \eta(\mathcal{L}^{\text{eff}} \varphi)$, $\mathcal{L}^{\text{eff}} :\eta(\varphi)\eta(\psi): = :\eta(\mathcal{L}^{\text{eff}}\varphi)\eta(\psi): + :\eta(\varphi)\eta(\mathcal{L}^{\text{eff}}\psi):$
- 1^{st} & 2^{nd} chaos enough because process is Gaussian

Tightness Easy, but explains why $\lambda_\varepsilon = \begin{cases} \frac{1}{\sqrt{|\log \varepsilon^2|}} & d=2 \\ \varepsilon^{\frac{d}{2}-1} & d\geq 3. \end{cases}$ 44

Enough to prove that $\{\eta^\varepsilon(\varphi)\}_\varepsilon$ is tight for every $\varphi \in \mathcal{S}(\mathbb{T}^2)$.

By Kolmogorov's criterion, we want uniform bound like $\underset{\text{in } \varepsilon}{\downarrow}$

$$\mathbf{E}\left(\eta^\varepsilon_t(\varphi) - \eta^\varepsilon_s(\varphi)\right)^\alpha \leq C(\varphi)\,|t-s|^{1+\beta} \qquad \alpha,\beta>0$$

(take $s=0$ by stationarity)

Integrate the SBE in time:

$$\eta^\varepsilon_t(\varphi) - \eta^\varepsilon_0(\varphi) = \underbrace{\int_0^t \eta^\varepsilon_u\left(\frac{\Delta\varphi}{2}\right)du}_{\mathrm{I}} + \underbrace{\int_0^t \underbrace{N^\varepsilon_\varphi(\eta^\varepsilon_u)}_{\text{Non-linearity tested against } \varphi} du}_{\mathrm{II}} + \underbrace{\int_0^t \xi(du, (-\Delta)^{\frac{1}{2}}\varphi)}_{\mathrm{III}}$$

III : ε-independent, Gaussian ✓

I & II : use "Itô trick" (Gubinelli - Jara '13)

Itô trick Let $F \in \bigoplus_{j=1}^{n} \mathcal{H}_j$, $p \geq 2$. $\exists C = C(p,n)$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t F(\eta_s^\varepsilon)\,ds\right|^p\right] \leq C\, T^{p/2} \left\|(-\mathcal{L}_0)^{-\frac{1}{2}} F\right\|^p$$

NB $\frac{p}{2} > 1$, $p > 2$

[Notation: $\|F\|^2 := \mathbb{E}(F^2)$]

for I, $F(\eta) = \eta(\Delta\varphi)$. Recalling that $\mathcal{L}_0 = \frac{(-\Delta)}{2}$,

$$\|(-\mathcal{L}_0)^{-\frac{1}{2}}\eta(\Delta)\| = \|\eta((-\Delta)^{-\frac{1}{2}}(-\Delta)\varphi)\| = \|\eta((-\Delta)^{\frac{1}{2}}\varphi)\| = \|(-\Delta)^{\frac{1}{2}}\varphi\|_{L^2}$$

for II $\qquad F = N_\varphi^\varepsilon(\eta)$

Note that

$$N_\varphi^\varepsilon(\eta) = i\lambda_\varepsilon \sum_{\ell,m} J^\varepsilon_{\ell,m}\, \omega\cdot(\ell+m)\, \hat\varphi(-\ell-m) :\hat\eta(\ell)\,\hat\eta(m):$$

so that

$$\left\| (-\mathcal{L}_0)^{-\frac12} N_\varphi^\varepsilon(\eta) \right\| = \lambda_\varepsilon^2 \sum_{\ell,m} \frac{(\omega\cdot(\ell+m))^2}{\underbrace{|\ell|^2+|m|^2}_{\text{from } (-\mathcal{L}_0)^{-\frac12}}} |\hat\varphi(\ell+m)|^2$$

$$= \sum_k (\omega\cdot k)^2 |\hat\varphi(k)|^2 \left(\lambda_\varepsilon^2 \sum_{\ell+m=k} \frac{J^\varepsilon_{\ell,m}}{|\ell|^2+|m|^2} \right)$$

Exercise: with λ_ε as above, $(\dots)$ uniformly bdd.

Identifying the limit

Recall: we want

$$F(\eta_t) - F(\eta_0) - \int_0^t \mathcal{L}^{\text{eff}} F(\eta_s)\,ds \equiv M_t^F \quad \text{to be a martingale}$$

↳ sub-sequential limit of η^ε.

$\forall$ F of the form $\begin{cases} F(\eta) = \eta(\varphi) \in \mathcal{H}_1 \\ F(\eta) = :\eta(\varphi)\eta(\psi): \in \mathcal{H}_2 \end{cases}$

$w \mapsto \frac{1}{2}(\nabla\cdot w)^2$

where $\mathcal{L}^{\text{eff}}$ acts diagonally on each coordinate as $\mathcal{L}_0 + c(w)\mathcal{L}_0$

That is: $$(\mathcal{L}^{\text{eff}} f)(k_{1:n}) = \sum_{j=1}^{n}\left[-\frac{|k_j|^2}{2} - \frac{c(w)}{2}(k_j\cdot w)^2\right] f(k_{1:n})$$

$f \in \mathcal{H}_n$

Consider, for simplicity, $F = \eta(\varphi) \in \mathcal{H}_1$.

Starting point:

$$\eta^\varepsilon_t(\varphi) - \eta^\varepsilon_0(\varphi) - \int_0^t \eta^\varepsilon_s\left(\frac{\Delta}{2}\varphi\right) ds - \int_0^t \left(\mathcal{A}^\varepsilon \eta^\varepsilon_s(\varphi)\right) ds = \mathcal{M}^{F,\varepsilon}_t$$

↳ Martingale

Rewrite:

$$\eta^\varepsilon_t(\varphi) - \eta^\varepsilon_0(\varphi) - \int_0^t \eta^\varepsilon_s\left(\frac{\Delta}{2}\varphi\right) ds + \int_0^t c(\omega)\, \eta^\varepsilon_s\left((\omega\cdot\nabla)^2 \varphi\right) ds$$

$$- \int_0^t c(\omega)\, \eta^\varepsilon_s\left((\omega\cdot\nabla)^2 \varphi\right) ds - \int_0^t \left(\mathcal{A}^\varepsilon \eta^\varepsilon_s(\varphi)\right) ds = \mathcal{M}^{F,\varepsilon}_t$$

Note: $\mathcal{A}^\varepsilon F = \mathcal{A}^\varepsilon_+ F + \mathcal{A}^\varepsilon_- F = \mathcal{A}^\varepsilon_+ F$ because $\mathcal{A}^\varepsilon_- : \mathcal{H}_1 \to 0$

$$\eta^\varepsilon_t(\varphi) - \eta^\varepsilon_0(\varphi) - \int_0^t \eta^\varepsilon_s\left(\frac{\Delta}{2}\varphi\right) ds - \int_0^t c(\omega)\, \eta^\varepsilon_s\left(\frac{(\omega\cdot\nabla)^2}{2}\varphi\right) ds$$

$$+ \int_0^t \Big[\underbrace{c(\omega)\, \eta^\varepsilon_s\left(\frac{(\omega\cdot\nabla)^2}{2}\varphi\right) ds}_{\mathcal{H}_1} - \underbrace{\left(A^\varepsilon_+ \eta^\varepsilon_s(\varphi)\right)}_{\mathcal{H}_2} \Big] ds = \mathcal{M}^{F,\varepsilon}_t$$

No way that $\int_0^t \cdots ds$ can be small

BUT

it can be $\underset{\varepsilon\to 0}{\approx}$ martingale $\mathcal{U}^\varepsilon_t$ $\longrightarrow$ new noise in the limit eq.

Suppose that we can find $c = c(\omega)$ (constant) and $V^\varepsilon \in L^2(\mathbb{P})$:

or $\approx$ as $\varepsilon \to 0$

$$A_+^\varepsilon \eta(\varphi) - \frac{1}{2}\eta\left(c(\omega)(\omega\cdot\nabla)^2\varphi\right) \simeq -\mathcal{L}^\varepsilon V^\varepsilon$$

"Fluctuation - dissipation relation"

(Landim - Yau '97
Komorowski - Olla - Landim (BOOK) '12

and in addition

$$\| V^\varepsilon \| \xrightarrow[\varepsilon \to 0]{} 0 \qquad (*)$$

Then,

$$V^\varepsilon(\eta_t^\varepsilon) - V^\varepsilon(\eta_0^\varepsilon) - \int_0^t \mathcal{L}^\varepsilon V^\varepsilon(\eta_s^\varepsilon)\,ds = M_t^{\varepsilon,F} \quad \text{martingale}$$

$$U^\varepsilon(\eta^\varepsilon_t) - U^\varepsilon(\eta^\varepsilon_0) - \int_0^t \mathcal{L}^\varepsilon U^\varepsilon(\eta^\varepsilon_s)\,ds = \mathcal{M}^{\varepsilon,F}_t \quad \text{martingale}$$

$\downarrow 0$ by $(*)$ $\qquad$ $\downarrow 0$ by $(*)$

$$+\int_0^t \left[A^\varepsilon_+ \eta^\varepsilon_s(\varphi) - \frac{1}{2}\eta^\varepsilon_s\left(c(\omega)(\nabla\cdot\omega)^2\varphi\right)\right] ds = \mathcal{M}^{\varepsilon,F}_t$$

as desired.

Summarizing, we need to find U^ε and $c = c(\omega)$ s.t.

- $A^\varepsilon_+ \eta(\varphi) = -\mathcal{L}^\varepsilon U^\varepsilon + \frac{1}{2}\eta\left(c\,(\nabla\cdot\omega)^2\varphi\right)$ + small error as $\varepsilon \to 0$
- $\|U^\varepsilon\| \to 0$ as $\varepsilon \to 0$
- $\mathcal{M}^{\varepsilon,F}_t$ (and $\mathcal{M}^F_t$) remain martingales as $\varepsilon \to 0$

How to approach

$$\underbrace{A_+^{\varepsilon}\eta(\varphi)}_{\mathcal{H}_2} = -\mathcal{L}^{\varepsilon}\underbrace{V^{\varepsilon}}_{\text{unknown},\ \in \bigoplus_{n=1}^{\infty}\mathcal{H}_n} + \frac{1}{2}\eta\Big(\overbrace{\underbrace{c}_{\text{unknown}}\,(\nabla\cdot\omega)^{\perp}\varphi}^{\in\mathcal{H}_1}\Big) \quad ?$$

$$V^{\varepsilon} = (V_j^{\varepsilon})_{j\geq 1}$$

Why difficult? ∞-many coupled equations

Project on $\mathcal{H}_1$

$$0 = -\mathcal{L}_0 V_1^{\varepsilon} - A_-^{\varepsilon} V_2^{\varepsilon} + \frac{1}{2}\eta(\dots\dots)$$

$$A_+^{\varepsilon}\eta(\varphi) = -\mathcal{L}_0 V_2^{\varepsilon} - A_-^{\varepsilon} V_3^{\varepsilon} - A_+^{\varepsilon} V_1^{\varepsilon}$$

$$\vdots$$

Idea: ~~find the true~~ $\cancel{v^\varepsilon}$ $\rightarrow$ guess an approximate v^ε

How?

Assume $\exists\, H^\varepsilon : \bigoplus_n \mathcal{H}_n \rightarrow \bigoplus_n \mathcal{H}$ positive, self-adjoint:

$$H^\varepsilon = -A_-^\varepsilon(-\mathcal{L}_0 + H^\varepsilon)^{-1} A_+^\varepsilon$$

(fixed-point operator eq)

&

$$H^\varepsilon \eta(\varphi) \rightarrow -\eta\Big(c(\omega)\underbrace{(\omega\cdot\nabla)^2}_{2}\varphi\Big) \quad \text{as } \varepsilon \to 0$$

Then, we can find v^ε

In fact, define $v^\varepsilon = (v_j^\varepsilon)_{j\geq 1}$ as $v_1^\varepsilon = 0$ &

$$v^\varepsilon \doteq (-\mathcal{L}_0 + H^\varepsilon)^{-1} A_+^\varepsilon v^\varepsilon + (-\mathcal{L}_0 + H^\varepsilon)^{-1} A_+^\varepsilon \eta(\varphi)$$

Let's compute

$$-\mathcal{L}^\varepsilon v^\varepsilon - A_+^\varepsilon \eta(\varphi) + \frac{1}{2}\eta(c(\nabla\cdot w)^2 \varphi) =$$

$$= (-\mathcal{L}_0 + H^\varepsilon) v^\varepsilon - H^\varepsilon v^\varepsilon - A_+^\varepsilon v^\varepsilon - A_-^\varepsilon v^\varepsilon - A_+^\varepsilon \eta(\varphi) + \frac{1}{2}\eta(\dots)$$

$$= \cancel{A_+^\varepsilon v^\varepsilon} + \cancel{A_+^\varepsilon \eta(\varphi)} - H^\varepsilon v^\varepsilon - \cancel{A_+^\varepsilon v^\varepsilon} - A_-^\varepsilon v^\varepsilon - \cancel{A_+^\varepsilon \eta(\varphi)} + \frac{1}{2}\eta(\dots)$$

$$= -H^\varepsilon v^\varepsilon - A_-^\varepsilon \cancel{(-\mathcal{L}_0 + H^\varepsilon)^{-1}} A_+^\varepsilon v^\varepsilon - A_-^\varepsilon (-\mathcal{L}_0 + H^\varepsilon)^{-1} A_+^\varepsilon \eta(\varphi) + \frac{1}{2}\eta(\dots)$$

$$= H^\varepsilon \eta(\varphi) + \frac{1}{2}\eta\left(c(w)(w\cdot\nabla)^2 \varphi\right) \longrightarrow 0$$

$$H^\varepsilon = -A_-^\varepsilon(-\mathcal{L}_0 + H^\varepsilon)^{-1}A_+^\varepsilon$$ Fixed-point operator equation

Difficulty : finding (or even proving $\exists$!) fixed point : hard (= no clue)

Way out : sufficient to find an approximate fixed point

The "replacement Lemma" (aka: the approximate fixed point)

Define

- $G(x) := \frac{1}{|\omega|^2}\left[\left(\frac{3|\omega|^2}{2\pi}x + 1\right)^{\frac{2}{3}} - 1\right]$

NB: $G(1) = c(\omega)$

- $L^\varepsilon(x) := \frac{1}{|\log \varepsilon^2|}\log\left(1 + \frac{1}{\varepsilon^2}x\right)$

NB: $L^\varepsilon(\cdot) \xrightarrow{\varepsilon\to 0} 1$ pointwise

- $\mathcal{G}^\varepsilon := G(L^\varepsilon(-\mathcal{L}_0))$

N.B: $\mathcal{G}^\varepsilon \psi \to c(\omega)\psi$ $\forall$ fixed $\psi \in L^2(\mathbb{R})$

- $\mathcal{N}$: $\mathcal{N}\psi = n\psi$ if $\psi \in \mathcal{H}_n$

"Number operator"

Lemma $\exists\ C > 0 : \ \forall\ \psi_1, \psi_2 \in L^1(\mathbb{P})$

$$\left| \left\langle \left[-A_-^\varepsilon \left(-\mathcal{L}_0 - \mathcal{L}_0^\omega \mathcal{G}^\varepsilon \right)^{-1} A_+^\varepsilon + \mathcal{L}_0^\omega \mathcal{G}^\varepsilon \right] \psi_1, \psi_2 \right\rangle \right| \leq$$

$$\leq C \lambda_\varepsilon^2 \left\| \mathcal{N} (-\mathcal{L}_0)^{1/2} \psi_1 \right\| \left\| \mathcal{N} (-\mathcal{L}_0)^{1/2} \psi_2 \right\|$$

In words : $-\mathcal{L}_0^w \mathcal{G}^\varepsilon \approx \mathcal{H}^\varepsilon$ as $\varepsilon \to 0$

$$-\mathcal{L}_0^w \mathcal{G}^\varepsilon \eta(\varphi) \to -\tfrac{1}{2} \eta\left(c(\omega)(\omega \cdot \nabla)^2 \varphi \right)$$

Replacement Lemma: idea of the proof

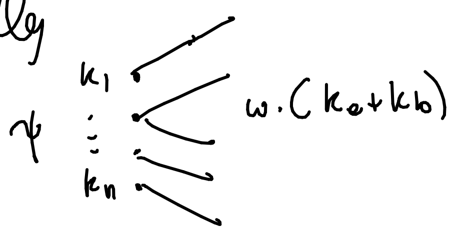
We want to compute

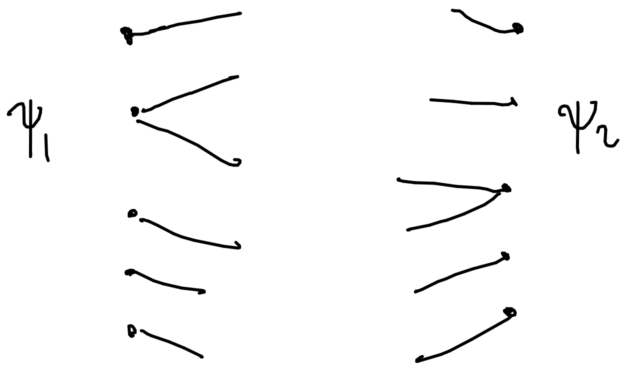
$$\left\langle -\mathcal{A}_-^{\varepsilon}\left(-\mathcal{L}_0+\mathcal{L}^{\omega}\mathcal{G}^{\varepsilon}\right)^{-1}\mathcal{A}_+^{\varepsilon}\psi_1,\ \psi_2\right\rangle = \left\langle \left(-\mathcal{L}_0+\mathcal{L}^{\omega}G\left(L^{\varepsilon}(-\mathcal{L}_0)\right)\right)^{-1}\mathcal{A}_+^{\varepsilon}\psi_1,\ \mathcal{A}_+^{\varepsilon}\psi_2\right\rangle$$

Recall how $\mathcal{A}_+^{\varepsilon}$ acts. Say, $\psi \in \mathcal{H}_n$

$$\mathcal{A}_+^{\varepsilon}\psi(k_{1:n+1}) = -\frac{i}{n+1}\lambda_{\varepsilon}\sum_{1\le a<b\le n+1}\omega\cdot(k_a+k_b)\,\mathbb{J}^{\varepsilon}_{k_a,k_b}\,\psi\left(k_a+k_b,\ k_{1:n+1\setminus\{a,b\}}\right)$$

Schematically





$\langle \qquad \rangle$ = Sum over momenta

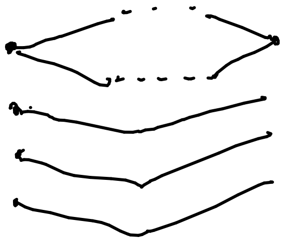
$k_1 \dots k_{n+1} \to l, m, k_{1:n}$

$l + m = k_1$

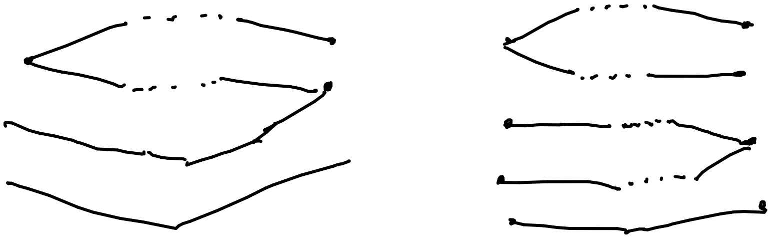
$\downarrow$

by symm.

"Diagonal diagrams"



"Off diagonal"



Turns out : off-diagonal terms are negligible

Diagonal :

from $(\omega \cdot (\ell + m))^2$ $\quad \ell + m = k_1$

$$m \sum_{k_1} (\omega \cdot k_1)^2 \, \psi_1(-k_{1:n}) \, \psi_2(k_{1:n}) \cdot \lambda_\varepsilon^2 \sum_{\ell + m = k_1} \frac{J^\varepsilon_{\ell, m}}{\Gamma + \Gamma^\omega G(L^\varepsilon(\Gamma))}$$

from choice of whom to split

from $(-\mathcal{L}_0 + \mathcal{L}^\omega G^\varepsilon)^{-1}$

$$m (\omega \cdot k_1)^2 \to \sum_1^n (k_i \cdot \omega)^2 = (-\mathcal{L}_0^W)$$

$$\Gamma = |\ell|^2 + |m|^2 + |k_{2:n}|^2$$

$$\Gamma^\omega = (\ell \cdot \omega)^2 + (m \cdot \omega)^2 + (k_{2:n} \cdot \omega)^2$$

After some amount of work,

$$\text{red term} \approx \frac{1}{\pi}\int_0^{L^\varepsilon(\frac{1}{2}|k_{1:n}|^2)} \frac{dy}{\sqrt{1+|\omega|^2 G(y)}} = G\left(L^\varepsilon\left(\tfrac{1}{2}|k_{1:n}|^2\right)\right)$$

$\Leftrightarrow$ G as in the Lemma

so that

$$\left\langle \left(-\mathcal{L}_0 + \mathcal{L}^\omega G\left(L^\varepsilon(-\mathcal{L}_0)\right)\right)^{-1} A_+^\varepsilon \psi_1, A_+^\varepsilon \psi_2 \right\rangle \approx \sum_{k_{1:n}} (\omega \cdot k_{1:n})^2 \psi_1(-k_{1:n})\, \psi_2(k_{1:n})\, G\left(L^\varepsilon\left(\tfrac{1}{2}|k_{1:n}|^2\right)\right)$$

$$= \left\langle (-\mathcal{L}_0)^\omega\, \mathcal{G}^\varepsilon \psi_1, \psi_2 \right\rangle \quad \text{as desired}$$

"$\square$"

Points swept under the carpet:

- Role of N [in Itô trick & Replacement lemma]

  Solution $\quad v^{\varepsilon} \to v^{\varepsilon, n} \in \bigoplus_{i}^{n} \mathcal{H}_j$

  $\varepsilon \to 0$ first

  $n \to \infty$ later

- for characterisation of SHE, need also $F = :\eta(\varphi)\eta(\psi):$, not just $F = \eta(\varphi)$

- the martingales $M^{\varepsilon, F}_{\cdot}$, $\mathcal{M}^{F,\varepsilon}_{\cdot}$ (page 20) tend to martingales as $\varepsilon \to 0$

  (not difficult, control on moments)

End of part II

Reminder: Stochastic Burgers Equation (SBE), $d \geq 3$

$\eta^\varepsilon = \eta^\varepsilon(t,x)$ $t \geq 0$, $x \in \mathbb{T}^d$ (d-dim Torus, side 2π)

$$\partial_t \eta^\varepsilon = \frac{1}{2}\Delta \eta^\varepsilon + \lambda_\varepsilon w \cdot \Pi_{\frac{1}{\varepsilon}} \nabla \left(\Pi_{\frac{1}{\varepsilon}} \eta^\varepsilon\right)^2 + (-\Delta)^{\frac{1}{2}} \xi$$

↳ space-time Gaussian white noise

$$\lambda_\varepsilon = \varepsilon^{\frac{d}{2}-1} \to 0 \quad \text{as } \varepsilon \to 0$$

$\varepsilon > 0$

$\eta^\varepsilon(0)$ = space white noise with 0 space average

$$\eta(\varphi) \stackrel{d}{=} N(0, \|\varphi\|_2^2)$$

$$\int_{\mathbb{T}^d} \varphi(x)\,dx = 0$$

Notation:

$\mathbb{P}, \mathbb{E}$: refers to space white noise

$\mathbf{P}, \mathbf{E}$: refers to law of process. Depends on ε

Reminder: the theorem for $d \geq 3$

Th. Let η^ε be the solution of

$$\begin{cases} \partial_t \eta^\varepsilon = \frac{1}{2}\Delta \eta^\varepsilon + \lambda_\varepsilon\, w\cdot \Pi_{\frac{1}{\varepsilon}} \nabla \left(\Pi_{\frac{1}{\varepsilon}} \eta^\varepsilon\right)^2 + (-\Delta)^{\frac{1}{2}}\xi \\ \eta^\varepsilon(0) = \text{white noise} \end{cases}$$

One has $\eta^\varepsilon \Rightarrow \eta$ (in $C([0,T], \mathcal{S}'(\mathbb{T}^d))$, $\forall\, T>0$),

where η solves

$$\partial_t \eta = \mathcal{L}^{\text{eff}} \eta + \left(-2\mathcal{L}^{\text{eff}}\right)^{\frac{1}{2}} \xi \qquad \mathcal{L}^{\text{eff}} = \overbrace{\frac{\Delta}{2}}^{\mathcal{L}_0 \,=} + c(w) \overbrace{\frac{(w\cdot\nabla)^2}{2}}^{\mathcal{L}_0^w \,\equiv}$$

(S.H.E.)

$c(w) > 0$

not explicit

Scheme of proof

- Tightness: already shown ✓
- characterisation of limit: exactly like for $d=2$: we want

$$F(\eta_t) - F(\eta_0) - \int_0^t \mathcal{L}^{\mathrm{eff}} F(\eta_s)\, ds \equiv M_t^F \quad \text{to be a martingale}$$

↳ sub-sequential limit of η^ε.

$\forall$ F of the form $\begin{cases} F(\eta) = \eta(\varphi) \in \mathcal{H}_1 \\ F(\eta) = :\eta(\varphi)\eta(\psi): \in \mathcal{H}_2 \end{cases}$

$\omega \mapsto \frac{1}{2}(\nabla\cdot\omega)^2$

where $\mathcal{L}^{\mathrm{eff}}$ acts diagonally on each coordinate as $\mathcal{L}_0 + c(\omega)\mathcal{L}_0$

- Like for $d=2$ we need to find U^ε and $c = c(\omega)$ s.t.

- $A_+^\varepsilon \eta(\varphi) = -\mathcal{L}^\varepsilon U^\varepsilon + \frac{1}{2}\eta\left(c\,(\nabla\cdot\omega)^2\varphi\right)$ + small error as $\varepsilon \to 0$

  $\|(-\mathcal{L}_0)^{-1}\text{error}\| \to 0$

- $\|U^\varepsilon\| \to 0$ as $\varepsilon \to 0$

From here on, the two proofs take different paths

$\nexists$ analog of "Replacement lemma" and of the Ansatz for U^ε

we need to find U^ε and $c = c(\omega)$ s.t. (say $F = \eta(\varphi)$)

- $A_+^\varepsilon \eta(\varphi) = -\mathcal{L}^\varepsilon U^\varepsilon + \frac{1}{2}\eta'(c(\nabla\cdot\omega)^2\varphi)$ + small error as $\varepsilon \to 0$

  $\|(-\mathcal{L}_0)^{-1}\text{error}\| \to 0$
- $\|U^\varepsilon\| \to 0$ as $\varepsilon \to 0$

Idea: let (*) $\mathcal{L}_{\geq 2} := P_{\geq 2}\mathcal{L}P_{\geq 2}$ $\qquad P_{\geq 2} : \bigoplus_{0}^{\infty} \mathcal{H}_j \to \bigoplus_{2}^{\infty} \mathcal{H}_j$

and U^ε be solution of

$$-\mathcal{L}_{\geq 2} U^\varepsilon = A_+ \eta(\varphi)$$

(*) major cheating here

Note that $U^\varepsilon = (U_j^\varepsilon)_{j\geq 2}$

$$-\mathcal{L}_{\geq 2} U^\varepsilon = A_+^\varepsilon \eta(\varphi)$$

$$-\mathcal{L}^\varepsilon U^\varepsilon = -\mathcal{L}_{\geq 2} U^\varepsilon - A_-^\varepsilon U_2^\varepsilon \in \mathcal{H}_1$$

$\Rightarrow$

$$-\mathcal{L}^\varepsilon U^\varepsilon + A_-^\varepsilon U_2^\varepsilon = A_+^\varepsilon \eta(\varphi)$$

So we need:

- $\| U^\varepsilon \| \xrightarrow{\varepsilon\to\infty} 0$
- $\left\| (-\mathcal{L}_0)^{-\frac{1}{2}} \left[A_-^\varepsilon U_2^\varepsilon - \frac{1}{2} \eta\left(c(\omega)(\nabla\cdot\omega)^2 \varphi\right) \right] \right\| \xrightarrow{\varepsilon\to\infty} 0$

for some $c(\omega) > 0$

Heuristic idea of the proof

We want $(-\mathcal{L}_0)^{-\frac{1}{2}} \mathcal{A}^\varepsilon_- \mathcal{V}^\varepsilon_2 \approx \frac{1}{2}(-\mathcal{L}_0)^{-\frac{1}{2}} \eta\big(c(\omega)(\nabla\cdot\omega)^2\varphi\big) =$

say, $\varphi = e_k = e^{ikx}$

$= -\frac{1}{2}\left(\frac{2}{|k|^2}\right)^{\frac{1}{2}} c(\omega)\,(\omega\cdot k)^2\,\hat{\eta}(k)$

Let us compute.

$$(-\mathcal{L}_0)^{-\frac{1}{2}} \mathcal{A}^\varepsilon_- \mathcal{V}^\varepsilon_2 = (-\mathcal{L}_0)^{-\frac{1}{2}} \mathcal{A}^\varepsilon_- \mathcal{V}^\varepsilon \Big|_{\text{component in } \mathcal{H}_1}$$

$$(-\mathcal{L}_0)^{-\frac{1}{2}} \mathcal{A}^\varepsilon_- \mathcal{V}^\varepsilon = (-\mathcal{L}_0)^{-\frac{1}{2}} \mathcal{A}^\varepsilon_- (-\mathcal{L}_{\geq 2})^{-1} \mathcal{A}^\varepsilon_+ \eta(\varphi)$$

$$= (-\mathcal{L}_0)^{-\frac{1}{2}} \mathcal{A}^\varepsilon_- \left(-\mathcal{L}_0 - \mathcal{A}^\varepsilon_{\geq 2}\right)^{-1} \mathcal{A}^\varepsilon_+ \eta(\varphi)$$

$$= (-\mathcal{L}_0)^{-\frac{1}{2}} \mathcal{A}_-^{\varepsilon} \left(-\mathcal{L}_0 - \mathcal{A}_{\geq 2}^{\varepsilon}\right)^{-1} \mathcal{A}_+^{\varepsilon} \overbrace{\eta(\varphi)}^{\hat{\eta}(k)}$$

$$= (-\mathcal{L}_0)^{-\frac{1}{2}} \mathcal{A}_-^{\varepsilon} (-\mathcal{L}_0)^{-\frac{1}{2}} \Big(I - \underbrace{(-\mathcal{L}_0)^{-\frac{1}{2}} \mathcal{A}_{\geq 2}^{\varepsilon} (-\mathcal{L}_0)^{-\frac{1}{2}}}_{= \mathcal{T}_{\geq 2}^{\varepsilon}}\Big)^{-1} (-\mathcal{L}_0)^{-\frac{1}{2}} \mathcal{A}_+^{\varepsilon} \hat{\eta}(k)$$

$$\mathcal{T}_{\pm}^{\varepsilon} = (-\mathcal{L}_0)^{-\frac{1}{2}} \mathcal{A}_{\pm}^{\varepsilon} (-\mathcal{L}_0)^{-\frac{1}{2}}$$

$$\mathcal{T}^{\varepsilon} = \mathcal{T}_-^{\varepsilon} + \mathcal{T}_+^{\varepsilon}$$

$$\mathcal{T}_{\geq 2}^{\varepsilon} = P_{\geq 2} \mathcal{T}^{\varepsilon} P_{\geq 2}$$

$$= \frac{|k|}{\sqrt{2}} \mathcal{T}_-^{\varepsilon} \left(1 - \mathcal{T}_{\geq 2}^{\varepsilon}\right)^{-1} \mathcal{T}_+^{\varepsilon} \hat{\eta}(k)$$

Now, <u>Pretend</u>(*) that we can expand

$$\left(1 - \mathcal{T}_{\geq 2}^{\varepsilon}\right)^{-1} = \sum_{n \geq 0} \left(\mathcal{T}_{\geq 2}^{\varepsilon}\right)^n = \sum_{n \geq 0} \left(\mathcal{T}_{+,\geq 2}^{\varepsilon} + \mathcal{T}_{-,\geq 2}^{\varepsilon}\right)^n$$

(*) the actual proof does not use this

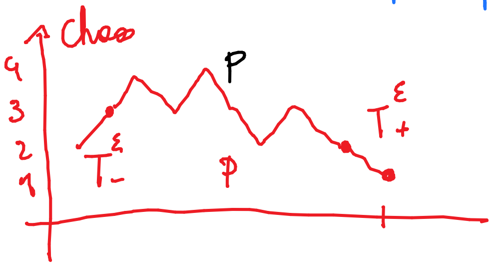
$$\left(T^{\varepsilon}_{+,\geq 2} + T^{\varepsilon}_{-,\geq 2}\right)^n \to \text{expand} \to \text{products} \quad T^{\varepsilon}_{+,\geq 2}\, T^{\varepsilon}_{+,\geq 2}\, T^{\varepsilon}_{-,\geq 2}\, T^{\varepsilon}_{+,\geq 2}$$

e.g.
$n = 4$

$$\overset{\text{III}}{T^{\varepsilon}_{p,\geq 2}}$$

↳ labels the sequence of + & −

$$\frac{|k|}{\sqrt{2}} T^{\varepsilon}_{-} T^{\varepsilon}_{p,\geq 2} T^{\varepsilon}_{+} \hat{\eta}(k) \Big|_{\text{component in } \mathcal{H}_1}$$



$= 0$ unless : p "balanced"

p does not go below 2

Π : collection of such paths

After some work: $\forall\, p \in \Gamma$

$$T_-^{\varepsilon}\, \mathcal{T}_p^{\varepsilon}\, T_+^{\varepsilon}\, \hat{\eta}(k) \xrightarrow{\varepsilon \to 0} c(p)\, \frac{(\omega\cdot k)^2}{|k|^2}\, \hat{\eta}(k)$$

constant, depending on p

$\Rightarrow$ $c(\omega)$ "obtained" by summing the $c(p)$, $p \in \Gamma$ (∞ many)

Recall $T_\pm^{\varepsilon} = (-\mathcal{L}_0)^{-\frac{1}{2}} A_\pm^{\varepsilon} (-\mathcal{L}_0)^{-\frac{1}{2}}$ and $A_\pm^{\varepsilon}$ explicit.

No inverses of $A_\pm^{\varepsilon}$ involved!

Comments

- Actual proof uses cutoff in chaos $\quad \mathcal{L}^{\varepsilon}_{\geq 2} \to \mathcal{L}^{\varepsilon}_{2,n} = P_{2,n} \mathcal{L} P^{\varepsilon}_{2,n}$

  $\varepsilon \to 0$ first, $\quad n \to \infty$ later

  $T^{\varepsilon}_{\geq 2} \to T^{\varepsilon}_{2,n}$

- actual proof does not use $(I - T^{\varepsilon}_{2,n})^{-1} = \sum_{j \geq 0} (T^{\varepsilon}_{2,n})^{j}$

  bounded, but not $\|T^{\varepsilon}_{2,n}\| < 1$

Thanks