Quenched local limit theorem for RWRE admitting bounded cycle representation

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joint work with

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Martin Slowik *(U Mannheim)*

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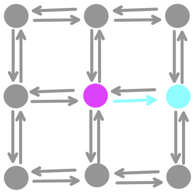
Berlin-Oxford
IRTG 2544

Technische Universität Berlin

RWRE

Background



RWRE

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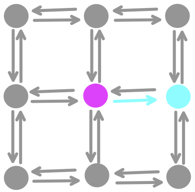

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  $\omega \equiv \{\omega_z(x) : z \in \mathcal{N}, x \in \mathbb{Z}^d\}$ (non-negative)

RWRE

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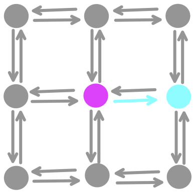

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- Continuous time random walk (**CTRW**):
  $(X_t^\omega)_{t\geq 0}$
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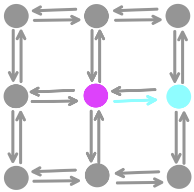

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- generator and Dirichlet energy
  - **quenched**

$$L^\omega f(x) = \sum_{z\in\mathcal{N}} \omega_z(x)\nabla_z f(x) \qquad \mathcal{E}^\omega(f,g) = \langle f, -L^\omega g\rangle$$

  - **annealed**

$$(\mathcal{L}\phi)(\omega) = \sum_{z\in\mathcal{N}} \omega_z(0) D_z\phi(\omega) \qquad \mathcal{E}(\phi,\varphi) = \mathbb{E}[\phi(-\mathcal{L}\varphi)]$$

RWRE

Model variations

RWRE

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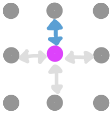

symmetric

RWRE

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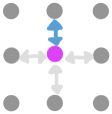

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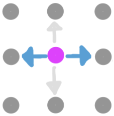

balanced

RWRE

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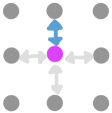

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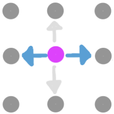

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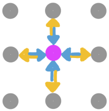

doubly stochastic

RWRE

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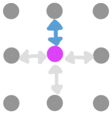

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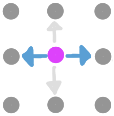

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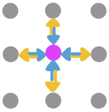

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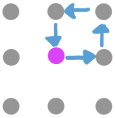

cyclic

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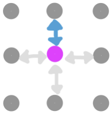

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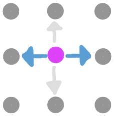

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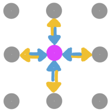

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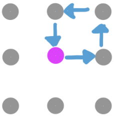

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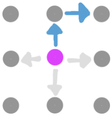

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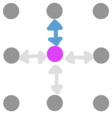

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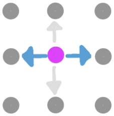

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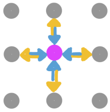

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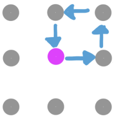

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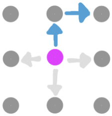

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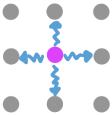

perturbed

RWRE

Previous Results on QFCLT and QLLT

- **Symmetric case**: random conductance model under (p-q)**moment condition**
  - **Quenched invariance principle**(Andres, Deuschel, Slowik '15)
    For $(\Omega, \mathcal{F}, \mathbb{P})$ ergodic wrt. $(\tau_x)_{x\in\mathbb{Z}^d}$, $\mathbb{P}-a.s.$

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Answer: **Bounded cycle!**

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$$\omega \equiv \{\omega(x,y) : (x,y) \in \vec{E}_d\}$$

- **bounded cycle representation (BCR)**: ω is almost surely generated by a collection of nearest-neighbor prototype cycles $\mathcal{C}$ of bounded length N shifting over the space with corresponding random cycle weights $\mathbb{P}$-a.s.

$$\omega(x,y) = \sum_{C \in \mathcal{C}} \omega_C(x,y) = \sum_{C \in \mathcal{C}} \sum_{z \in \mathbb{Z}^d} W_C(\tau_z \omega) \cdot \mathbb{1}_{C+z}(x,y).$$

BCR Model

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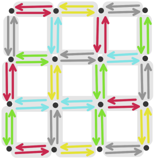
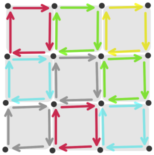


2-cycles
(conductance model)



4-cycles

Conditions

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(p-q) ***p-q*** **moment condition**: for all $(x, y) \in \vec{E}_d$, and $p, q \in (1, \infty]$,

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- Remark: for QFCLT the moment condition can be relaxed to ${}^1/_p + {}^1/_q < {}^2/_{(d-1)}$. See (Bella and Schäffner '19)

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- **Dirichlet energy** $\mathcal{E}_C$

$$\mathcal{E}_C(\varphi, \phi) = \mathbb{E}[\varphi \cdot \mathcal{L}_C \phi]$$

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We obtain following nice properties:

- **sector condition**

$$\mathcal{E}_C(\phi, \varphi)^2 \lesssim_{n_C} \mathcal{E}_C(\phi, \phi)\mathcal{E}_C(\varphi, \varphi)$$

- **local drift as bounded operator**

$$\mathbb{E}[V_C^i \varphi]^2 \lesssim_{n_C} \mathbb{E}[W_C]\mathcal{E}_C(\varphi, \varphi)$$

Bounded Cycle Representation

Results

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Theorem 1 (QFCLT)

(Deuschel, Slowik, W. 23+) Under (ERG) and (p-q), $\mathbb{P}-a.s.$*, it holds that for* $d \geq 2$

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with Σ *non-degenerate.*

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Theorem 2 (QLLT)

(Deuschel, Slowik, W. 23+) Under (ERG) and (p-q), $\mathbb{P}-a.s.$, it holds that for $d \geq 2$, $T_2 > T_1 > 0$, $K > 0$,

$$p_t^{(n)}(x, y) = n^d p_{n^2 t}(\lfloor xn \rfloor, \lfloor yn \rfloor) \longrightarrow p_t^{\Sigma}(x, y)$$

uniformly for $(x, t), (y, t) \in B(K) \times [T_1, T_2]$.

QFCLT

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QFCLT

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- **martingale decomposition** for $i = 1, \ldots, d$

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$$M^{(n)} = (M_t^{(n)})_{t \geq 0} \implies B(0, \Sigma) \qquad \text{in } P_0^\omega.$$

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Good news: Construction of corrector uses **sector condition**

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- **sublinearity of the corrector** for $\mathbb{P}$-a.e. ω

$$\sup_{x \in B(n)} \frac{1}{n} |\chi^i(\omega, x)| \xrightarrow{n \to \infty} 0.$$

- **Helland(1982)'s martingale FCLT** Set $M_t^{i,(n)} := \frac{1}{n} \Phi^i(\cdot, X_{n^2 t}), t \geq 0$. For $\mathbb{P}$-a.e. ω,

$$M^{(n)} = (M_t^{(n)})_{t \geq 0} \implies B(0, \Sigma) \qquad \text{in } P_0^\omega.$$

  - use $\langle M^{i,(n)}, M^{j,(n)} \rangle_t \to t\Sigma_{i,j}$ ergodicity of $t \to \tau_{X_t}(\omega)$

Good news: Construction of corrector uses **sector condition**

Most challenging: sublinearity of the corrector!

QFCLT

Key estimates for sublinearity

For f: non-negative and L^ω-subharmonic, and η cut-off on B

QFCLT

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- **Quenched energy estimate** (QEE)

$$\frac{\mathcal{E}^\omega(\eta f)}{|B|} \lesssim N^2 \|\nabla\eta\|^2_{\ell^\infty(\vec{E}_d)} \|\mu^\omega\|_{p,B} \|f^2\|_{p_*,B}$$

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- **Quenched maximum inequality** (QMI)

weighted Sobolev inequality (WSI) + energy estimate + **Moser iteration**

$$\max_{x \in B_z(n)} |f(x)| \lesssim \left(1 \vee N^2 \|\mu^\omega\|_{p,B_z(2n)} \left\|\nu^{\omega^S}\right\|_{q,B_z(2n)}\right)^\kappa \|f\|_{2p_*,B_z(2n)}$$

QFCLT

Useful inequalities for sublinearity

Graph inequalities

QFCLT

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Graph inequalities

(WSI) Weighted Sobolev inequality For $q \in [1, \infty)$, $\rho \equiv \rho(d, q) := \frac{d}{(d-2)+d/q}$

$$\|(\eta f)\|^2_{2\rho,B} \leq \mathbf{C}_{\mathsf{SI}}\mathsf{rad}(B)^2 \left\|\nu^S\right\|_{q,B} \frac{\mathcal{E}^{\omega^S}(\eta f)}{|B|}.$$

QFCLT

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(LPI) Local Poincaré inequality

$$\left\|f - \bar{f}_B\right\|_{2\rho,B}^2 \le \mathbf{C}_{\mathsf{PI}}\mathsf{rad}(B)^2 \left\|\nu^S\right\|_{q,B} \frac{\mathcal{E}_B^{\omega^S}(f)}{|B|}$$

QFCLT

Proof scheme for sublinearity

- (QEE) + (WSI) $\xrightarrow{\text{Moser}}$ (QMI)

QFCLT

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QLLT

Strategy

- **Decomposition of error** (Barlow, Hambly '09)

$$\boxed{J = J_1 + J_2 + J_3}$$

  - Total error: $x \in \mathbb{R}^d, \delta > 0$

$$J(t,n) = P_0^\omega\Big(X_t^{(n)} \in B_x(\delta)\Big) - \int_{B_x(\delta)} p_t^\Sigma(y)\mathrm{d}y$$

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$$J_2(t,n) = |nB_x(\delta)| \cdot p^\omega_{n^2 t}(0,\lfloor nx \rfloor) - \text{vol}(B_x(\delta)) \cdot p_t^\Sigma(x)\mathrm{d}y$$

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- Continuous error: center to neighbor

$$J_3(t,n) = \int_{B_x(\delta)} \left(p_t^\Sigma(x) - p_t^\Sigma(y)\right)\mathrm{d}y$$

QLLT

Strategy

- **Control on J, J_1, J_3: uniformly bounded by $o(\delta^d)$, as $\delta \to 0$**

QLLT

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(NDB$_u$) **Near-diagonal heat kernel upper bounds**

QLLT

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(HC) **Hölder continuity**
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In particular,

$$\text{(PHI)} \implies \text{(HC)} + (\text{NDB}_u)$$

Harnack Principle

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Theorem 3 (EHI)

(D, Slowik, Weng 23+) For any $x_0 \in \mathbb{Z}^d$ *and* $n \in \mathbb{N}$*. Let* $u > 0$ *be* L^ω *harmonic on* $B'(x_0, 2n) \equiv B(x_0, 2n + {}^N\!/_2 - 1)$*. Then, under (ERG) and (p-q),* $\mathbb{P} - a.s.$*, for* $d \geq 2$*, there exists constant* C_{EH}*, such that*

$$\max_{x \in B(x_0, n/2)} u(x) \leq C_{EH} \min_{x \in B(x_0, n/2)} u(x).$$

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Theorem 4 (PHI)

(D, Slowik, Weng 23+) For any $x \in \mathbb{Z}^d$, $t_0 \geq 0$, $n \in \mathbb{N}$. *Let* $u > 0$ *be* L^ω *caloric on* $Q'(2n) \equiv [t_0, t + n^2] \times B'(2n)$, *i.e.* $\partial_t u - L^\omega u = 0$. *Then, under (ERG) and (p-q),* $\mathbb{P} - a.s.$, *for* $d \geq 2$, *there exists constant* C_{PH}, *such that*

$$\max_{(t,x) \in Q_-} u(x) \leq C_{PH} \min_{(t,x) \in Q_+} u(x),$$

for $Q_- = [t_0 + \frac{1}{4}n^2, t_0 + \frac{1}{2}n^2] \times B(x_0, n/2)$, *and* $Q_+ = Q_- \circ \tau^{\text{time}}_{n^2/2}$.

The "Main Road": Bombieri and Giusti

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Lemma (BG)

Let $\{U_\sigma : \sigma \in (0,1]\}$ be a collection of subsets of a fixed measure space endowed with a measure m, and $U_{\sigma'} \subset U_\sigma$ if $\sigma' < \sigma$. Fix $0 < \sigma < 1$, $0 < \alpha^* \leq \infty$ and let f be a positive function on $U := U_1$. Suppose that there exists constants C_{BG_1}, C_{BG_2}

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(C1) *there exists* $\kappa > 0$ *such that for all* $\delta \leq \sigma' < \sigma < 1$ *and* $0 < \alpha \leq \min\{1, \alpha^*/2\}$*,*

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Bad news: to show (C1) and (C2) are atisfied is **complicated**!

The Bridge to (BG-C1)

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Lemma (QMI)

Fix $\omega \in \Omega$. let $f : \mathbb{Z}^d \to \mathbb{R}$ be non-negative and L^ω-subharmonic on $B \equiv B(n)$. Then for any $p, q \in (1, \infty]$ with $1/p + 1/q < \frac{2}{d}$, there exists $\kappa = \kappa(d,p,q)$ and $C = C(d,p,q,N)$ such that for all $1/2 \le \sigma' < \sigma \le 1$, we have

$$\|f\|_{\infty, B_{\sigma'}} \le C \left(\frac{1 \vee \|\mu\|_{p,B} \left\|\nu^S\right\|_{q,B}}{(\sigma - \sigma')^2} \right)^\kappa \|f\|_{2p_*, B_\sigma}, \tag{1}$$

where $p_* = p/(p-1)$ is the Hölder conjugate of p.

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Lemma (QMI$_{sp.t}$)

Fix $\omega \in \Omega$. let $f : \mathbb{Z}^d \to \mathbb{R}$ be non-negative and L^ω-subcaloric on $Q \equiv Q(n) = [t_0, t_0 + n^2] \times B(n)$. Then for any $p, q \in (1, \infty]$ with $1/p + 1/q < \frac{2}{d}$, $\forall \varepsilon \in (0, \frac{1}{4})$, s', s'' are chosen such that $s' - t_0 > \varepsilon n^2, t_0 + n^2 - s'' \geq \varepsilon n^2$, there exists $\kappa = \kappa(d, p, q)$ and $C = C(d, p, q, N)$ such that for all $1/2 \leq \sigma' < \sigma \leq 1$, we have

$$\|f\|_{\infty, Q_{\sigma'}} \leq C\left(\frac{(1 \vee \|\mu\|_{p,B})(1 \vee \left\|\nu^S\right\|_{q,B})}{\varepsilon(\sigma - \sigma')^2}\right)^\kappa \|f\|_{2, Q_\sigma}, \qquad (2)$$

where $Q_\sigma = I_\sigma \times B_\sigma$, and $I_\sigma = [\sigma t_0 + (1-\sigma)s', (1-\sigma)s'' + \sigma(t_0 + n^2)]$.

The Bridge to (BG-C2)

"Front wheel": Cut-off control

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Proposition 1 (Cut-off control)

(Deuschel, Slowik, W. '23+) Let $\eta_n(x) := \left(1 - \frac{1}{n} d(x, B'(n))\right) \vee 0$*, be a cut-off with linear decay, and define* $\eta_{\gamma,n}$ *be the geometric mean of* η_n *along the cycle. Then*

$$\limsup_{n\to\infty} n^{2-d} \sum_{x\in B''(2n)\setminus B(n)} W_{C+x}\eta^2_{C+x,n} M^2_{C+x,n} \leq C(N,d)\mathbb{E}[W_C] \qquad (3)$$

where $M_{\gamma,n} := \max_{x\in\gamma} \left|\ln \frac{\eta_n^2(x)}{\eta^2_{\gamma,n}}\right|$.

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Proof Sketch:

$$\begin{aligned}
&\limsup_{n\to\infty} n^{2-d} \sum_{x\in B(2n-5K)\setminus B(n)} W_{C+x}\eta^2_{C+x,n} M^2_{C+x,n} \\
&\le \left(\limsup_{n\to\infty} n^{-d} \sum_{B(2n-5K)\setminus B(n)} W_{C+x}\right) \left(\limsup_{n\to\infty} n^2 \eta_n^2(x) M^2_{C+x,n}\right) \\
&\le (2\mathbf{C}_{\text{vol}})^d \, 4e\,(N-2)^2\, \mathbb{E}[W_C]\cdot C \\
&=: C(N,d)\mathbb{E}[W_C]
\end{aligned}$$

the bridge to (BG-C2)

"Back wheel": Energy inequality for the logarithm

Proposition 2 (Deuschel, Kumagai ’13)

[Prop.3.7 (ii) $n = 1$] For each $l \in \mathbb{N}$ and $M > 0$, there exists $c_1, c_2 > 0$ such that

$$\sum_{j=1}^{l}(e^{\alpha_j} - 1)e^{\bar{w}_j} + c_2 - c_1 \sum_{j=1}^{l} \alpha_j^2 \geq 0,$$

for all $(\alpha_1, \cdots, \alpha_l), (\bar{w}_1, \cdots, \bar{w}_l) \in \mathbb{R}^l$ with $\sum_j \alpha_j = \sum_j \bar{w}_j = 0$.

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Proposition 2 (Deuschel, Kumagai '13)

[Prop.3.7 (ii) $n = 1$] For each $l \in \mathbb{N}$ and $M > 0$, there exists $c_1, c_2 > 0$ such that

$$\sum_{j=1}^{l} (e^{\alpha_j} - 1)e^{\bar{w}_j} + c_2 - c_1 \sum_{j=1}^{l} \alpha_j^2 \geq 0,$$

for all $(\alpha_1, \cdots, \alpha_l), (\bar{w}_1, \cdots, \bar{w}_l) \in \mathbb{R}^l$ with $\sum_j \alpha_j = \sum_j \bar{w}_j = 0$.

Lemma (Energy inequality for the logarithm)

Fix $\gamma = (x_1, \cdots, x_{n_\gamma})$. Then for $\eta, u > 0$, there exists $c_{1,\gamma}$, $c_{2,\gamma,\eta} > 0$, such that

$$\mathcal{E}_\gamma(\eta^2 u^{-1}, u) \leq c_{2,\gamma,\eta} \eta_\gamma^2 - c_{1,\gamma} \eta_\gamma^2 \mathcal{E}_\gamma(\ln u), \tag{4}$$

where $\eta_\gamma := \prod_{x \in \gamma} \eta(x)^{\frac{1}{n_\gamma}}$.

Proof: Apply above proposition with above choices and $\alpha_j = \ln u(x_{j+1}) - \ln u(x_j)$, and $\bar{w}_j = 2(\ln \eta_n(x_j) - \ln \eta_{\gamma,n})$.

Ride "two wheels" to EHI

For non-negative $L^\omega-$ harmonic function u on $B'(2n) \supset B(n)$:

First combine two ingredients, get (QEE$_{\ln}$): $\limsup_{n\to\infty} \frac{n^2 \cdot \mathcal{E}^{\omega\,S}_{B(n)}(\ln u)}{|B(n)|} \le C(N,d)\mathbb{E}[\mu]$.

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- Put things together, obtain (EHI): for $\sigma' < \sigma$

$$\max_{B_{\sigma'}} u(x) \lesssim_{N,d,(p-q)} \|u\|_{2p^*, B_\sigma} = \|f\|_{2p*, B_\sigma} \cdot e^{(\ln u)_B} \lesssim_{A_1, A_2} \min_{B_{\sigma'}} u(x)$$

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\text{Energy ineq. for } \ln \text{ (4)} \quad &\le \sum_{\substack{\gamma \subset B''(2n) \\ \gamma \not\subset B'(n)}} W_\gamma(\omega) \eta_{n,\gamma}^2 M_{\gamma,n}^2 - C_1 \sum_{\gamma \subset B''(2n)} \eta_{n,\gamma}^2 \mathcal{E}_\gamma^\omega(\ln u_t) + O(n^{d-2}) \\
\text{Cut-off control (3)} \quad &\le O(n^{d-2}) - C_1 \mathcal{E}^\omega_{B(n)}(\ln u_t) \qquad (5)
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- (QMI)$_{sp.t}$: need (QEE) in space time, a consequence of (QEE) in space, by using the Moser iteration, implies (BG-C2).
- the rest is similar as in (EHI).

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Open Problem

- cycles of **unbounded length** but weighted.

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