THE “STARSHIP ENTERPRISE” PROBLEM

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Abstract. We study the following problem taken from [1], Exercises E10.10. The starship Enterprise is initially at distance R from the Sun. The navigation system of the starship is broken: one can only fix a distance to be traveled, then the starship makes a jump of that size in a uniformly chosen direction. The aim is to reach the Solar System, represented by a ball of radius $r < R$ centered at the Sun. We show that, whatever strategy is chosen, the probability of ever reaching the Solar System is strictly smaller than (r/R). This bound is shown to be optimal, in the sense that for each $\varepsilon > 0$ there is a strategy for which the probability of reaching the Solar System exceeds $(r/R) - \varepsilon$.

1. Preliminaries

The electric potential generated by a charge uniformly distributed on the surface $\partial B(\vec{0}, 1)$ of the unit sphere centered in the origin $\vec{0} \in \mathbb{R}^3$, is given (up to multiplicative constants) by

$$\varphi(\vec{x}) \;:=\; \int_{\partial B(\vec{0},1)} \frac{1}{|\vec{x}+\vec{r}|}\,\mathrm{d}\sigma(\vec{r}) \;=\; \begin{cases} 1 & \text{if } |\vec{x}| \le 1 \\ \dfrac{1}{|\vec{x}|} & \text{if } |\vec{x}| > 1 \end{cases}, \tag{1}$$

where $\sigma(\cdot)$ denotes the normalized uniform measure on $\partial B(\vec{0}, 1)$. This is a consequence of the celebrated Gauss Theorem for the electric field.†

If $\vec{\Theta}$ is a random variable uniformly distributed on the surface of the unit sphere in $\mathbb{R}^3$, then relation (1) yields

$$\mathbb{E}\left[\frac{1}{\vec{x}+c\,\vec{\Theta}}\right] \;=\; \int_{\partial B(0,1)} \frac{1}{|\vec{x}+c\,\vec{r}|}\,\mathrm{d}\sigma_1(\vec{r}) \;=\; \frac{1}{c}\,\varphi\!\left(\frac{\vec{x}}{c}\right) \;=\; \begin{cases} \dfrac{1}{c} & \text{if } |\vec{x}| \le c \\ \dfrac{1}{|\vec{x}|} & \text{if } |\vec{x}| > c \end{cases}. \tag{2}$$

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†One can prove relation (1) using the *mean value property* and the *maximum principle* of harmonic functions. The function $r \mapsto q(\vec{r}) := 1/|\vec{r}|$ is harmonic on $\mathbb{R}^3 \setminus \{\vec{0}\}$, that is $\Delta q(\vec{r}) := \sum_{i=1}^{3} \partial^2 q/\partial r_i^2(\vec{r}) = 0$ for all $\vec{r} \neq \vec{0}$. If $B(\vec{x}, 1) \subseteq \mathbb{R}^3 \setminus \{\vec{0}\}$, that is if $|\vec{x}| > 1$, the *mean value property* of harmonic functions says that the average of $q(\cdot)$ over $\partial B(\vec{x}, 1)$ equals $q(\vec{x})$: this is precisely the second line of (1).

Next we observe that $\varphi(\cdot)$ is continuous on the whole $\mathbb{R}^3$, by passing in the limit inside the integral in (1), which can be justified by uniform integrability (e.g. showing the boundedness of the integrand in $L^{1+\varepsilon}$ for some $\varepsilon > 0$, i.e. $\int_{\partial B(\vec{0},1)} q(\vec{x}+\vec{r})^{1+\varepsilon}\,\mathrm{d}\sigma(\vec{r}) \le (const.)$). From the second line of (1) we thus deduce that $\varphi(\vec{x}) = 1$ for $|\vec{x}| = 1$. For any x in the interior of $B(0, 1)$ we can differentiate inside the integral in (1) thanks to dominated convergence (the incremental ratios are uniformly bounded). Since $q(\cdot)$ is harmonic in $\mathbb{R}^3 \setminus \{0\}$, it follows that $\varphi(\cdot)$ is harmonic in the interior of $B(0, 1)$, and continuous until the boundary. Being constantly equal to 1 on $\partial B(0, 1)$, it follows from the *maximum principle* that $\varphi(\vec{x}) = 1$ for $|\vec{x}| \le 1$.

2. The solution

A probabilistic model for our problem can be built as follows. All the random variables are defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

- Let $(\vec{\Theta}_n)_{n\geq 1}$ be a sequence of i.i.d. random variables, uniformly distributed on the surface of the unit sphere in $\mathbb{R}^3$, which give the directions of the space-hops. We introduce the natural filtration $\mathcal{F}_n := \sigma(\vec{\Theta}_1, \ldots, \vec{\Theta}_n)$ and we set $\mathcal{F}_0 := \{\emptyset, \Omega\}$.
- The sequence of distances for the space-hops, that we call the *strategy*, is modeled by a nonnegative *previsible* process $(C_n)_{n\geq 1}$, i.e. C_n is $\mathcal{F}_{n-1}$–measurable for every n.
- The position of the starship is then described by the $\mathbb{R}^3$–valued process $(\vec{X}_n)_{n\geq 0}$ defined by

$$\vec{X}_0 \;:=\; (R, 0, 0) \qquad\qquad \vec{X}_n \;:=\; \vec{X}_{n-1} + C_n \vec{\Theta}_n \,. \tag{3}$$

The goal is to reach the Solar System, that is $B(\vec{0}, r)$, so we define the stopping time

$$\tau := \inf\{n \in \mathbb{N} : |\vec{X}_n| \leq r\} \in \mathbb{N} \cup \{+\infty\} \,. \tag{4}$$

We want to investigate the probability of ever reaching the solar system, that is $\mathbb{P}[\tau < \infty]$.

It turns out to be very useful to introduce the process $(Z_n)_{n\geq 0}$ defined by

$$Z_n := \frac{1}{|\vec{X}_n|} \,. \tag{5}$$

Proposition 2.1. *Whatever strategy (C_n) is adopted, the process (Z_n) is a supermartingale. If in addition $C_n \leq |\vec{X}_{n-1}|$ $\forall n$, then (Z_n) is a martingale.*

Proof. The process (Z_n) is clearly adapted. Being nonnegative, the conditional expectation of Z_n is well defined. Since $\vec{X}_{n-1}$ and C_n are $\mathcal{F}_{n-1}$ measurable while $\vec{\Theta}_n$ is independent of $\mathcal{F}_{n-1}$, it follows from (5), (3) and (2) that

$$\begin{aligned}
\mathbb{E}\big[Z_n \big| \mathcal{F}_{n-1}\big] \;&=\; \mathbb{E}\bigg[\frac{1}{|\vec{X}_{n-1} + C_n \vec{\Theta}_n|}\bigg|\mathcal{F}_{n-1}\bigg] \\
&=\; \mathbb{E}\bigg[\frac{1}{|\vec{x} + c\,\vec{\Theta}_n|}\bigg]\bigg|_{\vec{x} = \vec{X}_{n-1}, c = C_n} \;=\; \begin{cases} \dfrac{1}{|\vec{X}_{n-1}|} & \text{if } C_n \leq |\vec{X}_{n-1}| \\[2ex] \dfrac{1}{C_n} & \text{if } C_n > |\vec{X}_{n-1}| \end{cases} \,.
\end{aligned}$$

It follows that $\mathbb{E}\big[Z_n \big| \mathcal{F}_{n-1}\big] \leq 1/|\vec{X}_{n-1}| =: Z_{n-1}$, and the equality holds if (and only if) $C_n \leq |\vec{X}_{n-1}|$. It only remains to show that $Z_n \in L^1$ for every $n \in \mathbb{N}$. Trivially $Z_0 = 1/R \in L^1$; assuming that $\mathbb{E}(Z_{n-1}) < \infty$, the monotonicity of conditional expectation yields $\mathbb{E}(Z_n) = \mathbb{E}(\mathbb{E}(Z_n|\mathcal{F}_{n-1})) \leq \mathbb{E}(Z_{n-1}) < \infty$, hence the proof is completed by induction. □

Now we can prove our main result.

Theorem 2.2. *Whatever strategy (C_n) is adopted, the probability of ever reaching the solar system is strictly smaller than r/R:*

$$\mathbb{P}\big[\tau < \infty\big] \;<\; \frac{r}{R} \,. \tag{6}$$

Proof. We start showing that (6) holds with the weak inequality. Since the process (Z_n) is a nonnegative supermartingale, from the Martingale Convergence Theorem it follows that

$$\mathbb{P}\text{–a.s.} \quad \exists \lim_{n\to\infty} Z_n \in [0,+\infty)\,.$$

Also the stopped process $(Z_n^\tau := Z_{\tau\wedge n})$ is a supermartingale, hence

$$\mathbb{E}\big[Z_n^\tau\big] \le \mathbb{E}\big[Z_0^\tau\big] = \mathbb{E}\big[Z_0\big] = \frac{1}{R}\,. \tag{7}$$

Now notice that $\mathbb{P}$–a.s.

$$\lim_{n\to\infty} Z_n^\tau = Z_\tau\, 1_{\{\tau<\infty\}} + \Big(\lim_{n\to\infty} Z_n\Big)\, 1_{\{\tau=\infty\}} \ge Z_\tau\, 1_{\{\tau<\infty\}} \ge \frac{1}{r}\, 1_{\{\tau<\infty\}}\,, \tag{8}$$

where in the last inequality we have used the fact that, on the event $\{\tau < \infty\}$, by definition $|\vec{X}_\tau| \le r$ and therefore $Z_\tau \ge 1/r$. Taking the expected value on both sides we finally get

$$\frac{1}{r}\,\mathbb{P}\big[\tau<\infty\big] \le \mathbb{E}\Big[\lim_{n\to\infty} Z_n^\tau\Big] \le \liminf_{n\to\infty} \mathbb{E}\big[Z_n^\tau\big] \le \frac{1}{R}\,, \tag{9}$$

where we have applied Fatou's Lemma and we have used (7).

To prove that the bound (6) is actually strict, we claim that

$$|\vec{X}_\tau| < r \quad \text{on the event } \{\tau<\infty\}\,, \quad \mathbb{P}\text{–a.s.}\,. \tag{10}$$

In fact the following stronger statement holds true: for any strategy (C_n)

$$|\vec{X}_k| \ne r \quad \forall k \in \mathbb{N}\,, \quad \mathbb{P}\text{–a.s.}\,,$$

see Appendix A.1 for a proof. Now assume that the event $\{\tau < \infty\}$ has positive probability (otherwise there is nothing to prove). Then from equation (10) it follows that the last inequality in (8) is strict with positive probability. By taking expectation it follows that the first inequality in (9) is strict, hence $\mathbb{P}[\tau < \infty] < r/R$ for any strategy (C_k). □

Finally we show that the bound in (6) is optimal.

Theorem 2.3. *For every $\varepsilon \in (0, r)$ there exists a strategy (C_n^ε) such that*

$$\mathbb{P}\big[\tau<\infty\big] \ge \frac{r-\varepsilon}{R}\,.$$

Proof. We define (C_n^ε) in the following way:

$$C_n^\varepsilon := \begin{cases} |\vec{X}_{n-1}| - (r-\varepsilon) & \text{if } |\vec{X}_{n-1}| > r \\ 0 & \text{if } |\vec{X}_{n-1}| \le r \end{cases}\,. \tag{11}$$

It is easy to check that with this definition $Z_n^\tau = Z_n$ for all n. We are going to show that the fundamental inequalities in the proof of Theorem 2.2 now become equalities.

Since $|C_n^\varepsilon| \le |\vec{X}_{n-1}|$ for all n, from Proposition 2.1 it follows that this time the process (Z_n) is a nonnegative martingale, hence

$$\mathbb{E}\big[Z_n\big] = \mathbb{E}\big[Z_0\big] = \frac{1}{R}\,. \tag{12}$$

Again the Martingale Convergence Theorem yields that

$$\mathbb{P}\text{–a.s.} \quad \exists \lim_{n\to\infty} Z_n =: Z_\infty \in [0,+\infty)\,. \tag{13}$$

By the definition (11) of (C_n^ε), it is easy to check that for all n we have

$$Z_n \le \frac{1}{r-\varepsilon}\,, \tag{14}$$

therefore (Z_n) is a bounded process and we can apply the Dominated Convergence Theorem:

$$\mathbb{E}\big[Z_\infty\big] \;=\; \lim_{n\to\infty} \mathbb{E}\big[Z_n\big] \;=\; \frac{1}{R}\,, \tag{15}$$

where in the second equality we have used (12).

It remains to identify the limit Z_∞. Arguing as in (8) we have that $\mathbb{P}$–a.s.

$$Z_\infty \;=\; Z_\tau\, 1_{\{\tau<\infty\}} \;+\; \Big(\lim_{n\to\infty} Z_n\Big)\, 1_{\{\tau=\infty\}}\,,$$

and this time the observation that the second term in the r.h.s. is nonnegative does not suffice anymore. However one can show that this term is indeed zero, that is

$$\mathbb{P}\text{–a.s.} \quad \big\{\tau=\infty\big\} \;\subseteq\; \Big\{\lim_{n\to\infty} Z_n \;=\; 0\Big\}\,, \tag{16}$$

see Appendix A.2. Therefore $Z_\infty = Z_\tau\, 1_{\{\tau<\infty\}}$, $\mathbb{P}$–a.s., and by (15) we get

$$\frac{1}{R} \;=\; \mathbb{E}\big[Z_\tau\, 1_{\{\tau<\infty\}}\big] \;\leq\; \frac{1}{r-\varepsilon}\, \mathbb{P}\big[\tau<\infty\big]\,,$$

because $Z_\tau \leq 1/(r-\varepsilon)$, as it follows from (14). This completes the proof. □

Appendix A. Some technical points

A.1. We are going to show that equation (10) holds true. More precisely, we prove that for any strategy (C_n)

$$|\vec{X}_k| \neq r,\; |\vec{X}_k| \neq 0 \quad \forall k\in\mathbb{N}\,, \quad \mathbb{P}\text{–a.s.}\,. \tag{17}$$

We proceed by induction on k. The $k=0$ case holds trivially since $|\vec{X}_0| = R > r > 0$. Now assume that (17) holds for $k \leq n-1$: by equation (3), the variable $\vec{X}_n$ conditionally on $\mathcal{F}_{n-1}$ is uniformly distributed on the surface of the sphere $\partial B(\vec{X}_{n-1}, C_n)$. Given that by the inductive hypothesis $\vec{X}_{n-1} \neq 0$, $\vec{X}_{n-1} \neq r$, it follows that $\mathbb{P}$–a.s. $\mathbb{P}\big[|\vec{X}_n| = x \,\big|\, \mathcal{F}_{n-1}\big] = 0$ for both $x=r$ and $x=0$, and by the Tower Property of the conditional expectation one has that $\mathbb{P}\big[|\vec{X}_n| = x\big] = 0$ for $x=r$ and $x=0$, that is the inductive step.

A.2. We give a proof of equation (16), or equivalently

$$\mathbb{P}\text{–a.s.} \quad \big\{\tau=\infty\big\} \;\subseteq\; \Big\{\lim_{n\to\infty} |\vec{X}_n| \;=\; \infty\Big\}\,. \tag{18}$$

To prove this fact, let us consider the sequence of events

$$\mathcal{A}_n := \{|\vec{X}_n| \geq |\vec{X}_{n-1}| + \varepsilon/2\}\,,$$

where ε is the parameter appearing in the definition of the strategy (C_n^ε). It follows by (13) that $\lim_{n\to\infty} |\vec{X}_n|$ exists in $(0,+\infty]$, $\mathbb{P}$–a.s.. Using this fact, it is easy to see that

$$\mathbb{P}\text{–a.s.} \quad \big\{\mathcal{A}_n \text{ i.o.}\big\} \;\subseteq\; \Big\{\lim_{n\to\infty} |\vec{X}_n| \;=\; \infty\Big\}\,,$$

where $\big\{\mathcal{A}_n \text{ i.o.}\big\} := \big\{\limsup_{n\to\infty} \mathcal{A}_n\big\}$. Therefore, we are left with showing that

$$\mathbb{P}\text{–a.s.} \quad \big\{\tau=\infty\big\} \;\subseteq\; \big\{\mathcal{A}_n \text{ i.o.}\big\}\,. \tag{19}$$

The idea behind the proof of (19) is quite simple. Let us define the sequence of events

$$\mathcal{B}_n := \{\cos(\vec{\Theta}_n, \vec{X}_{n-1}) \geq 1/2\},$$

where by $\cos(\vec{a}, \vec{b})$ we mean the cosine of the angle between the two vectors $\vec{a}$ and $\vec{b}$. Thus if $\mathcal{B}_n$ happens it means that the direction $\vec{\Theta}_n$ of the n–th movement of the Enterprise is close (in a quantitative way) to the direction of $\vec{X}_{n-1}$, hence we are getting more distant from the Sun. In fact, by (3) we have

$$\mathcal{B}_n \subseteq \big\{ |\vec{X}_n| > |\vec{X}_{n-1}| + C_n^\varepsilon/2 \big\} . \tag{20}$$

From (11) it is clear that, on the event $\{\tau = \infty\}$, one has $C_n^\varepsilon \geq \varepsilon$ for all n, therefore

$$\{\tau = \infty\} \cap \mathcal{B}_n \subseteq \big\{ |\vec{X}_n| > |\vec{X}_{n-1}| + \varepsilon/2 \big\} = \mathcal{A}_n , \qquad \forall n \in \mathbb{N} ,$$

from which it follows that

$$\{\tau = \infty\} \cap \{\mathcal{B}_n \text{ i.o.}\} \subseteq \{\mathcal{A}_n \text{ i.o.}\} . \tag{21}$$

If we prove that $\mathrm{P}(\mathcal{B}_n \text{ i.o.}) = 1$, relation (19) follows. For this, by the Borel-Cantelli lemma, is enough to show that $(\mathcal{B}_n)_{n\in\mathbb{N}}$ are independent events with the same probability $p > 0$.

Let us prove the independence of $(\mathcal{B}_n)_{n\in\mathbb{N}}$, which is not obvious.[†] Since the distribution of $\vec{\Theta}_n$ is rotation invariant, $\mathbb{P}(\cos(\vec{\Theta}_n, \vec{v}) \geq \frac{1}{2}) = \mathbb{P}(\cos(\vec{\Theta}_n, \vec{w}) \geq \frac{1}{2})$ for all $\vec{v}, \vec{w} \in \mathbb{R}^3 \setminus \{0\}$. Choosing $\vec{w} = (0, 0, 1)$, we obtain

$$\mathbb{P}\big[\cos(\vec{\Theta}_n, \vec{v}) \geq \tfrac{1}{2}\big] = \mathbb{P}\big[\vec{\Theta}_1 \in \{(x, y, z) \in \mathbb{R}^3 : \ z \geq 1/2\}\big] = \frac{1}{4}.$$

Since $\vec{X}_{n-1}$ is $\mathcal{F}_{n-1}$-measurable, while $\vec{\Theta}_n$ is independent of $\mathcal{F}_{n-1}$, we have

$$\mathbb{P}\big[\mathcal{B}_n \,\big|\, \mathcal{F}_{n-1}\big] = \mathbb{P}\big[\cos(\vec{\Theta}_n, \vec{v}) \geq \tfrac{1}{2}\big]\big|_{\vec{v}=\vec{X}_{n-1}} = \frac{1}{4}.$$

For $k \in \mathbb{N}$ and $n_1 < n_2 < \ldots < n_k$, since $\mathcal{B}_i \in \mathcal{F}_j$ for $i \leq j$,

$$\mathbb{P}\big[\mathcal{B}_{n_1} \cap \ldots \cap \mathcal{B}_{n_k}\big] = \mathbb{E}\big[\mathbb{1}_{\mathcal{B}_{n_1} \cap \ldots \cap \mathcal{B}_{n_{k-1}}} \mathbb{P}\big[\mathcal{B}_{n_k} \,\big|\, \mathcal{F}_{n_k - 1}\big]\big] = \frac{1}{4}\, \mathbb{P}\big[\mathcal{B}_{n_1} \cap \ldots \cap \mathcal{B}_{n_{k-1}}\big],$$

hence, by induction, $\mathbb{P}\big[\mathcal{B}_{n_1} \cap \ldots \cap \mathcal{B}_{n_k}\big] = (\frac{1}{4})^k$. This shows that that the events $(\mathcal{B}_n)_{n\in\mathbb{N}}$ are independent with $\mathbb{P}[\mathcal{B}_n] = \frac{1}{4}$, completing the proof.

References

[1] D. Williams (1991), *Probability with martingales*, Cambridge University Press

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