

STRONG DISORDER FOR STOCHASTIC HEAT FLOW AND 2D DIRECTED POLYMERS

QUENTIN BERGER, FRANCESCO CARAVENNA, AND NICOLA TURCHI

ABSTRACT. The critical 2D Stochastic Heat Flow (SHF) is a universal measure-valued process providing a notion of solution to the ill-defined 2D stochastic heat equation. We investigate the SHF in the regime of large time and large disorder strength, proving a sharp form of *local extinction*: we identify the rate at which the distribution collapses to zero. We also identify the spatial scale governing the transition from vanishing to diverging mass, and from extinction to an averaged behavior. Corresponding results are established for the partition functions of 2D directed polymers, which yield precise free energy estimates. Our proof refines classical change of measure and coarse-graining techniques, introducing new ideas of independent interest.

Our findings provide novel insight into the 2D stochastic heat equation regularized via space-time discretization: for any regime of supercritical disorder strength β , including the case where $\beta > 0$ is kept fixed, the solution exhibits fluctuations on a superdiffusive scale.

1. INTRODUCTION AND MAIN RESULTS ON THE STOCHASTIC HEAT FLOW

The critical 2D *Stochastic Heat Flow (SHF)* with disorder strength $\vartheta \in \mathbb{R}$ is a stochastic process $\mathcal{Z}^\vartheta = (\mathcal{Z}_{s,t}^\vartheta(dx, dy))_{0 \leq s < t < \infty}$ of random measures on $\mathbb{R}^2 \times \mathbb{R}^2$. It was introduced in [CSZ23a] as the universal scaling limit of 2D directed polymer partition functions, which we recall below, under a critical rescaling of disorder strength. It is also the limit of solutions of the 2D stochastic heat equation with mollified noise, see [Tsa24], where an axiomatic definition is provided.

The fact that space dimension two is *critical* for the stochastic heat equation and directed polymers makes the SHF especially interesting, as a rare example of a *non-Gaussian scaling limit in the critical dimension and at the critical point*. A brief overview of the literature on the SHF is presented in Section 1.2. We refer to the lecture notes [CSZ24] for an extended discussion, as well as additional background and connections to singular SPDEs.

1.1. Overview of our contribution. We focus on the one-time marginal of the SHF:

$$\mathcal{Z}_t^\vartheta(dx) := \mathcal{Z}_{0,t}^\vartheta(\mathbb{R}^2, dx)$$

which is a random measure on \mathbb{R}^2 . We investigate both regimes of *strong disorder* $\vartheta \rightarrow \infty$ and *large time* $t \rightarrow \infty$, where a phenomenon of *local extinction* occurs, *i.e.* $\mathcal{Z}_t^\vartheta(dx) \rightarrow 0$ locally.

Our main results identify the *decay rate of the SHF distribution* (see Theorem 1.1) as well as the *growth rate of the spatial scale* at which a transition in the mass of the SHF occurs, from a regime of vanishing mass to a regime of diverging mass (see Theorem 1.5). Both rates are shown to be exponential in time t and doubly-exponential in the disorder strength ϑ .

We present in the next Section 2 corresponding results for 2D directed polymers, which have an independent interest as they are valid across *all regimes of disorder strength* (see Theorem 2.2). This allows us to derive *refined estimates on the free energy* (see Theorem 2.8) which improve on the best available bounds in the literature [Lac10, BL17].

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We also discuss in Section 2.2 the implication of our results for the 2D stochastic heat equation regularized through space-time discretization. We allow the disorder strength β to vary arbitrarily in the super-critical regime, including the case where $\beta > 0$ is kept fixed as the regularization is removed. We show that the solution exhibits a transition from local extinction to an averaged behavior at an explicit *superdiffusive scale* (see Theorem 2.3). This identifies the regime where non-trivial fluctuations of the solution can be observed.

The strategy of our proof builds upon the by-now classical approach based on *change of measure* and *coarse-graining*, introduced in the seminal works [DGLT09, GLT10a, GLT10b] and subsequently applied in various contexts. A key difficulty in successfully implementing this strategy lies in the choice of a suitable *proxy* for the random variable of interest, in our case the partition function of 2D directed polymers, which must be tractable enough while remaining sufficiently close to the original partition function.

The core novelty of our approach is the identification of a *new proxy*, arising from a *coarse-grained chaos expansion* (see Section 4.3), for which we only need moment bounds at fixed (large) $\vartheta \in \mathbb{R}$, despite ultimately deriving results in the strong-disorder regime $\vartheta \rightarrow \infty$. This allows us to implement somehow optimal change of measure and coarse-graining arguments. The main ideas are illustrated in Sections 3 and 4, where we also develop *change of scale arguments* of independent interest. We believe that our strategy is sufficiently robust and transparent to be broadly applicable, and we expect it to be useful in other contexts.

1.2. A quick overview of the SHF literature. Many features of the SHF have been investigated, in particular its moments. The second moment was first studied in [BC98] in the context of solutions to the 2D stochastic heat equation, exploiting a connection with the delta-Bose gas from [ABD95]; refined results, also in the setting of directed polymers, were later obtained in [CSZ19a]. The third moment was obtained in [CSZ19b], and all integer moments were later derived in [GQT21]; see also [Che24] for further connections with the delta-Bose gas.

The asymptotic analysis of moments is challenging, due to their intricate structure. Important progress has been obtained recently in [GN25], where a sharp lower bound on their growth rate was established through a novel connection between moments of the SHF and the Gaussian Free Field; note that the moments grow with a doubly-exponential rate, a feature that also appears in our results. Let us also mention [LZ24], where small-scale asymptotics were derived, extending the approach developed by [CZ23] in the sub-critical regime.

Concerning the properties of the SHF as a random measure, estimates on its singularity and regularity were obtained in [CSZ25]. It was also proved in [CSZ23b] that the SHF is *not a Gaussian Multiplicative Chaos (GMC)* on \mathbb{R}^2 via comparison of moments. Very recently, the SHF was shown in [CT25] to enjoy a *conditional GMC structure* on path space, which yields as corollaries the full support property (strict positivity), also obtained independently in [Nak25b], and the local extinction of mass for strong disorder, discussed below.

Other features of the SHF include a Chapman–Kolmogorov property and the construction of associated polymer measures [CM24], continuity in time and the already mentioned characterization [Tsa24]. Let us also mention the black noise property [GT25] and an enhanced noise sensitivity property for directed polymer partition functions [CD25], which both yield independence between SHF and white noise. Recent progress on a martingale description of the SHF has also been obtained [Nak25a, Che25].

These results mostly refer to the SHF with a finite time-horizon and fixed disorder strength $\vartheta \in \mathbb{R}$. Some results are also available in the weak-disorder regime $\vartheta \rightarrow -\infty$, such as Edwards–Wilkinson (Gaussian) fluctuations [CCR25, Theorem 1.2] and an asymptotic log-normality for small scales [CSZ25, Theorem 1.2]. Corresponding results, and many others, hold for directed polymers and the stochastic heat equation in the sub-critical regime, for which we refer again

to [CSZ24]. Here, on the other hand, we investigate the properties of the SHF in the large time $t \rightarrow \infty$ or strong disorder regime $\vartheta \rightarrow +\infty$.

1.3. Main results for the SHF. The first moment $\mathbb{E}[\mathcal{Z}_t^\vartheta(dx)] = dx$ of the SHF is simply the Lebesgue measure on \mathbb{R}^2 . In particular, using the functional notation

$$\mathcal{Z}_t^\vartheta(\varphi) := \int_{\mathbb{R}^2} \varphi(x) \mathcal{Z}_t^\vartheta(dx),$$

we have $\mathbb{E}[\mathcal{Z}_t^\vartheta(\varphi)] = 1$ for any probability density φ on \mathbb{R}^2 (we call φ the *initial condition*).

It turns out that the second moment diverges for strong disorder: for any probability density φ

$$\lim_{\vartheta \rightarrow \infty} \mathbb{E}[\mathcal{Z}_t^\vartheta(\varphi)^2] = \infty.$$

This implies that higher moments diverge quickly: namely, one can easily deduce by size-biasing and Jensen's inequality (see [CSZ25, Remark 1.14]) that

$$\forall h > 2: \quad \frac{\mathbb{E}[\mathcal{Z}_t^\vartheta(\varphi)^h]}{\mathbb{E}[\mathcal{Z}_t^\vartheta(\varphi)^2]^{\frac{h}{2}}} \xrightarrow{\vartheta \rightarrow \infty} \infty,$$

which expresses a form of *intermittent behavior*. Similar asymptotics also hold as $t \rightarrow \infty$.

1.3.1. Strong disorder and local extinction. In view of the intermittent behavior described above, it is natural to expect that $\mathcal{Z}_t^\vartheta(\varphi)$ vanishes for strong disorder or large time:

$$\mathcal{Z}_t^\vartheta(\varphi) \longrightarrow 0 \quad \text{in probability as } t \rightarrow \infty \text{ or } \vartheta \rightarrow \infty. \quad (1.1)$$

The large-time convergence was obtained in [CSZ25], while the strong disorder convergence was very recently proved in [CT25], as a consequence of a *conditional GMC structure*.

We establish in this paper a *quantitative version* of this convergence, valid in any regime of strong disorder $\vartheta \rightarrow \infty$ and/or large time $t \rightarrow \infty$. We allow for varying initial conditions φ with possibly *diverging support*, and we establish *optimal bounds*, displaying an exponential decay rate in time t and a doubly-exponential decay rate in the disorder strength ϑ . Let us denote by $\mathcal{M}_1(r)$ the set of probability densities with support in the ball of radius r :

$$\mathcal{M}_1(r) = \left\{ \varphi : \mathbb{R}^2 \rightarrow [0, \infty) \text{ s.t. } \int_{\mathbb{R}^2} \varphi(x) dx = 1, \quad \varphi(x) = 0 \text{ for } |x| > r \right\}. \quad (1.2)$$

We can now state our first main result, which we prove in Section 7.

Theorem 1.1 (Strong disorder and large time for the SHF). *There exist universal constants $c_0, c_1, c_2 \in (0, \infty)$ such that, for any $t > 0$ and $\vartheta \in \mathbb{R}$,*

$$\frac{1}{c_1} e^{-c_1 t e^\vartheta} \leq \sup_{\varphi \in \mathcal{M}_1(e^{c_0 t e^\vartheta} \sqrt{t})} \mathbb{E}[\mathcal{Z}_t^\vartheta(\varphi) \wedge 1] \leq \frac{1}{c_2} e^{-c_2 t e^\vartheta}. \quad (1.3)$$

The same bounds hold replacing $\mathbb{E}[\mathcal{Z}_t^\vartheta(\varphi) \wedge 1]$ by a fractional moment $\mathbb{E}[\mathcal{Z}_t^\vartheta(\varphi)^\gamma]$ with $\gamma \in (0, 1)$, for suitable constants $c_i = c_i(\gamma)$.

Correspondingly, we can bound the right tail probability of $\mathcal{Z}_t^\vartheta(\varphi)$: for any $\varepsilon \in (0, 1)$ there are constants $C_{1,\varepsilon}, C_{2,\varepsilon} \in (0, \infty)$ such that

$$C_{1,\varepsilon} e^{-c_1 t e^\vartheta} \leq \sup_{\varphi \in \mathcal{M}_1(e^{c_0 t e^\vartheta} \sqrt{t})} \mathbb{P}(\mathcal{Z}_t^\vartheta(\varphi) \geq \varepsilon) \leq C_{2,\varepsilon} e^{-c_2 t e^\vartheta}. \quad (1.4)$$

The core of Theorem 1.1 is the upper bound in (1.3), which we derive from a corresponding result for 2D directed polymers; see Theorem 2.2 below. The upper bound in (1.4) follows by Markov's inequality, since $\mathbb{P}(Z \geq \varepsilon) \leq (\varepsilon \wedge 1)^{-1} \mathbb{E}[Z \wedge 1]$ for any random variable $Z \geq 0$, while the lower bounds in (1.3) and (1.4) are obtained via the second moment method.

Remark 1.2 (Lower bounds and second moment). We prove the lower bounds in (1.3) and (1.4) by the Paley–Zygmund inequality coupled with a variance upper bound; see Proposition 2.10 below. In both (1.3) and (1.4), the sup can be removed if we choose $\varphi = \mathcal{U}_{\sqrt{t}}$ to be uniform in the ball of radius \sqrt{t} , where we set

$$\mathcal{U}_r(x) := \frac{1}{\pi r^2} \mathbf{1}_{B(0,r)}(x) \quad \text{with} \quad B(0,r) := \{x \in \mathbb{R}^2: |x| \leq r\}. \quad (1.5)$$

Remark 1.3 (Truncated vs. fractional moments). The *truncated mean* $\mathbb{E}[\mathcal{Z}_t^\vartheta(\varphi) \wedge 1]$ appearing in (1.3) may also be written as $\mathbb{P}(\mathcal{Z}_t^\vartheta(\varphi) > U)$ with U an independent uniform random variable on $(0, 1)$. This quantity can be compared with *fractional moments* $\mathbb{E}[\mathcal{Z}_t^\vartheta(\varphi)^\gamma]$ with $\gamma \in (0, 1)$ (see Lemma 3.2 below) and it also measures the *total variation distance* between the law \mathbb{P} and its *size-biased version* with respect to $\mathcal{Z}_t^\vartheta(\varphi)$ (see Remark 4.2). In our proofs, we will use both the truncated mean and fractional moments, since each quantity has its own advantages and limitations; see Section 4 for a discussion.

Remark 1.4 (Scaling covariance, strong disorder and large time). The dependence of our bounds (1.3), (1.4) on the parameters t and ϑ agrees with the *scaling covariance property* of the SHF [CSZ23a, Theorem 1.2], which states that for any t, ϑ, φ we have the equality in distribution

$$\forall a > 0: \quad \mathcal{Z}_{at}^\vartheta(\varphi_{\sqrt{a}}) \stackrel{d}{=} \mathcal{Z}_t^{\vartheta+\log a}(\varphi) \quad \text{where we set} \quad \varphi_{\sqrt{a}}(x) := \frac{1}{a} \varphi\left(\frac{x}{\sqrt{a}}\right). \quad (1.6)$$

This property connects strong-disorder and large-time regimes: replacing t by t/a and setting $a = e^{-\vartheta}$ (resp. $a = t$) lets us set $\vartheta = 0$ (resp. $t = 1$), which yields

$$\mathcal{Z}_t^{\vartheta}(\varphi_{\sqrt{e^{-\vartheta}}}) \stackrel{d}{=} \mathcal{Z}_{te^{\vartheta}}^0(\varphi), \quad \mathcal{Z}_t^{\vartheta}(\varphi_{\sqrt{t}}) \stackrel{d}{=} \mathcal{Z}_1^{\vartheta+\log t}(\varphi). \quad (1.7)$$

Since it was proved in [CSZ25, Theorem 1.4] that $\mathcal{Z}_T^{\vartheta_0}(\varphi) \rightarrow 0$ in probability as $T \rightarrow \infty$ for fixed ϑ_0 , we could deduce from the first relation in (1.7) that $\mathcal{Z}_t^{\vartheta}(\varphi_{\sqrt{e^{-\vartheta}}}) \rightarrow 0$ as $\vartheta \rightarrow \infty$. We stress, however, that this property is *much weaker* than (1.1), and even more than (1.3), (1.4) and (1.8), because shrinking the support of the initial condition φ helps convergence to zero*.

Similarly, from property (1.1) and the second relation in (1.7), we could deduce that $\mathcal{Z}_t^{\vartheta}(\varphi_{\sqrt{t}}) \rightarrow 0$ in probability as $t \rightarrow \infty$ for fixed ϑ . However, our bounds (1.3), (1.4) are much stronger, since the space scale is increased by a factor $e^{cte^{\vartheta}}$.

1.3.2. *Transition for the mass of the SHF on large spatial scales.* From Theorem 1.1 we deduce the behavior of the mass of the SHF in balls $B(0, r)$ with large radius $r \rightarrow \infty$. Even though $\mathbb{E}[\mathcal{Z}_t^\vartheta(B(0, r))] = \pi r^2 \rightarrow \infty$, a *transition occurs on the spatial scale* $r = e^{cte^{\vartheta}} \sqrt{t}$ as either $\vartheta \rightarrow \infty$ or $t \rightarrow \infty$: with high probability, the mass *vanishes* for small $c > 0$, while it *diverges* for large c . The following is our second main result, proved in Section 7.

Theorem 1.5 (Transition for the SHF mass in large balls). *There are constants $\delta > 0$ and $0 < c' < c'' < \infty$ such that the following holds for any $t > 0$ and $\vartheta \in \mathbb{R}$:*

$$\text{with probability at least } 1 - \frac{1}{\delta} e^{-\delta te^{\vartheta}}: \quad \begin{cases} \mathcal{Z}_t^\vartheta(B(0, e^{c' te^{\vartheta}} \sqrt{t})) \leq t e^{-\delta te^{\vartheta}}, \\ \mathcal{Z}_t^\vartheta(B(0, e^{c'' te^{\vartheta}} \sqrt{t})) \geq t e^{+\delta te^{\vartheta}}. \end{cases} \quad (1.8)$$

We prove the first line of (1.8) by exploiting the upper bound in (1.3), while for the second line we use the second moment method with a variance bound from Proposition 2.10. We naturally expect the transition to be sharp, in the following sense.

*For instance, by [CSZ25, Theorem 1.1], we have $\mathcal{Z}_t^\vartheta(\varphi_{\sqrt{a}}) \rightarrow 0$ in probability as $a \downarrow 0$ even for fixed t, ϑ .

Conjecture 1.6. *There exists some $\tilde{c} > 0$ such that we have the following convergence in probability: as $t \rightarrow \infty$ and/or $\vartheta \rightarrow \infty$,*

$$\mathcal{Z}_t^\vartheta(B(0, e^{cte^\vartheta} \sqrt{t})) \xrightarrow{\mathbb{P}} \begin{cases} 0 & \text{if } c < \tilde{c}, \\ +\infty & \text{if } c > \tilde{c}. \end{cases}$$

Since $\mathbb{E}[\mathcal{Z}_t^\vartheta(dx)] = dx$, it is natural to compare the mass of the SHF on a large spatial scale with the mass of the Lebesgue measure, somehow capturing the “escape of mass to infinity” of the SHF $\mathcal{Z}_t^\vartheta(dx)$. Theorem 1.5 already shows that on scales $e^{cte^\vartheta} \sqrt{t}$ with $c < c'$ the SHF mass vanishes. On the other hand, by spatial ergodicity, we expect some averaging to occur on larger scales. Let us thus define a spatially rescaled version of the SHF $\mathcal{Z}_t^\vartheta(dx)$:

$$\hat{\mathcal{Z}}_t^{\vartheta,c}(dx) := \frac{\mathcal{Z}_t^\vartheta(d(e^{cte^\vartheta} \sqrt{t}x))}{(e^{cte^\vartheta} \sqrt{t})^2} \quad \text{for } c \in (0, \infty), \quad (1.9)$$

where the normalization ensures $\mathbb{E}[\hat{\mathcal{Z}}_t^{\vartheta,c}(dx)] = dx$. We then prove the following transition, from extinction to an averaged behavior, proved in Section 7.

Theorem 1.7 (Supercritical rescaling of SHF). *There are constants $0 < c' < c'' < \infty$ (possibly different from those in Theorem 1.5) such that, for any $t > 0$ and $\varphi \in C_c(\mathbb{R}^2)$ with $\int \varphi = 1$, we have as $t \rightarrow +\infty$ and/or $\vartheta \rightarrow +\infty$,*

$$\hat{\mathcal{Z}}_t^{\vartheta,c}(\varphi) = \int_{\mathbb{R}^2} \varphi(x) \hat{\mathcal{Z}}_t^{\vartheta,c}(dx) \xrightarrow{d} \begin{cases} 0 & \text{if } c < c', \\ 1 & \text{if } c > c''. \end{cases} \quad (1.10)$$

As in Conjecture 1.6, we expect that this transition is sharp.

Conjecture 1.8. *There exists a critical constant $\hat{c} \in (0, \infty)$ (possibly different from \tilde{c} in Conjecture 1.6) such that the convergence in distribution (1.10) still holds with c', c'' both replaced by \hat{c} . For this critical choice of $c = \hat{c}$, we have the following convergence in distribution (possibly after lower order corrections to the rescaling in (1.9)): for any $t > 0$ and $\varphi \in C_c(\mathbb{R}^2)$*

$$\hat{\mathcal{Z}}_t^{\vartheta,c}(\varphi) \xrightarrow[\vartheta \rightarrow \infty]{d} \int_{\mathbb{R}^2} \varphi(x) \mathcal{U}_t(dx) \quad (1.11)$$

where $\mathcal{U}_t(dx)$ is a non-trivial (i.e. random) measure-valued process on \mathbb{R}^2 .

We may interpret Theorems 1.5 and 1.7 as manifestations of an *intermittent* behavior of the SHF. The mass of the SHF vanishes on scales $e^{cte^\vartheta} \sqrt{t}$ with $c < c'$, ensuring that there are typically no “high peaks” in the distribution at this scale; however, since $\mathbb{E}[\mathcal{Z}_t^\vartheta(dx)] = dx$, high peaks could occur, but with very small probability. On the other hand, at larger spatial scales $e^{cte^\vartheta} \sqrt{t}$ with $c > c''$ an averaging behavior occurs, meaning that we have encountered sufficiently many of the (rare but high) peaks. Theorems 1.5 and 1.7 thus give information on the spatial scale at which the high peaks appear. With this picture in mind, and in view of the exponential scaling, we expect the limit \mathcal{U}_t in (1.11) to be a suitable Poisson Point Process on \mathbb{R}^2 , corresponding to the location of (very high) peaks in the SHF.

1.4. Structure of the paper. In Section 2 we present our main results for 2D Directed Polymers and the Stochastic Heat Equation, see in particular Theorems 2.2 and 2.3.

In Section 3 we describe the proof of Theorem 2.2: by coarse-graining and change of scale arguments, we can reduce it to a key Proposition 3.1, which is the core of our paper.

Section 4 contains the main ideas for the proof of the key Proposition 3.1, which involve size-biasing and change of measure arguments. This leads to some explicit moment estimates, proved in the following Sections 5 and 6.

Section 7 collects the proof of the other main results: Theorems 1.1, 1.5 and 1.7 on the SHF, Theorem 2.3 on the Stochastic Heat Equation and Theorem 2.8 on Directed Polymers.

Further technical results (which follow well-established paths) are postponed to the appendices.

1.5. Notation. \mathbb{N} denotes the set of non-negative integers. For a point $x = (x_1, x_2)$ in \mathbb{R}^2 we let $|x| = \sqrt{x_1^2 + x_2^2}$ and $|x|_\infty = \max\{|x_1|, |x_2|\}$ denote its Euclidean and ℓ^∞ norms. For technical convenience, in the proofs we will slightly modify the families $\mathcal{M}_1(r)$ and $\mathcal{M}_1^{\text{disc}}(r)$, see (1.2) and (2.12), replacing $|\cdot|$ with $|\cdot|_\infty$ in their definition; this only affects constants.

Given two positive sequences $(a_N)_{N \in \mathbb{N}}$ and $(b_N)_{N \in \mathbb{N}}$, we write $a_N \sim b_N$ if $\lim_{N \rightarrow \infty} a_N/b_N = 1$ and $a_N \ll b_N$ if $\lim_{N \rightarrow \infty} a_N/b_N = 0$, $a_N \gg b_N$ if $\lim_{N \rightarrow \infty} a_N/b_N = +\infty$. For $M \leq N \in \mathbb{N}$ we write $\llbracket M, N \rrbracket$ for the set $\{M, M+1, \dots, N\}$. When A is a set we indicate with $|A|$ its cardinality.

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2. MAIN RESULTS FOR 2D DIRECTED POLYMERS AND STOCHASTIC HEAT EQUATION

The SHF was obtained in [CSZ23a] as the limit of 2D directed polymer partition functions in an appropriate *critical window* of disorder strength. We are going to derive our Theorem 1.1 from a corresponding result for the 2D directed polymer model, that we in fact prove in the whole *super-critical regime*. We discuss afterwards some consequences on the 2D stochastic heat equation regularized by space-time discretization, again in the whole super-critical regime.

2.1. Strong disorder for 2D directed polymers. Let us start by recalling the definition of the directed polymer model. Let $S = (S_n)_{n \in \mathbb{N}}$ be the simple (symmetric, nearest-neighbor) random walk on \mathbb{Z}^2 , and denote \mathbf{P}_x its law when $S_0 = x \in \mathbb{Z}^2$; let \mathbf{E}_x denote the corresponding expectation. We simply write \mathbf{P}, \mathbf{E} for $\mathbf{P}_0, \mathbf{E}_0$. Additionally, consider a collection $\omega = (\omega(n, x))_{n \in \mathbb{N}, x \in \mathbb{Z}^2}$ of i.i.d. random variables, independent of S , with law denoted by \mathbb{P} . With a slight abuse of notation to denote by ω a generic random variable $\omega_{n,x}$, we assume that for some $\bar{\beta} > 0$

$$\mathbb{E}[\omega] = 0, \quad \mathbb{E}[\omega^2] = 1, \quad \lambda(\beta) := \log \mathbb{E}[e^{\beta\omega}] < +\infty \quad \text{for all } 0 < \beta < \bar{\beta}. \quad (2.1)$$

For $N \in \mathbb{N}$ and $\beta > 0$, the point-to-plane (1+2-dimensional) directed polymer model is defined as the Gibbs measure with Hamiltonian (up to a sign) $H_N^{\beta,\omega}(S) := \sum_{n=1}^N (\beta\omega(n, S_n) - \lambda(\beta))$. We are interested in the *point-to-plane partition function* started at $x \in \mathbb{Z}^2$, defined by

$$Z_N^{\beta,\omega}(x) := \mathbf{E}_x \left[e^{H_N^{\beta,\omega}(S)} \right] \quad \text{with} \quad H_N^{\beta,\omega}(S) := \sum_{n=1}^N (\beta\omega(n, S_n) - \lambda(\beta)). \quad (2.2)$$

We view $(Z_N^{\beta,\omega}(x))_{x \in \mathbb{Z}^2}$ as a random field: for a function $f \in \ell^1(\mathbb{Z}^2)$ we define

$$Z_N^{\beta,\omega}(f) := \sum_{x \in \mathbb{Z}^2} f(x) Z_N^{\beta,\omega}(x), \quad (2.3)$$

which is the integral of f with respect to the random measure $\sum_{x \in \mathbb{Z}^2} Z_N^{\beta,\omega}(x) \delta_x$. The main result of [CSZ23a] shows that this measure, diffusively rescaled, converges to a unique limit, which they named Critical 2D Stochastic Heat Flow (SHF), provided the disorder strength β is rescaled in the so-called *critical window*, which we now define.

Recalling (2.1), we define for $\beta \geq 0$ and $N \in \mathbb{N}$ the key quantities

$$\sigma^2(\beta) := \text{Var}[e^{\beta\omega - \lambda(\beta)}] = e^{\lambda(2\beta) - 2\lambda(\beta)} - 1, \quad R_N := \sum_{n=1}^N \mathbf{P}(S_{2n} = 0). \quad (2.4)$$

Note that $\sigma^2(\beta) \sim \beta^2$ as $\beta \downarrow 0$ and we can write, see [CSZ19a, Proposition 3.2],

$$R_N = \frac{1}{\pi}(\log N + \alpha_N) \quad \text{with} \quad \lim_{N \rightarrow \infty} \alpha_N = \alpha := 4 \log 2 + \gamma - \pi \simeq 0.208, \quad (2.5)$$

where $\gamma := -\int_0^\infty e^{-x} \log x \, dx \simeq 0.577$ is the Euler–Mascheroni constant. Then, the critical window corresponds to taking

$$\beta = \beta_N(\vartheta) \downarrow 0 \quad \text{such that} \quad \sigma^2(\beta) = \frac{1}{R_N} \left(1 + \frac{\vartheta + o(1)}{\log N} \right), \quad (2.6)$$

where $\vartheta \in \mathbb{R}$ is a fixed parameter, called the *disorder strength* in the critical regime.

We now recall the main result of [CSZ23a]. To match the random walk periodicity, we set

$$\mathbb{Z}_{\text{even}}^d := \{x = (x^1, \dots, x^d) \in \mathbb{Z}^d : x^1 + \dots + x^d \text{ is even}\}. \quad (2.7)$$

Given an integrable function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$, we define its rescaled version $\varphi^{(N)} : \mathbb{Z}_{\text{even}}^2 \rightarrow \mathbb{R}$ by

$$\varphi^{(N)}(x) := \frac{N}{2} \int_{|y - \frac{x}{\sqrt{N}}|_1 \leq \frac{1}{\sqrt{N}}} \varphi(y) \, dy \quad (2.8)$$

where $|\cdot|_1$ denotes the ℓ^1 norm. We then have the convergence in distribution

$$\forall \vartheta \in \mathbb{R} : \quad Z_{\lfloor Nt \rfloor}^{\beta_N, \omega}(\varphi^{(N)}) \xrightarrow[N \rightarrow \infty]{d} \mathcal{Z}_t^\vartheta(\varphi) \quad \text{for } \beta_N = \beta_N(\vartheta) \text{ as in (2.6)}. \quad (2.9)$$

In this article, we go *beyond the critical window*, *i.e.* we derive estimates which hold for *arbitrary* $\beta > 0$ (small enough) and $N \in \mathbb{N}$. This means that ϑ in (2.6) needs not be fixed, but may vary with N and β . To this purpose, we refine the correspondence (2.6) as follows:

$$\sigma^2(\beta) = \frac{1}{R_N} \left(1 - \frac{\vartheta + o(1)}{\pi R_N} \right)^{-1} = \frac{1}{R_N - \frac{\vartheta + o(1)}{\pi}}. \quad (2.10)$$

Note that relations (2.10) and (2.6) are equivalent for any fixed $\vartheta \in \mathbb{R}$, in view of (2.5) (they are also equivalent for $|\vartheta| = o(\sqrt{\log N})$, but *not* for $|\vartheta| \geq \text{constant} \sqrt{\log N}$).

Remark 2.1. For $\beta \rightarrow 0$ and $N \rightarrow \infty$, we can rewrite (2.10) more elegantly as

$$\sigma^2(\beta) = \frac{1}{R_{\lfloor N/e^\vartheta \rfloor} + o(1)}.$$

Given any $\beta > 0$ and $N \in \mathbb{N}$, we therefore quantify the disorder strength by a parameter $\vartheta = \vartheta(N, \beta)$ that we extract from (2.10): ignoring the $o(1)$ term, we define explicitly

$$\vartheta(N, \beta) := \pi R_N - \frac{\pi}{\sigma^2(\beta)}. \quad (2.11)$$

Intuitively, for $\beta = \beta_N(\vartheta)$ in the critical window (2.6) we have $\vartheta(N, \beta) \rightarrow \vartheta$ as $N \rightarrow \infty$, while taking any regime of $\beta > 0$ beyond the critical window corresponds to $\vartheta(N, \beta) \rightarrow \infty$.

We introduce a discrete analogue of (1.2) for mass functions:

$$\mathcal{M}_1^{\text{disc}}(r) = \left\{ f : \mathbb{Z}^2 \rightarrow [0, 1] \quad \text{s.t.} \quad \sum_{z \in \mathbb{Z}^2} f(z) = 1, \quad f(z) = 0 \quad \text{for } |z| > r \right\}. \quad (2.12)$$

We can now state our main result for 2D directed polymers. The strategy of the proof is presented in Section 3, while the details are developed in the subsequent sections.

Theorem 2.2 (Strong disorder for 2D directed polymers). *There are constants $\beta_0, c_0, c_1, c_2 \in (0, \infty)$ such that, uniformly over $N \in \mathbb{N}$ and $\beta \in (0, \beta_0)$, we can bound*

$$\frac{1}{c_1} \exp\left(-c_1 e^{\vartheta(N, \beta)}\right) \leq \sup_{f \in \mathcal{M}_1^{\text{disc}}(e^{c_0} e^{\vartheta(N, \beta)} \sqrt{N})} \mathbb{E}[Z_N^{\beta, \omega}(f) \wedge 1] \leq \frac{1}{c_2} \exp\left(-c_2 e^{\vartheta(N, \beta)}\right), \quad (2.13)$$

where we define $\vartheta(N, \beta)$ as in (2.11) (see also (2.10)). Note that, in view of (2.5), we have

$$e^{\vartheta(N, \beta)} = e^{\alpha_N N} e^{-\frac{\pi}{\sigma^2(\beta)}} = (1 + o(1)) e^{\alpha N} e^{-\frac{\pi}{\sigma^2(\beta)}} \quad \text{as } N \rightarrow \infty. \quad (2.14)$$

In particular, for any sequence of mass functions $f_N \in \mathcal{M}_1^{\text{disc}}(e^{c_0} e^{\vartheta(N, \beta)} \sqrt{N})$, we have the convergence in distribution $Z_N^{\beta, \omega}(f_N) \rightarrow 0$ as $N \rightarrow \infty$.

We will deduce Theorem 1.1 for the SHF from Theorem 2.2 by taking $\beta = \beta_N(\vartheta)$ in the critical regime (2.6) so that $\vartheta(N, \beta) \rightarrow \vartheta$ (see Section 7). We stress, however, that Theorem 2.2 is *stronger* since it allows for $\vartheta(N, \beta) \rightarrow \infty$. This lets us also deduce interesting information on the *free energy*, which we discuss in Section 2.3 below.

2.2. On the supercritical 2D Stochastic Heat Equation. The 2D stochastic heat equation (SHE) is the singular stochastic PDE formally given for $t > 0$ and $x \in \mathbb{R}^2$ by

$$\begin{cases} \partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) + \beta \xi(t, x) u(t, x) \\ u(0, x) = 1 \quad (\text{for simplicity}) \end{cases} \quad (2.15)$$

where ξ is space-time white noise and $\beta > 0$ tunes the disorder strength. In mild formulation, this reads as

$$u(t, x) = 1 + \int_0^t \int_{\mathbb{R}^2} g_{t-s}(x-y) u(s, y) \beta \xi(s, y) \, ds \, dy \quad (2.16)$$

where $g_t(x) = \frac{1}{2\pi t} e^{-\frac{|x|^2}{2t}}$ is the heat kernel. Let us stress that this equation is ill-defined in dimension two and higher, since the solution u is expected to be a genuine distribution, hence the singular product $u \cdot \xi$ has no clear meaning.

A natural way to remove the singularity is to suitably regularize the equation, so that a well-defined solution $u_1^\beta(t, x)$ exists (the subscript 1 indicates the regularization scale). Many regularizations are possible including mollification in space, discretization in space or space-time, truncation in Fourier space, etc. We focus on space-time discretization, as in [CSZ23a], turning the integrals in (2.16) into Riemann sums on the even lattice (see (2.7))

$$\mathbb{T}_1 := (\mathbb{N}_0 \times \mathbb{Z}^2)_{\text{even}}.$$

Replacing white noise by i.i.d. random variables and the heat kernel by the random walk transition kernel*, the solution $u_1^\beta(t, x)$ is nothing but the *partition function* $Z_t^{\omega_t}(x)$ from (2.2) with time-reversed environment ω_t : upon piecewise constant extension, we have for $t \geq 0$, $x \in \mathbb{R}^2$,

$$u_1^\beta(t, x) = Z_{[t]}^{\beta, \omega^{[t]}}([x]) \quad \text{with} \quad \omega_m = (\omega_m(n, z) := \omega(m-n, z))_{n \in \mathbb{N}, z \in \mathbb{Z}^2}, \quad (2.17)$$

where we denote by $([t], [x])$ the closest point in \mathbb{T}_1 to $(t, x) \in [0, \infty) \times \mathbb{R}^2$.

We are interested in the *large-scale scaling properties of* $u_1^\beta(t, x)$. In view of the parabolic nature of the SHE, it is natural to consider for $N \in \mathbb{N}$ the diffusively rescaled solution

$$u_N^\beta(t, x) := u_1^\beta(Nt, \sqrt{N}x) = Z_{[Nt]}^{\beta, \omega^{[Nt]}}([\sqrt{N}x]), \quad (2.18)$$

so that the regime $N \rightarrow \infty$ corresponds to *zooming out space and time*. Also note that $u_N^\beta(t, x)$ is the solution of the SHE (2.16) discretized on the finer lattice

$$\mathbb{T}_N := \left(\frac{1}{N} \mathbb{N}_0 \times \frac{1}{\sqrt{N}} \mathbb{Z}^2\right)_{\text{even}}$$

which approximates space-time $[0, \infty) \times \mathbb{R}^2$ as $N \rightarrow \infty$. For this reason, finding a *non-trivial (random) limit of* $u_N^\beta(t, x)$ as $N \rightarrow \infty$ provides a notion of solution to the ill-defined 2D SHE. This is the viewpoint taken in [CSZ23a], where it was shown that for $\beta = \beta_N(\vartheta)$ in the *critical window* (2.6) the solution $u_N^\beta(t, x)$ converges to the SHF $\mathcal{Z}_t^\vartheta(dx)$, see (2.9).

A natural question is the behavior of $u_N^\beta(t, x)$ *beyond the critical window*, that is for disorder strength $\beta = \beta_N$ such that $\vartheta(N, \beta) \rightarrow \infty$, see (2.11). We call this range of (β, N) the *supercritical*

*More precisely, for $(t, x) \in \mathbb{T}_1$ we replace $\beta \xi(t, x)$ by $e^{\beta \omega(t, x) - \lambda(\beta)} - 1$ and $g_t(x)$ by $q_t(x) := \mathbf{P}(S_t = x)$.

regime^{*}, which includes in particular the case when $\beta > 0$ is kept fixed as $N \rightarrow \infty$. A direct consequence of our Theorem 2.2 is the *local extinction of $u_N^\beta(t, x)$ averaged in space*:

$$\forall t > 0, \forall \varphi \in C_c(\mathbb{R}^2): \quad \int_{\mathbb{R}^2} \varphi(x) u_N^\beta(t, x) dx \xrightarrow[N \rightarrow \infty]{d} 0. \quad (2.19)$$

We now present quantitative refinements of this result.

Since rescaling space-time diffusively in (2.18) leads to the degenerate limit (2.19), we expect the solution $u_1^\beta(t, x)$ to display non-trivial fluctuations on a *super-diffusive scale*. To capture this phenomenon, it is natural to modify the space rescaling in (2.18), replacing \sqrt{N} therein by $\sqrt{D_N N}$ for a suitable diverging factor $D_N \rightarrow \infty$, with the aim of finding a non-trivial limit. This viewpoint was taken, for instance, in [CMT25], where the regularized 2D stochastic Burgers equation was shown to admit non-trivial fluctuations for the choice $D_N \propto (\log N)^{2/3}$.

In our setting of the 2D SHE, it turns out that the space rescaling in (2.18) needs to be modified by an *exponential factor*:

$$\hat{u}_N^{\beta,c}(t, x) := u_1^\beta(Nt, \rho_{Nt}^{\beta,c} \sqrt{Nt} x) \quad \text{with} \quad \rho_{Nt}^{\beta,c} := e^{c e^{\vartheta(Nt, \beta)}}, \quad c \in (0, \infty). \quad (2.20)$$

In the supercritical regime $\vartheta(N, \beta) \rightarrow \infty$ we have $\rho_{Nt}^{\beta,c} \rightarrow \infty$ for any $t > 0$ (note that by (2.10) we have $\vartheta(Nt, \beta) = \vartheta(N, \beta) + \log t + o(1)$). Also note that, by (2.14), we can write

$$\rho_{Nt}^{\beta,c} = e^{c e^\alpha Nt f_\beta(1+o(1))} \quad \text{with} \quad f_\beta := e^{-\frac{\pi}{\sigma^2(\beta)}}, \quad (2.21)$$

which shows that $\rho_{Nt}^{\beta,c}$ grows *exponentially in N* for fixed $\beta > 0$ (see Remark 2.5 for $\beta \downarrow 0$).

The following result, proved in Section 7, shows that a transition from extinction to an averaged behavior takes place for the rescaled solution $\hat{u}_N^{\beta,c}$, as the constant c is varied.

Theorem 2.3 (Supercritical 2D Stochastic Heat Equation). *There are constants $\beta_0 > 0$ and $0 < c' < c'' < \infty$ such that the following holds: given $N \in \mathbb{N}$ and $\beta_N \in (0, \beta_0)$ in the supercritical regime $\vartheta(N, \beta_N) \rightarrow \infty$, for any $t > 0$ and $\varphi \in C_c(\mathbb{R}^2)$ with $\int \varphi = 1$ we have*

$$\int_{\mathbb{R}^2} \varphi(x) \hat{u}_N^{\beta_N, c}(t, x) dx \xrightarrow[N \rightarrow \infty]{d} \begin{cases} 0 & \text{if } c < c', \\ 1 & \text{if } c > c''. \end{cases} \quad (2.22)$$

This holds, in particular, if $\beta_N \equiv \beta \in (0, \beta_0)$ is kept fixed as $N \rightarrow \infty$.

Let us discuss the significance of this result for the 2D SHE (2.15)-(2.16) regularized by space-time discretization. While the critical regime (2.6) is the correct rescaling of β to obtain a non-trivial limit, the SHF [CSZ23a], it is natural to ask *what happens beyond this critical window* (including the case when $\beta > 0$ is fixed, *i.e.* not rescaled at all). Theorem 2.3 shows that non-trivial fluctuations of the solution can only be observed on the *super-diffusive scale* $\rho_{Nt}^{\beta,c} \sqrt{N}$ from (2.20), for some $c \in (c', c'')$. We expect a similar phenomenon to occur also for the SHE in one spatial dimension, see Remark 2.7.

Checking that the scale $\rho_{Nt}^{\beta,c} \sqrt{N}$ provides an *upper bound* for space fluctuations is not difficult: a variance computation yields the second line of (2.22), which shows that an averaged behavior takes place for suitable (large) $c > 0$. However, in view of the intermittent nature of the solution, it is not at all clear whether the spatial scale identified by the variance is of the correct order. Our result shows that this is indeed the case, for suitable (small) $c > 0$, thanks to a *lower bound* on the scale of space fluctuations provided by the first line in (2.22).

Analogously to Conjecture 1.8, we expect that the transition in (2.22) is sharp.

*The subcritical regime when $\beta \ll \beta_N(\vartheta)$ is rescaled *below the critical window* (2.6) has been studied in depth in the literature, see e.g. [CSZ17, CSZ20, CZ23, CZ24, CCR25, CD25, CNZ25].

Conjecture 2.4. *There exists a critical constant $\hat{c} \in (0, \infty)$ such that the convergence in distribution (2.22) still holds with both c', c'' replaced by \hat{c} . For this critical choice of $c = \hat{c}$, we have the following convergence in distribution (possibly after lower-order corrections to the scaling in (2.20)): for any $t > 0$ and $\varphi \in C_c(\mathbb{R}^2)$*

$$\int_{\mathbb{R}^2} \varphi(x) \hat{u}_N^{\beta_N, \hat{c}}(t, x) \, dx \xrightarrow[N \rightarrow \infty]{d} \int_{\mathbb{R}^2} \varphi(x) \mathcal{U}_t(dx) \quad (2.23)$$

where $\mathcal{U}_t(dx)$ is a non-trivial (i.e. random) measure-valued process on \mathbb{R}^2 (possibly the same process which appears in Conjecture 1.8).

We conclude with a few remarks.

Remark 2.5 (Stretched exponential scale). The rescaling factor $\rho_N^{\beta, c}$ in (2.21) is truly exponential in N when $\beta > 0$ is kept fixed, while it is slower than exponential for vanishing β . A natural way to interpolate between the critical regime (2.6) and the fixed $\beta > 0$ case is to consider

$$\beta \sim \frac{\hat{\beta}}{\sqrt{R_N}} \quad \text{such that} \quad \sigma^2(\beta) = \frac{\hat{\beta}^2}{R_N} \quad \text{for} \quad \hat{\beta} \in (1, \infty).$$

This choice yields the *stretched exponential scale*

$$\rho_N^{\beta, c} = e^{c(e^\alpha N)^{1-\hat{\beta}^{-2}}}$$

which recovers the pure exponential scale as $\hat{\beta} \rightarrow \infty$.

Remark 2.6 (On the mollified 2D Stochastic Heat Equation). We can also consider the 2D SHE regularized by mollification in space, whose solution is known to converge to the SHF in the critical window, see [Tsa24]. Adapting the techniques of the present paper, we can establish a version of Theorems 2.2 and 2.3 in this setting, which we will do in a forthcoming work.

Remark 2.7 (1D Stochastic Heat Equation). In space dimension $d = 1$, the SHE (2.15)-(2.16) has a well-defined solution $u^\beta(t, x)$ (with no need of regularization). Defining, as in (2.18),

$$u_N^\beta(t, x) := u^\beta(Nt, \sqrt{N}x),$$

one can check from (2.15) that $u_N^\beta(t, x) \stackrel{d}{=} u^{\beta N^{1/4}}(t, x)$. It follows that there is a *critical regime* $\beta \sim \hat{\beta} N^{-1/4}$ which leaves u_N^β invariant in law.

In the *supercritical regime* $\beta \gg N^{-1/4}$ we expect the same result as in Theorem 2.3 for an *exponentially rescaled solution* $\hat{u}_N^{\beta, c}$, defined as in (2.20)-(2.21) with f_β replaced by β^4 :

$$\hat{u}_N^{\beta, c}(t, x) := u^\beta(Nt, \rho_{Nt}^{\beta, c} \sqrt{Nt} x) \quad \text{with} \quad \rho_{Nt}^{\beta, c} := e^{c Nt \beta^4}, \quad c \in (0, \infty).$$

We can give a heuristic explanation for the need of an exponential rescaling as follows. Let us assume the conjectured convergence of the KPZ solution $\log u^\beta(t, x)$ to the Airy process $\mathcal{A}_1(\cdot)$, which we may write as follows (we set $\beta = 1$ for simplicity): for some constants $F, a, b > 0$

$$u^{\beta=1}(t, x) \stackrel{d}{\approx} \exp(-Ft + at^{1/3} \mathcal{A}_1(\frac{x}{bt^{2/3}}) + o(t^{1/3})) \quad \text{as } t \rightarrow \infty. \quad (2.24)$$

The Airy process has a light tail $\mathbb{P}(\mathcal{A}_1(z) > u) \approx \exp(-cu^{3/2})$ and good spatial mixing properties, which yield slow spatially growing extrema: $\max_{|z| \leq R} \mathcal{A}_1(z) \approx (\log R)^{2/3}$. This implies that we need *exponentially large* $|x| \approx \exp(Ct)$ for the second term $t^{1/3} \mathcal{A}_1(\frac{x}{bt^{2/3}})$ in (2.24) to overcome the leading term $-Ft$, in order to prevent $u^{\beta=1}(t, x)$ from vanishing as $t \rightarrow \infty$.

2.3. Further results for directed polymers. In the space dimension $d = 2$, the point-to-plane partition function $Z_N^{\beta,\omega} := Z_N^{\beta,\omega}(0)$ converges a.s. to 0 as $N \rightarrow \infty$ for any fixed disorder strength $\beta > 0$, and it does so exponentially fast, as shown in [Lac10]. Its exponential decay rate to 0 is called (up to a sign) the free energy (or pressure) and it is defined as

$$\mathbf{F}(\beta) := \lim_{N \rightarrow \infty} \frac{1}{N} \log Z_N^{\beta,\omega} = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E}[\log Z_N^{\beta,\omega}] \in (-\infty, 0], \quad (2.25)$$

where the limit is known to exist a.s. and in $L^1(\mathbb{P})$, see e.g. [Com17, Thm. 2.1]. We point out that free energy is related to some localization properties of the polymer, see e.g. [CH06, CSY03].

It was shown in [Lac10] that $\mathbf{F}(\beta) < 0$ for any $\beta > 0$ with some explicit bounds; a few years later, [BL17] refined the bounds and showed that

$$\mathbf{F}(\beta) = -\exp\left(-\left(1 + o(1)\right)\frac{\pi}{\beta^2}\right) \quad \text{as } \beta \downarrow 0.$$

Our next result substantially improves these bounds: we identify the *exact exponential decay rate* as $\pi/\sigma^2(\beta)$, rather than simply* π/β^2 , and we “bring the $o(1)$ out of the exponential”.

Theorem 2.8 (Improved free energy bounds). *There are constants $c, c' \in (0, \infty)$ such that*

$$\forall \beta \in (0, 1): \quad -\frac{c'}{\sigma^2(\beta)^4} \exp\left(-\frac{\pi}{\sigma^2(\beta)}\right) \leq \mathbf{F}(\beta) \leq -c \exp\left(-\frac{\pi}{\sigma^2(\beta)}\right) \quad (2.26)$$

where we recall that $\sigma^2(\beta) = e^{\lambda(2\beta) - 2\lambda(\beta)} - 1$.

The upper bound in (2.26) is the main novelty: we deduce it from Theorem 2.2, more precisely from the upper bound in (2.13). We refer to Section 7 for the proof. The lower bound in (2.26) follows closely the strategy of [BL17, §4] based on super-additivity and concentration arguments for $\log Z_N^{\beta,\omega}$; a few new estimates are needed, see Appendix B for details.

Remark 2.9. We do not expect either bound in (2.26) to be optimal, but the upper bound should be quite sharp (the prefactor $\sigma^2(\beta)^{-4}$ is due to a limitation of the current techniques). As discussed in Appendix B, the precise asymptotic behavior might be

$$\mathbf{F}(\beta) \sim -c \log\left(\frac{1}{\sigma^2(\beta)}\right) e^{-\frac{\pi}{\sigma^2(\beta)}} \quad \text{as } \beta \downarrow 0.$$

Our last result concerns *variance estimates* for the directed polymer partition function, which will be used in Section 3 to prove the lower bound in (2.13). Let us focus on the initial condition given by the uniform distribution on the discrete ball of radius $\rho\sqrt{N}$ for some $\rho > 0$, namely

$$\mathcal{U}_{\rho\sqrt{N}}^{\text{disc}}(x) := \frac{1}{|B(0, \rho\sqrt{N}) \cap \mathbb{Z}^2|} \mathbf{1}_{B(0, \rho\sqrt{N}) \cap \mathbb{Z}^2}(x). \quad (2.27)$$

The proof is given in Section 5.3.

Proposition 2.10 (Variance estimates for directed polymers). *Define $\vartheta(N, \beta)$ as in (2.11). There is a constant $c_3 > 0$ such that, uniformly over $N \in \mathbb{N}$, $\beta \in (0, 1)$ and $\rho \in (0, \infty)$, we have*

$$\text{Var}[Z_N^{\beta,\omega}(\mathcal{U}_{\rho\sqrt{N}}^{\text{disc}})] \leq c_3 \frac{\exp(c_3 e^{\vartheta(N, \beta)})}{\rho^2}. \quad (2.28)$$

This can be sharpened for $N \rightarrow \infty$, $\beta \downarrow 0$ such that $\vartheta(N, \beta) \rightarrow \infty$: in this regime, there is $o(1) \rightarrow 0$ such that, uniformly over $\rho \in (0, \infty)$, we have

$$\text{Var}[Z_N^{\beta,\omega}(\mathcal{U}_{\rho\sqrt{N}}^{\text{disc}})] \leq \frac{\exp((e^{-\gamma} + o(1)) e^{\vartheta(N, \beta)})}{\rho^2} \quad (2.29)$$

where $\gamma := -\int_0^\infty e^{-x} \log x \, dx \simeq 0.577$ is the Euler–Mascheroni constant.

*Note that $\lambda(\beta) = \frac{1}{2}\beta^2 + \frac{\kappa_3}{3!}\beta^3 + \frac{\kappa_4}{4!}\beta^4 + O(\beta^5)$ as $\beta \downarrow 0$, where κ_3, κ_4 are the third and fourth cumulants of the disorder distribution. It follows that we have $e^{-\pi/\beta^2} \sim (cst.) e^{-\pi/\sigma^2(\beta)}$ only when $\kappa_3 = 0$.

Remark 2.11. We believe the upper bound (2.29) to be sharp. In other words, for (say) $\rho = 1$, one should also be able to prove that as $N \rightarrow \infty$, $\beta \downarrow 0$ with $\vartheta(N, \beta) \rightarrow \infty$ we have

$$\text{Var}[Z_N^{\beta_N, \omega}(\mathcal{U}_{\sqrt{N}}^{\text{disc}})] \geq \exp((e^{-\gamma} + o(1))e^{\vartheta(N, \beta)}).$$

Since this lower bound is not useful to us, we have decided to skip its proof. One could also try to improve these estimates by “bringing the $o(1)$ out of the exponential”.

3. STRATEGY OF THE PROOF OF THEOREM 2.2

In this section, we present the strategy of the proof of Theorem 2.2. We first discuss the upper bound in (2.13) which is the core of the paper, see Section 3.1. We then prove the lower bound exploiting the variance estimates in Proposition 2.10, see Section 3.2.

3.1. Strategy of the upper bound in Theorem 2.2. Our proof for the upper bound in Theorem 2.2 follows three main steps.

- (1) First, we formulate a key result, Proposition 3.1, which is suboptimal (with respect to Theorem 2.2) in several senses: (i) the starting point is on the basic diffusive scale \sqrt{N} without the factor $e^{c_0 e^\vartheta}$ in (2.13); (ii) the parameter $\vartheta \in [3, \infty)$ is fixed so Proposition 3.1 is proved only in the critical window; (iii) the bound we obtain is polynomial in ϑ instead of doubly-exponential.
- (2) Second, we use *coarse-graining techniques* to extend the previous bound to any $\beta > 0$ (small) and $N \in \mathbb{N}$, meanwhile also obtaining an exponential decay in N , see Proposition 3.3. By now, this is a well-established method, which dates back to [Lac10] (in the context of directed polymers); technical details are postponed to Appendix A.
- (3) Third, we use a change of scale argument to quantify the effect of enlarging the scale of the starting point: this leads to Proposition 3.6, which completes the proof of Theorem 2.2.

As a consequence, the upper bound in Theorem 2.2 is reduced to the key Proposition 3.1, whose proof is given afterwards in Section 4. Let us now give some details on the steps outlined above.

Step 1: Key (suboptimal) result. We first state a weaker version of Theorem 2.2 formulated in the next key proposition. This is actually the core of the paper; we will present the key ideas of its proof in Section 4.

Proposition 3.1 (Key proposition). *There is a universal constant $C > 0$ such that, for any given $\vartheta \in [3, \infty)$, if we consider any $\beta_N = \beta_N(\vartheta)$ in the critical regime (2.10), we have*

$$\limsup_{N \rightarrow \infty} \sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{N})} \mathbb{E}[Z_N^{\beta_N, \omega}(f) \wedge 1] \leq \frac{C}{\vartheta}. \quad (3.1)$$

We view Proposition 3.1 as an estimate in the (upper) critical window, *i.e.* we will use it for ϑ large but fixed, in order to apply a “finite-volume criterion” in the next step. Let us now deduce from (3.1) a corresponding bound on a fractional moment of $Z_N^{\beta_N, \omega}(f)$:

$$\limsup_{N \rightarrow \infty} \sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{N})} \mathbb{E}[Z_N^{\beta_N, \omega}(f)^{1/2}] \leq \frac{\sqrt{2C}}{\sqrt{\vartheta}}, \quad (3.2)$$

thanks to the following general result.

Lemma 3.2. *For any random variable $Z \geq 0$ with $\mathbb{E}[Z] = 1$ and any $\gamma \in (0, 1)$ we have*

$$\mathbb{E}[Z \wedge 1] \leq \mathbb{E}[Z^\gamma] \leq 2^\gamma \mathbb{E}[Z \wedge 1]^{\gamma \wedge (1-\gamma)}.$$

In particular

$$\mathbb{E}[Z \wedge 1] \leq \mathbb{E}[Z^{1/2}] \leq \sqrt{2} \mathbb{E}[Z \wedge 1]^{1/2}.$$

Proof. The first inequality simply uses that $x \leq x^\gamma$ for $x \in [0, 1]$ to get that $\mathbb{E}[Z \wedge 1] \leq \mathbb{E}[(Z \wedge 1)^\gamma] \leq \mathbb{E}[Z^\gamma]$. For the second one, consider first $\gamma \leq \frac{1}{2}$. By $Z = (Z \wedge 1)(Z \vee 1)$ we have

$$\mathbb{E}[Z^\gamma] = \mathbb{E}[(Z \wedge 1)^\gamma (Z \vee 1)^\gamma] \leq \mathbb{E}[Z \wedge 1]^\gamma \mathbb{E}[(Z \vee 1)^{\frac{\gamma}{1-\gamma}}]^{1-\gamma},$$

by Hölder's inequality. Since $\frac{\gamma}{1-\gamma} \leq 1$ for $\gamma \leq \frac{1}{2}$ we have by Jensen $\mathbb{E}[(Z \vee 1)^{\frac{\gamma}{1-\gamma}}] \leq \mathbb{E}[Z \vee 1]^{\frac{\gamma}{1-\gamma}}$ and note that $\mathbb{E}[Z \vee 1] \leq \mathbb{E}[Z] + 1 = 2$.

We next consider $\gamma > \frac{1}{2}$: since $\frac{\gamma}{1-\gamma} > 1$, by $x^{\frac{\gamma}{1-\gamma}} \leq x$ for $x \in [0, 1]$ we can bound, again by Hölder's inequality,

$$\mathbb{E}[Z^\gamma] = \mathbb{E}[(Z \wedge 1)^\gamma (Z \vee 1)^\gamma] \leq \mathbb{E}[(Z \wedge 1)^{\frac{\gamma}{1-\gamma}}]^{1-\gamma} \mathbb{E}[Z \vee 1]^\gamma \leq \mathbb{E}[Z \wedge 1]^{1-\gamma} \mathbb{E}[Z \vee 1]^\gamma,$$

which completes the proof since $\mathbb{E}[Z \vee 1] \leq 2$. \square

Step 2: Finite-volume criterion via a coarse-graining procedure. Let us now upgrade the result from the key Proposition 3.1, more precisely the fractional moment version (3.2), by extending it to arbitrary $N \in \mathbb{N}$ and $\beta > 0$ small (*i.e.* not necessarily in the critical window) and improving the bound with an exponential decay in N .

Recalling (2.11) and (2.5), for any $\beta > 0$ and $\vartheta \in \mathbb{R}$ we define $N_\beta(\vartheta) \in \mathbb{N}$ which inverts asymptotically the relation (2.10):

$$N_\beta(\vartheta) := \lfloor e^{-\alpha} e^\vartheta e^{\frac{\pi}{\sigma^2(\beta)}} \rfloor = (1 + o(1)) e^{-\alpha} e^\vartheta e^{\frac{\pi}{\sigma^2(\beta)}} \quad \text{as } \beta \downarrow 0. \quad (3.3)$$

Proposition 3.3 (Improved bound). *There exist constants $\beta_0, \hat{\vartheta} \in (0, \infty)$ such that the following holds: defining $\hat{N}_\beta := N_\beta(\hat{\vartheta})$ by (3.3), we have*

$$\forall \beta \in (0, \beta_0), \forall N \in \mathbb{N}: \quad \sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{\hat{N}_\beta})} \mathbb{E}[Z_N^{\beta, \omega}(f)^{1/2}] \leq 3e^{-N/\hat{N}_\beta}. \quad (3.4)$$

Proof. We first rewrite the relation (3.2) inverting the roles of β and N : if we let $\beta \downarrow 0$, we have that β satisfies (2.10) with $N = N_\beta(\vartheta) \rightarrow \infty$ from (3.3). Then (3.2) can be rewritten as follows:

$$\limsup_{\beta \downarrow 0} \sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{N_\beta(\vartheta)})} \mathbb{E}[Z_{N_\beta(\vartheta)}^{\beta, \omega}(f)^{1/2}] \leq \frac{\sqrt{2C}}{\sqrt{\vartheta}}.$$

In particular, given any $\vartheta \geq 1$, we can fix a suitable $\tilde{\beta}(\vartheta) > 0$ small enough such that (say)

$$\forall \beta \in (0, \tilde{\beta}(\vartheta)): \quad \sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{N_\beta(\vartheta)})} \mathbb{E}[Z_{N_\beta(\vartheta)}^{\beta, \omega}(f)^{1/2}] \leq \frac{2\sqrt{C}}{\sqrt{\vartheta}}. \quad (3.5)$$

We next improve this estimate allowing $N \in \mathbb{N}$ to be arbitrary. The core idea is the following *coarse-graining* result, which gives a finite-size volume criterion for the exponential decay of the partition function: it shows that if the fractional moment is small at some scale L , then it starts decreasing exponentially in N/L . This result is somehow classical in the literature, but we provide a self-contained proof in Appendix A below. We recall that $\mathcal{M}_1^{\text{disc}}(r)$ is defined in (2.12), where we replace for convenience $|\cdot|$ with $|\cdot|_\infty$.

Proposition 3.4 (Coarse-graining). *If there exist $L \in \mathbb{N}, \beta > 0$ such that*

$$\sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{L})} \mathbb{E}[Z_L^{\beta, \omega}(f)^{1/2}] \leq \frac{1}{113},$$

then for all $N \in \mathbb{N}$, we have

$$\sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{L})} \mathbb{E}[Z_N^{\beta, \omega}(f)^{1/2}] \leq 3e^{-N/L}. \quad (3.6)$$

Remark 3.5. It is enough to prove (3.6) for $N \geq L$, since for $N < L$ we have $\mathbb{E}[Z_N^{\beta,\omega}(f)^{1/2}] \leq \mathbb{E}[Z_N^{\beta,\omega}(f)]^{1/2} = 1$. Also, let us stress that the constant $\frac{1}{113}$ depends on the distribution of the random walk (we have simply taken a number that works for the simple random walk).

Recalling (3.5), we now fix $\hat{\vartheta} = (2 \cdot 113)^2 C$ so that $\frac{2\sqrt{C}}{\sqrt{\hat{\vartheta}}} \leq \frac{1}{113}$. If we correspondingly define $\beta_0 := \tilde{\beta}(\hat{\vartheta}) > 0$ and $\hat{N}_\beta := N_\beta(\hat{\vartheta})$, then (3.5) yields

$$\forall \beta \in (0, \beta_0): \quad \sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{\hat{N}_\beta})} \mathbb{E}[Z_{\hat{N}_\beta}^{\beta,\omega}(f)^{1/2}] \leq \frac{1}{113}.$$

It only remains to apply Proposition 3.4 with $L = \hat{N}_\beta$ to complete the proof of Proposition 3.3. \square

Step 3. The change of scale argument. We finally show that the scale of the starting point in the bound (3.4) can be enlarged, to get the following result.

Proposition 3.6 (Large scale bound). *In the same setting as Proposition 3.3, we have*

$$\forall \beta \in (0, \beta_0), \quad \forall N \in \mathbb{N}: \quad \sup_{f \in \mathcal{M}_1^{\text{disc}}(e^{\frac{1}{2}N/\hat{N}_\beta}\sqrt{N})} \mathbb{E}[Z_N^{\beta,\omega}(f)^{1/2}] \leq 7e^{-\frac{1}{3}N/\hat{N}_\beta}. \quad (3.7)$$

This completes the proof of Theorem 2.2. Indeed, if we define $\vartheta = \vartheta(N, \beta)$ by (2.11), recalling (2.14) and $\hat{N}_\beta = N_\beta(\hat{\vartheta})$ from Proposition 3.3, with $N_\beta(\cdot)$ defined in (3.3), we can bound

$$\frac{N}{\hat{N}_\beta} \geq N e^\alpha e^{-\hat{\vartheta}} e^{-\frac{\pi}{\sigma^2(\beta)}} = e^{\alpha - \alpha N} e^{\vartheta(N, \beta) - \hat{\vartheta}} \geq c e^{\vartheta(N, \beta) - \hat{\vartheta}} \quad \text{with } c := \inf_{N \in \mathbb{N}} e^{\alpha - \alpha N} > 0.$$

Plugged into (3.7) and using Lemma 3.2, this yields (2.13) with $c_0 = \frac{c}{2}e^{-\hat{\vartheta}}$ and $c_2 = \min\{\frac{c}{3}e^{-\hat{\vartheta}}, \frac{1}{7}\}$.

Proof of Proposition 3.6. The key tool is the following general result, which allows to control fractional moments with starting points at two different scales.

Lemma 3.7 (Changing scales). *For any $1 \leq A \leq B$ and any $\gamma \in (0, 1)$, we have*

$$\sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{B})} \mathbb{E}[Z_N^{\beta,\omega}(f)^\gamma] \leq \left(4 \frac{B}{A}\right)^{1-\gamma} \sup_{g \in \mathcal{M}_1^{\text{disc}}(\sqrt{A})} \mathbb{E}[Z_N^{\beta,\omega}(g)^\gamma].$$

Proof. We can include the L^∞ ball of radius \sqrt{B} in the union of K disjoint L^∞ balls $(B_i)_{1 \leq i \leq K}$ of radius \sqrt{A} (we can estimate $K \leq 4\frac{B}{A}$). For any probability density $f \in \mathcal{M}_1^{\text{disc}}(\sqrt{B})$, we can decompose it as $f = \sum_{i=1}^K \alpha_i g_i$ where $\alpha_i := \int f \mathbf{1}_{B_i}$ and $g_i := \frac{1}{\alpha_i} f \mathbf{1}_{B_i}$ is just f conditioned on B_i . This way, we may write

$$Z_N^{\beta,\omega}(f) = \sum_{i=1}^K \alpha_i Z_N^{\beta,\omega}(g_i). \quad (3.8)$$

For $\gamma \in (0, 1)$, using the subadditive inequality $(\sum_i z_i)^\gamma \leq \sum_i z_i^\gamma$ for $z_i \geq 0$, we obtain that

$$\mathbb{E}[Z_N^{\beta,\omega}(f)^\gamma] \leq \sum_{i=1}^K \alpha_i^\gamma \mathbb{E}[Z_N^{\beta,\omega}(g_i)^\gamma] \leq \sup_{g \in \mathcal{M}_1^{\text{disc}}(\sqrt{A})} \mathbb{E}[Z_N^{\beta,\omega}(g)^\gamma] \sum_{i=1}^K \alpha_i^\gamma,$$

using also translation invariance. Now, using Hölder's inequality, we can bound $\sum_{i=1}^K \alpha_i^\gamma \leq K^{1-\gamma}$, so recalling that $K \leq 4B/A$ this concludes the proof. \square

Thanks to Lemma 3.7, we deduce from Proposition 3.3 that for all $\beta \in (0, \beta_0)$ and $N \in \mathbb{N}$

$$\sup_{f \in \mathcal{M}_1^{\text{disc}}(e^{\frac{1}{2}N/\hat{N}_\beta}\sqrt{N})} \mathbb{E}[Z_N^{\beta,\omega}(f)^{1/2}] \leq 2 \left(\frac{N}{\hat{N}_\beta} e^{N/\hat{N}_\beta}\right)^{1/2} 3 e^{-N/\hat{N}_\beta} = 6 \left(\frac{N}{\hat{N}_\beta}\right)^{1/2} e^{-\frac{1}{2}N/\hat{N}_\beta}.$$

This completes the proof of (3.7), since $6\sqrt{x} e^{-\frac{1}{2}x} \leq 7e^{-\frac{1}{3}x}$ for $x \geq 0$. \square

3.2. Proof of the lower bound in Theorem 2.2. We consider the following inequality, in the spirit of Paley–Zygmund: for any random variable $Z \geq 0$

$$\mathbb{E}[Z \wedge 1] \geq \frac{\mathbb{E}[Z]^2}{1 + \mathbb{E}[Z^2]} = \frac{\mathbb{E}[Z]^2}{1 + \mathbb{E}[Z]^2 + \text{Var}[Z]}. \quad (3.9)$$

The proof is simple: starting from the identity $Z = (Z \wedge 1)(Z \vee 1)$, we get by Cauchy–Schwarz

$$\mathbb{E}[Z]^2 \leq \mathbb{E}[(Z \wedge 1)^2] \mathbb{E}[(Z \vee 1)^2] = \mathbb{E}[Z^2 \wedge 1] \mathbb{E}[Z^2 \vee 1] \leq \mathbb{E}[Z \wedge 1] (1 + \mathbb{E}[Z^2]).$$

To prove the lower bound in (2.13), it suffices to apply (3.9) to $Z = Z_N^{\beta, \omega}(f)$ with $f = \mathcal{U}_{\sqrt{N}}^{\text{disc}}$, noting that $\mathbb{E}[Z] = 1$ and plugging in the estimate (2.28) from Proposition 2.10 (with $\rho = 1$). Overall, we see that the lower bound in (2.13) holds with $c_1 = c_3 + 2$. \square

4. PROOF OF PROPOSITION 3.1

We have shown in Section 3 how to reduce Theorem 2.2 to the weaker key Proposition 3.1. This section is devoted to the proof of Proposition 3.1: we outline the strategy and formulate general propositions, proved in the following subsections.

4.1. Size bias. We first introduce the key notion of *size-biased measure*.

Definition 4.1 (Size-biased measure). Denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space on which the disorder ω is defined, and recall the point-to-plane partition function $Z_N^{\beta, \omega}(x)$ from (2.2). For $x \in \mathbb{Z}^2$, we define the *size-biased measure* with starting point x by $d\tilde{\mathbb{P}}_x = Z_N^{\beta, \omega}(x) d\mathbb{P}$, *i.e.*

$$\tilde{\mathbb{P}}_x(A) = \tilde{\mathbb{P}}_{x, N}^\beta(A) := \mathbb{E}[\mathbf{1}_A Z_N^{\beta, \omega}(x)] \quad \text{for any } A \in \mathcal{F}. \quad (4.1)$$

More generally, the *size-biased measure with initial condition f* (a mass function) is

$$\tilde{\mathbb{P}}_f(\cdot) = \tilde{\mathbb{P}}_{f, N}^\beta(\cdot) := \mathbb{E}[\mathbf{1}_{(\cdot)} Z_N^{\beta, \omega}(f)].$$

Let us note that there is a nice interpretation of the size-biased measure $\tilde{\mathbb{P}}_x$. Indeed, simply using Fubini’s theorem and the definition (2.2)–(2.3) of $Z_N^{\beta, \omega}(f)$ we get that

$$\tilde{\mathbb{P}}_f(A) = \mathbf{E}_f \left[\mathbb{E} \left[e^{\sum_{n=1}^N \beta \omega_{n, s_n} - \lambda(\beta)} \mathbf{1}_A \right] \right] = \mathbf{E}_f [\tilde{\mathbb{P}}^{(S)}(A)] = \mathbf{E}_f [\tilde{\mathbb{P}}_N^{\beta, (S)}(A)], \quad (4.2)$$

where we have denoted $\mathbf{P}_f = \sum_{x \in \mathbb{Z}^2} f(x) \mathbf{P}_x$ the law of the simple random walk with initial distribution f , and $d\tilde{\mathbb{P}}_N^{\beta, (s)} = \prod_{n=1}^N e^{\beta \omega_{n, s_n} - \lambda(\beta)} d\mathbb{P}$ is the law of an environment *tilted along the path s* . Hence the size-biased measure $\tilde{\mathbb{P}}_f$ can be understood as a two-step procedure: first, draw a simple random walk path S with starting distribution f ; then tilt the environment (up to time N) by $e^{\beta \omega - \lambda(\beta)}$ along the path S . Such interpretation may be useful to build some intuition when comparing \mathbb{P} and $\tilde{\mathbb{P}}_f$, but in practice we will not use it in our proofs.

Remark 4.2 (Size bias and total variation distance). Fix any $Z \geq 0$ with $\mathbb{E}[Z] = 1$ and consider the size-biased measure $d\tilde{\mathbb{P}} = Z d\mathbb{P}$. For any event $A \in \mathcal{F}$ we have $\mathbb{E}[Z \wedge 1] \leq \mathbb{P}(A) + \tilde{\mathbb{P}}(A^c)$ (just bound $Z \wedge 1 \leq 1$ on A and $Z \wedge 1 \leq Z$ on A^c) and the inequality is sharp, since we can take $A = \{Z \geq 1\}$ to get equality. It follows that

$$\mathbb{E}[Z \wedge 1] = \inf_{A \in \mathcal{F}} \{\mathbb{P}(A) + \tilde{\mathbb{P}}(A^c)\} = 1 - d_{\text{TV}}(\mathbb{P}, \tilde{\mathbb{P}}),$$

where d_{TV} is the total variation distance between probability measures. Therefore, *showing that $\mathbb{E}[Z \wedge 1]$ is small corresponds to showing that $d_{\text{TV}}(\mathbb{P}, \tilde{\mathbb{P}})$ is close to 1.*

Remark 4.3 (Divergence under the size-biased probability). Since $\mathbb{E}[Z \wedge 1] = \tilde{\mathbb{E}}[\frac{1}{Z} \wedge 1]$, Theorem 2.2 can also be interpreted as a result on the divergence of $Z_N^{\beta, \omega}(f)$ under the size-biased measure $\tilde{\mathbb{P}}_f$, more precisely giving a rate of convergence to 0 for $1/Z_N^{\beta, \omega}(f)$ under $\tilde{\mathbb{P}}_f$.

Remark 4.4 (Anomalous path detection). In view of Remark 4.2, showing that $\mathbb{E}[Z_N^{\beta,\omega}(f) \wedge 1] \rightarrow 0$ amounts to a statistical problem of being able to find an appropriate event $A = A_N$ to discriminate between two environment distributions: \mathbb{P} and $\tilde{\mathbb{P}}_{f,N}^\beta$. Such “anomalous path detection problems” have been investigated (mostly in dimension 1), see for instance in [ACCHZ08, CZ18], or [ABBDL10] for a discussion on similar hypothesis testing problems.

4.2. Strategy of the proof of Proposition 3.1. We need to give an upper bound on $\mathbb{E}[Z_N^{\beta,\omega}(f) \wedge 1]$ inside the critical window, *i.e.* for a fixed $\vartheta \in [3, \infty)$, and for diffusive initial conditions f , *i.e.* supported in a ball of radius \sqrt{N} .

We combine a *change of scale* argument, that we use to reduce (again) the initial diffusive scale \sqrt{N} to a smaller scale $\sqrt{\tilde{N}}$, with a *change of measure* argument. The first step is the following result, proved in Section 4.4.

Proposition 4.5 (Change of scale and measure). *For any $\beta > 0$, for any $N, \tilde{N} \in \mathbb{N}$ with $\tilde{N} \leq N$, we can bound*

$$\sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{N})} \mathbb{E}[Z_N^{\beta,\omega}(f) \wedge 1] \leq 8 \sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{\tilde{N}})} \mathbb{E}\left[Z_N^{\beta,\omega}(f) \wedge \frac{N}{\tilde{N}}\right]. \quad (4.3)$$

Additionally, for any $f \in \mathcal{M}_1^{\text{disc}}(\sqrt{\tilde{N}})$ and any event A_N (that may depend on f), recalling (4.1) we have

$$\mathbb{E}\left[Z_N^{\beta,\omega}(f) \wedge \frac{N}{\tilde{N}}\right] \leq \frac{N}{\tilde{N}} \mathbb{P}(A_N) + \tilde{\mathbb{P}}_f(A_N^c). \quad (4.4)$$

For the bounds (4.3)-(4.4) to be useful, we must find an event $A_N = A_N(f)$ for which both $\mathbb{P}(A_N)$ and $\tilde{\mathbb{P}}_f(A_N^c)$ are small, *i.e.* $A_N(f)$ is atypical under \mathbb{P} but typical under the size-biased measure $\tilde{\mathbb{P}}_f$, uniformly for $f \in \mathcal{M}_1^{\text{disc}}(\sqrt{\tilde{N}})$.

For this, we refine ideas developed for point-to-plane partition functions (see e.g. the proof of Theorem 2.9 in [JL25]): this leads to the following result, which states that such an event A_N may be found.

Proposition 4.6 (Bounds for the event A_N). *Fix $1 \leq \eta < \vartheta < \infty$ with $\vartheta - \eta \geq 1$. For $N \in \mathbb{N}$ set $\tilde{N} = e^{-\eta}N$, and consider $\beta = \beta_N(\vartheta)$ in the critical regime (2.6), or equivalently (2.10) (note that $\tilde{\mathbb{P}}_f$ depends on β , see (4.1)). Then, for any $f \in \mathcal{M}_1^{\text{disc}}(\sqrt{\tilde{N}})$ we can find for each $N \in \mathbb{N}$ an event $A_N = A_N(f) \in \mathcal{F}$ such that*

$$\limsup_{N \rightarrow \infty} \sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{\tilde{N}})} \mathbb{P}(A_N(f)) \leq C_1 \frac{\vartheta - \eta}{\eta} e^{-(\vartheta - \eta)}, \quad (4.5)$$

$$\limsup_{N \rightarrow \infty} \sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{\tilde{N}})} \tilde{\mathbb{P}}_f(A_N^c(f)) \leq \frac{C_2}{\eta}, \quad (4.6)$$

where $C_1, C_2 > 0$ are universal constants.

The reason why the bounds (4.5) and (4.6) have the specified dependence on η, ϑ will be clear below (they are determined by the mean and variance of a suitable random variable X). For the moment, it suffices to note that plugging these bounds in (4.3)-(4.4) we obtain

$$\limsup_{N \rightarrow \infty} \sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{N})} \mathbb{E}[Z_N^{\beta,\omega}(f) \wedge 1] \leq 8C_1 \frac{\vartheta - \eta}{\eta} e^{2\eta - \vartheta} + \frac{8C_2}{\eta}.$$

Then, if we choose $\eta = \vartheta/3$, this concludes the proof of Proposition 3.1.

To prove Proposition 4.6, we need to find $A_N = A_N(f)$ which is atypical under \mathbb{P} , but which becomes typical under the size-biased measure $\tilde{\mathbb{P}}_f$. One could in theory take $A_N(f) = \{Z_N^{\beta,\omega}(f) > \varepsilon\}$ for some $\varepsilon > 0$ small enough: this would easily yield $\tilde{\mathbb{P}}_f(A_N^c(f)) \leq \varepsilon$ by definition

of $\tilde{\mathbb{P}}_f$, but the difficult part remains to show that $\mathbb{P}(A_N(f)) = \mathbb{P}(Z_N^{\beta N}(f) > \varepsilon)$ is small, so this does not simplify the original problem.

A more manageable solution is to find a *simpler random variable* $X = X(f)$ which acts as a proxy for $Z_N^{\beta, \omega}(f)$, for which we are able to estimate the expectation and variance under $\mathbb{P}, \tilde{\mathbb{P}}_f$, uniformly in $f \in \mathcal{M}_1^{\text{disc}}(\sqrt{N})$. We clarify this strategy in the following lemma.

Lemma 4.7 (Choice of the event A_N). *Consider some random variable $X(f) = X_N(f)$ such that*

$$\mathbb{E}[X(f)] = 0 \quad \text{and} \quad \tilde{\mathbb{E}}_f[X(f)] > 0$$

and define the event

$$A_N = A_N(f) = \left\{ X(f) \geq \frac{1}{2} \tilde{\mathbb{E}}_f[X(f)] \right\}.$$

Then, we get that

$$\mathbb{P}(A_N(f)) \leq 4 \frac{\text{Var}[X(f)]}{\tilde{\mathbb{E}}_f[X(f)]^2}, \quad \tilde{\mathbb{P}}_f(A_N^c(f)) \leq 4 \frac{\tilde{\text{Var}}_f[X(f)]}{\tilde{\mathbb{E}}_f[X(f)]^2}.$$

The proof of the lemma follows directly by Chebyshev's inequality, applied with threshold $\frac{1}{2} \tilde{\mathbb{E}}_f[X(f)]$ to $X(f)$ under \mathbb{P} and to $X(f) - \tilde{\mathbb{E}}_f[X(f)]$ under $\tilde{\mathbb{P}}_f$. It therefore only remains to find a suitable $X = X(f)$ such that $\text{Var}[X(f)], \tilde{\text{Var}}_f[X(f)] \ll \tilde{\mathbb{E}}_f[X(f)]^2$ uniformly for $f \in \mathcal{M}_1^{\text{disc}}(\sqrt{N})$.

The choice of $X = X(f)$ is the most delicate point in the strategy and the main novelty of our proof. We discuss this issue in Section 4.3, arriving at the explicit choice of X defined in the next (4.14), which may be described as the first (linear) term in a coarse-grained chaos expansion of the partition function over time intervals of length \tilde{N} .

We finally state our main estimates on $\text{Var}[X(f)], \tilde{\mathbb{E}}_f[X(f)], \tilde{\text{Var}}_f[X(f)]$ that will be proved in the next sections. The first two lemmas follow from second moment calculations and are proven in Section 5. The last estimate is more difficult and will be proven in Section 6.

Lemma 4.8 (Variance bound). *Let the assumptions of Proposition 4.6 prevail. Defining $X(f) = X_N(f)$ by (4.14) below, we have $\mathbb{E}[X(f)] = 0$ and*

$$\limsup_{N \rightarrow \infty} \sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{N})} \text{Var}[X(f)] \leq C \frac{\eta}{\vartheta - \eta} e^{\vartheta - \eta},$$

where $C < \infty$ is a universal constant.

Lemma 4.9 (Size-biased mean bound). *Let the assumptions of Proposition 4.6 prevail. Defining $X(f) = X_N(f)$ by (4.14) below, we have*

$$\liminf_{N \rightarrow \infty} \inf_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{N})} \tilde{\mathbb{E}}_f[X(f)] \geq c \frac{\eta}{\vartheta - \eta} e^{\vartheta - \eta},$$

where $c > 0$ is a universal constant.

Proposition 4.10 (Size-biased variance bound). *Let the assumptions of Proposition 4.6 prevail. Defining $X(f) = X_N(f)$ by (4.14) below, we have*

$$\limsup_{N \rightarrow \infty} \sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{N})} \tilde{\text{Var}}_f[X(f)] \leq C' \eta \left(\frac{1}{\vartheta - \eta} e^{\vartheta - \eta} \right)^2,$$

where $C' > 0$ is a universal constant.

Together with Lemma 4.7, these estimates readily show that

$$\limsup_{N \rightarrow \infty} \sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{N})} \mathbb{P}(A_N(f)) \leq Cc^{-2} \frac{\vartheta - \eta}{\eta} e^{-(\vartheta - \eta)},$$

$$\text{and } \limsup_{N \rightarrow \infty} \sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{N})} \tilde{\mathbb{P}}_f(A_N^c(f)) \leq C'c^{-2} \frac{1}{\eta},$$

which are the bounds announced in Proposition 4.6. (Of course, the reason why we stated the bounds in the precise form (4.5) and (4.6) was dictated by the mean and variance of X .)

Remark 4.11 (Size bias, reprise). Another approach to bound $\tilde{\mathbb{P}}_f(A_N^c(f))$ would be to use the size-biased representation (4.2), introducing some well-chosen (random walk) event $B \in \sigma\{S_n, n \leq N\}$ and then writing $\tilde{\mathbb{P}}_f(A_N^c(f)) \leq \mathbf{E}_f[\tilde{\mathbb{P}}_N^{\beta, (S)}(A_N^c(f)) \mathbf{1}_B] + \mathbf{P}_f(B^c)$. This is what is usually done in this setting, see for instance [BL17, §3] or [JL25, §6.2]. The advantage of this idea is that, once one has reduced to work on the event B , it possibly makes it easier to control $\tilde{\mathbb{E}}_N^{\beta, (S)}[X_N(f)]$ and $\tilde{\text{Var}}_N^{\beta, (S)}[X_N(f)]$, and thus $\tilde{\mathbb{P}}_N^{\beta, (S)}(A_N^c(f))$. We will not need such a strategy, since our choice for event $A_N(f)$ will already make the computation of $\tilde{\mathbb{E}}_f[X_N(f)]$, $\tilde{\text{Var}}_f[X_N(f)]$ manageable.

The remainder of this section is devoted to the choice of the proxy X (see Section 4.3) and to the proof of Proposition 4.5 (see Section 4.4).

4.3. Choice of a good proxy for the partition function. We next discuss the *choice of the proxy* $X(f) = X_N(f)$ for the partition function $Z_N^{\beta, \omega}(f)$. Let us introduce some useful notation. For $n, N \in \mathbb{N}$, $x \in \mathbb{Z}^2$, we denote by $q_n(x)$ the simple random walk transition probability, that is

$$q_n(x) = q(n, x) := \mathbf{P}(S_n = x).$$

We will also denote

$$q_n^{(f)}(x) := \mathbf{P}_f(S_n = x) = \sum_{z \in \mathbb{Z}^2} f(z) q_n(x - z). \quad (4.7)$$

A first approach: chaos expansion and L^2 projections. Let us define

$$\xi_{n,x} = \xi_{n,x}^{(\beta)} := e^{\beta\omega(n,x) - \lambda(\beta)} - 1$$

and notice that the $(\xi_{n,x})_{n \in \mathbb{N}, x \in \mathbb{Z}^2}$ are i.i.d. with $\mathbb{E}[\xi_{n,x}] = 0$ and $\mathbb{E}[\xi_{n,x}^2] = e^{\lambda(2\beta) - 2\lambda(\beta)} - 1 =: \sigma^2(\beta)$, see (2.4). Rewriting the partition function as $Z_N^{\beta, \omega}(f) = \mathbf{E}_f[\prod_{n=1}^N (1 + \xi_{n, S_n})]$ and expanding the product, we obtain the following *polynomial chaos expansion*

$$Z_N^{\beta, \omega}(f) = 1 + \sum_{k=1}^N \sum_{1 \leq n_1 < \dots < n_k \leq N} \sum_{x_0 \in \mathbb{Z}^2} f(x_0) \sum_{x_1, \dots, x_k \in \mathbb{Z}^2} \prod_{i=1}^k q_{n_i - n_{i-1}}(x_i - x_{i-1}) \prod_{i=1}^k \xi_{n_i, x_i}, \quad (4.8)$$

where we set by convention $n_0 = 0$. Let us notice that the terms $\prod_{i=1}^k \xi_{n_i, x_i}$ in the above expansion are orthogonal in L^2 . Therefore, one can reinterpret the above as the L^2 decomposition of $Z_N^{\beta, \omega}(f)$ over the linear subspace of L^2 generated by the orthogonal variables

$$\xi(A) = \prod_{z \in A} \xi_z \quad \text{for } A \subset \mathbb{N} \times \mathbb{Z}^2.$$

Then, gathering the space-time points in a subset $A = \{(n_i, x_i) : 1 \leq i \leq k\} \subset \llbracket 1, N \rrbracket \times \mathbb{Z}^2$, the chaos expansion above can be rewritten more compactly as follows:

$$Z_N^{\beta, \omega}(f) = \sum_{A \subset \llbracket 1, N \rrbracket \times \mathbb{Z}^2} q^{(f)}(A) \xi(A), \quad (4.9)$$

where we have set

$$q^{(f)}(A) := \mathbf{P}_f(A \subset \{(i, S_i)\}_{i \geq 1}) = \sum_{x_0 \in \mathbb{Z}^2} f(x_0) \prod_{i=1}^k q_{n_i - n_{i-1}}(x_i - x_{i-1}), \quad (4.10)$$

with $n_0 = 0$ by convention. Note that we have $q^{(f)}(\emptyset) := 1$ by definition; we also denote $q(A) = q^{(\delta_0)}(A)$ for simplicity.

A simple choice for a proxy X_N for $Z_N^{\beta,\omega} = Z_N^{\beta,\omega}(0)$ is to take the first term in the chaos expansion, namely

$$\sum_{n=1}^N \sum_{x \in \mathbb{Z}^2} q_n(x) \xi_{n,x}. \quad (4.11)$$

This corresponds to the L^2 projection of $Z_N^{\beta,\omega}$ on the linear subspace of L^2 generated by the $(\xi_{n,x})$. We refer for instance to [JL25, Section 6] where the functional (4.11) is used to show that the martingale critical point is equal to 0 as soon as $R_N \rightarrow \infty$. In fact, one needs a slightly finer strategy than simply use Chebyshev's inequality to bound $\tilde{\mathbb{P}}(A_N^c)$, but let us not dwell on details here. The method can be pushed to show that $Z_N^{\beta,\omega} \rightarrow 0$ in probability as soon as $\sigma^2(\beta)R_N \rightarrow \infty$. (This is not optimal, since the point-to-plane partition function $Z_N^{\beta,\omega}$ is known to converge to 0 as soon as $\liminf \sigma^2(\beta)R_N \geq 1$, see [CSZ17, Theorem 2.8].)

In [BL17] (and [BL18] for the disordered pinning model), the authors consider a more involved functional, namely (a slightly modified version of) the k -th order term in the chaos expansion (4.9) with $f = \delta_0$, that is

$$\sum_{A \subset \llbracket 1, N \rrbracket \times \mathbb{Z}^2, |A|=k} q(A) \xi(A). \quad (4.12)$$

They take $k = k_N \rightarrow \infty$ slowly (in fact $k_N = (\log \log N)^2$) to show that $Z_N^{\beta,\omega} \rightarrow 0$ in probability as soon as $\liminf \sigma^2(\beta)R_N > 1$ (which is closer to the optimal result mentioned above). The result of [BL17] is in fact stronger and the authors prove a bound on the free energy.

One could take an even more faithful approximation of $Z_N^{\beta,\omega}(f)$ than (4.12). A close to optimal proxy would indeed be to keep in the chaos expansion (4.8) all orders $1 \leq k \leq \log N$, namely

$$\sum_{A \subset \llbracket 1, N \rrbracket \times \mathbb{Z}^2, 1 \leq |A| \leq \log N} q^{(f)}(A) \xi(A), \quad (4.13)$$

since it bears a positive proportion of the variance of $Z_N^{\beta,\omega}(f)$ at criticality, *i.e.* when $\sigma^2(\beta)R_N = 1 + O(\frac{1}{\log N})$, see (2.6). However, this would make the analysis extremely technical: the calculations in [BL17] are already difficult, so dealing with variance terms in (4.13) would quickly turn into a computation nightmare.

A new approach: a coarse-grained version of the chaos expansion. Our new idea is to introduce a *coarse-graining on the intermediate scale* $\tilde{N} = e^{-\eta}N$ (recall Proposition 4.6): more precisely, to make expression (4.13) more manageable, we only consider subsets A with *time-width at most* \tilde{N} , and we further restrict the starting time of A to be larger than \tilde{N} (to forget about the initial condition) and smaller than $N - \tilde{N}$. This leads to our proxy for $Z_N^{\beta}(f)$:

$$X(f) := \sum_{A \in \mathcal{I}} q^{(f)}(A) \xi(A) \quad \text{with} \quad (4.14)$$

$$\mathcal{I} := \left\{ A \subset \llbracket 1, N \rrbracket \times \mathbb{Z}^2 : 1 \leq |A| \leq \log N, \text{width}(A) \leq \tilde{N}, \text{start}(A) \in \llbracket \tilde{N} + 1, N - \tilde{N} \rrbracket \right\}$$

where for $A = \{(n_1, x_1), \dots, (n_k, x_k)\}$ with $1 \leq n_1 < \dots < n_k$ we have defined the quantities $\text{start}(A) := n_1$ and $\text{width}(A) := n_k - n_1$.

To give some more insight, let us explain why $X(f)$ in (4.14) roughly corresponds to *the first (linear) term in a suitable coarse-grained version of the chaos expansion*. Assuming for simplicity that $M := N/\tilde{N}$ is an integer, we can write the Hamiltonian in (2.2) as a sum of terms corresponding to time intervals of size \tilde{N} :

$$H_N^{\beta,\omega}(S) = \sum_{j=1}^M \mathcal{H}_j^{\beta,\omega}(S) \quad \text{with} \quad \mathcal{H}_j^{\beta,\omega}(S) = \sum_{n \in \llbracket (j-1)\tilde{N}+1, j\tilde{N} \rrbracket} (\beta\omega(n, S_n) - \lambda(\beta)),$$

so that we can write

$$Z_N^{\beta,\omega}(f) = \mathbf{E}_f \left[\prod_{j=1}^M e^{\mathcal{H}_j^{\beta,\omega}(S)} \right].$$

Then, writing each term $e^{\mathcal{H}_j^{\beta,\omega}(S)} = 1 + (e^{\mathcal{H}_j^{\beta,\omega}(S)} - 1) =: 1 + \Xi_j(S)$ and expanding the product, we obtain a *coarse-grained version* of the chaos expansion:

$$Z_N^{\beta,\omega}(f) = 1 + \sum_{j=1}^M \mathbf{E}_f[\Xi_j(S)] + \sum_{1 \leq j_1 < j_2 \leq M} \mathbf{E}_f[\Xi_{j_1}(S) \Xi_{j_2}(S)] + \dots,$$

where we omit higher-order terms to lighten notation. The inspiration for our proxy $X(f)$ is *the first (linear) term* $\sum_{j=1}^M \mathbf{E}_f[\Xi_j(S)]$ in this coarse-grained expansion (which may be viewed as a generalization of the basic linear approximation (4.11) on the coarse-grained scale \tilde{N}). The actual choice (4.14) for our proxy corresponds to a slight modification of this idea, where we consider a “sliding” strip of width \tilde{N} . The restriction $1 \leq |A| \leq \log N$ will be needed for technical reasons, namely to obtain a good control on $\tilde{\text{Var}}_f[X(f)]$.

4.4. Change of scale and measure: proof of Proposition 4.5. We start with a *change of scale* for the starting point. To this purpose, we show an analogue of Lemma 3.7, except for the truncated mean $\mathbb{E}[Z_N^{\beta,\omega}(f) \wedge 1]$ rather than the fractional moment $\mathbb{E}[Z_N^{\beta,\omega}(f)^\gamma]$.

Lemma 4.12 (Change of scale). *For any $A, B \in \mathbb{N}$ with $A \leq B$, we have*

$$\sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{B})} \mathbb{E}[Z_N^{\beta,\omega}(f) \wedge 1] \leq 8 \sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{A})} \mathbb{E}\left[Z_N^{\beta,\omega}(f) \wedge \frac{B}{A}\right] \quad (4.15)$$

The proof is a direct consequence of the following general lemma with $K \leq 4\frac{B}{A}$, combined with the same decomposition as in the proof of Lemma 3.7, see (3.8).

Lemma 4.13. *Let $(\alpha_i)_{1 \leq i \leq K}$ be non-negative numbers with $\sum_{i=1}^K \alpha_i = 1$, and let $(Z_i)_{1 \leq i \leq K}$ be non-negative random variables. Then, if we set $Z = \sum_{i=1}^K \alpha_i Z_i$, we have*

$$\mathbb{E}[Z \wedge 1] \leq 2 \max_{1 \leq i \leq K} \mathbb{E}[Z_i \wedge K].$$

Proof. Define $A_i := \{Z_i > K\}$ and let $B = \bigcup_{i=1}^K A_i$. Then, bounding $Z \wedge 1 \leq 1$ on the event B and $Z \wedge 1 \leq Z$ on B^c , we have that

$$\mathbb{E}[Z \wedge 1] \leq \mathbb{P}(B) + \mathbb{E}[Z \mathbf{1}_{B^c}] \leq \sum_{i=1}^K \mathbb{P}(A_i) + \sum_{i=1}^K \alpha_i \mathbb{E}[Z_i \mathbf{1}_{A_i^c}],$$

where we have used sub-additivity for the first term and $B^c = \bigcap_{i=1}^K A_i^c \subseteq A_i^c$ for all i . Now, since $A_i = \{Z_i > K\}$, we have that $\mathbb{P}(A_i) \leq \frac{1}{K} \mathbb{E}[Z_i \wedge K]$ by Markov’s inequality and $\mathbb{E}[Z_i \mathbf{1}_{A_i^c}] \leq \mathbb{E}[Z_i \wedge K]$ by definition of A_i^c . Plugging this in the above gives that

$$\mathbb{E}[Z \wedge 1] \leq \sum_{i=1}^K \frac{1}{K} \mathbb{E}[Z_i \wedge K] + \sum_{i=1}^K \alpha_i \mathbb{E}[Z_i \wedge K],$$

which concludes the proof. \square

We next use a *change of measure* argument to estimate the right-hand side of (4.15). We state it both for the truncated mean $\mathbb{E}[Z_N^{\beta,\omega}(f) \wedge \frac{N}{\tilde{N}}]$ and for the fractional moment $\mathbb{E}[Z_N^{\beta,\omega}(f)^\gamma]$, since the proof we have is simplified with respect to what we found in the literature.

Lemma 4.14 (Change of measure). *Let $Z \geq 0$ be a non-negative random variable. For any $L > 0$ and any event $A \in \mathcal{F}$, we have*

$$\mathbb{E}[Z \wedge L] \leq L \mathbb{P}(A) + \mathbb{E}[Z \mathbf{1}_{A^c}].$$

If additionally, $\mathbb{E}[Z] = 1$ for any $\gamma \in (0, 1)$, we have, for any event $A \in \mathcal{F}$

$$\mathbb{E}[Z^\gamma] \leq \mathbb{P}(A)^{1-\gamma} + \mathbb{E}[Z\mathbf{1}_{A^c}]^\gamma.$$

The above improves and simplifies [JL24, Lem. 2.2], which controls the moment of order $1/2$; in fact, we simplify its proof and get a general fractional moment (note that [JL25, Lem. 3.2] also controls a fractional moment, but in a non-optimal way).

Proof. For the first inequality, we simply bound $Z \wedge L \leq L$ on A and $Z \wedge L \leq Z$ on A^c : this gives the desired bound.

For the fractional moment, we write $\mathbb{E}[Z^\gamma] = \mathbb{E}[Z^\gamma\mathbf{1}_A] + \mathbb{E}[Z^\gamma\mathbf{1}_{A^c}]$. For the first term, we use Hölder's inequality to get $\mathbb{E}[Z^\gamma\mathbf{1}_A] \leq \mathbb{E}[Z]^\gamma \mathbb{P}(A)^{1-\gamma} = \mathbb{P}(A)^{1-\gamma}$. For the second term, we use Jensen's inequality to get $\mathbb{E}[Z^\gamma\mathbf{1}_{A^c}] \leq \mathbb{E}[Z\mathbf{1}_{A^c}]^\gamma$. This concludes the proof. \square

Proposition 4.5 is a simple combination of Lemma 4.12 (for (4.3)) with $A = \tilde{N}$, $B = N$ and of Lemma 4.14 with $L = \frac{N}{\tilde{N}}$, $Z = Z_N^{\beta, \omega}(f)$ (for (4.4)), recalling that $\tilde{\mathbb{P}}_f(A_N^c) := \mathbb{E}[Z_N^{\beta, \omega}(f)\mathbf{1}_{A_N^c}]$. \square

5. SECOND MOMENT ESTIMATES

In this section, we prove Lemma 4.8 and Lemma 4.9, which mostly rely on second moment estimates. (Proposition 4.10 requires third-moment-type estimates and we will prove it in Section 6.) We also prove Proposition 2.10 in Section 5.3.

Throughout this section, we fix $1 \leq \eta < \vartheta < \infty$ with $\vartheta - \eta \geq 1$, for $N \in \mathbb{N}$ we set $\tilde{N} = e^{-\eta}N$, and we consider $\beta = \beta_N(\vartheta)$ in the critical regime (2.6), or equivalently (2.10).

5.1. Preliminary notation and variance estimate. Let us rewrite our proxy X from (4.14) in a way that will be more convenient for calculations: introducing the notation

$$\mathcal{I}_m := \{A \subset \mathbb{N} \times \mathbb{Z}^2 : \text{start}(A) = m, \text{width}(A) \leq \tilde{N}, 1 \leq |A| \leq \log N\}, \quad (5.1)$$

we decompose $X(f)$ into contributions of strips of width \tilde{N} that start at time m :

$$X(f) = \sum_{m=\tilde{N}+1}^{N-\tilde{N}} X_m(f) \quad \text{with} \quad X_m(f) := \sum_{A \in \mathcal{I}_m} q^{(f)}(A) \xi(A). \quad (5.2)$$

Remark 5.1 (Orthogonal projection). Denoting by $\Pi_{\mathcal{I}_m}$ the orthogonal projection onto the linear subspace of L^2 generated by the $\xi(A)$ with $A \in \mathcal{I}_m$, we can write $X_m(f) = \Pi_{\mathcal{I}_m} Z_N^{\beta, \omega}(f)$.

If (m, y) denotes the first point in A , we can write $q^{(f)}(A) = q_m^{(f)}(y) q(A')$ with $A' = A - (m, y)$ the set A translated by its first point (with this point being removed); see (4.10). This leads to the following decomposition of $X_m(f)$:

$$X_m(f) = \sum_{y \in \mathbb{Z}^2} q_m^{(f)}(y) \xi_{m,y} \hat{Z}_{\tilde{N}}^{\beta, \omega}(m, y), \quad (5.3)$$

where, in view of (5.1), $\hat{Z}_{\tilde{N}}^{\beta, \omega}(m, y)$ is a partition function starting from (m, y) with time-width at most \tilde{N} and restricted to chaos orders up to $\log N - 1$. More precisely, denoting by $\theta^{m,y}\omega = (\omega_{n+m,x+y})_{n \in \mathbb{N}, x \in \mathbb{Z}^2}$ the translated environment, we can write

$$\hat{Z}_{\tilde{N}}^{\beta, \omega}(m, y) = \hat{Z}_{\tilde{N}}^{\beta, \theta^{m,y}\omega} \quad \text{with} \quad \hat{Z}_{\tilde{N}}^{\beta, \omega} := \sum_{A \subset [1, \tilde{N}] \times \mathbb{Z}^2, |A| \leq \log N - 1} q(A) \xi(A). \quad (5.4)$$

We stress that $\hat{Z}_{\tilde{N}}^{\beta, \omega}(m, y)$ depends only on $(\xi_{n,z} : n \geq m+1)$, hence it is independent of $\xi_{m,y}$.

Looking back at (5.3), it follows that the $X_m(f)$ are (centered and) uncorrelated:

$$\text{Cov}(X_m(f), X_{m'}(f)) = \mathbf{1}_{\{m=m'\}} \text{Var}[X_m(f)]. \quad (5.5)$$

Similarly, recalling that $\mathbb{E}[(\xi_{n,x})^2] = \sigma^2(\beta)$, we can write

$$\mathrm{Var}[X_m(f)] = \sum_{y \in \mathbb{Z}^2} q_m^{(f)}(y)^2 \sigma^2(\beta) \mathbb{E}[\hat{Z}_{\tilde{N}}^{\beta,\omega}(m,y)^2] = q_{2m}(f,f) \sigma^2(\beta) \mathbb{E}[(\hat{Z}_{\tilde{N}}^{\beta,\omega})^2], \quad (5.6)$$

where we have used translation invariance and introduced the collision kernel

$$q_{2m}(f,g) := \sum_{x,x' \in \mathbb{Z}^2} f(x) q_{2m}(x-x') g(x') = \sum_{y \in \mathbb{Z}^2} q_m^{(f)}(y) q_m^{(g)}(y), \quad (5.7)$$

the last identity following from Chapman–Kolmogorov.

Notice that, by orthogonality of the $\xi(A)$ in the definition (5.4) of $\hat{Z}_{\tilde{N}}^{\beta,\omega}$, we have

$$\mathcal{V}_{\tilde{N}} = \mathcal{V}_{\tilde{N}}(\beta, N) := \mathbb{E}[(\hat{Z}_{\tilde{N}}^{\beta,\omega})^2] = \sum_{k=0}^{\log N-1} \sigma^2(\beta)^k \sum_{A \subseteq [1, \tilde{N}] \times \mathbb{Z}^2, |A|=k} q(A)^2. \quad (5.8)$$

Then, we have the following estimate, whose proof is postponed to Section 5.3 below.

Lemma 5.2. *Fix $1 \leq \eta < \vartheta < \infty$ with $\vartheta - \eta \geq 1$. For $N \in \mathbb{N}$ set $\tilde{N} = e^{-\eta} N$ and take $\beta = \beta_N(\vartheta)$ in the critical regime (2.6), or equivalently (2.10). For N sufficiently large we have*

$$\frac{c}{\vartheta - \eta} e^{\vartheta - \eta} \leq \sigma^2(\beta_N) \mathcal{V}_{\tilde{N}} \leq \frac{c'}{\vartheta - \eta} e^{\vartheta - \eta}, \quad (5.9)$$

where $c, c' \in (0, \infty)$ are universal constants (we can take any $c < \frac{\pi}{4} e^{-2\gamma}$ with γ the Euler–Mascheroni constant and any $c' > \pi$).

Remark 5.3. Even though we only need to apply Lemma 5.2 when ϑ is fixed, an inspection of the proof shows that we could allow for $\vartheta = \vartheta_N \rightarrow \infty$, as long as $\vartheta_N \ll \sqrt{\log N}$.

5.2. Variance and size-biased expectation: proof of Lemmas 4.8 and 4.9. Recall the collision kernel $q_{2m}(f,g) := \sum_{x,x'} f(x) q_{2m}(x-x') g(x')$, and introduce the weighted Green function between time $s \leq t$:

$$G_{s,t}(f,g) := \sum_{m=s+1}^t q_{2m}(f,g). \quad (5.10)$$

Since the $X_m(f)$ are centered and uncorrelated (see (5.5)), recalling the computation (5.6), we therefore end up with the following expression for $\mathrm{Var}[X(f)]$:

$$\mathrm{Var}[X(f)] = \sum_{m=\tilde{N}+1}^{N-\tilde{N}} \mathrm{Var}[X_m(f)] = G_{\tilde{N}, N-\tilde{N}}(f,f) \sigma^2(\beta) \mathcal{V}_{\tilde{N}}. \quad (5.11)$$

As far as the size-biased expectation of $X(f)$ is concerned, notice that by linearity we have that $\tilde{\mathbb{E}}_f[X(f)] = \sum_{m=\tilde{N}+1}^{N-\tilde{N}} \tilde{\mathbb{E}}_f[X_m(f)]$. Recalling the definition (4.1) of $\tilde{\mathbb{E}}_f$ and the orthogonal projection $\Pi_{\mathcal{I}_m}$ from Remark 5.1 (recall that $X_m(f) = \Pi_{\mathcal{I}_m} Z_N^{\beta,\omega}(f)$), we can write

$$\begin{aligned} \tilde{\mathbb{E}}_f[X_m(f)] &= \mathbb{E}[X_m(f) Z_N^{\beta,\omega}(f)] = \mathbb{E}[X_m(f) \Pi_{\mathcal{I}_m} Z_N^{\beta,\omega}(f)] \\ &= \mathbb{E}[X_m(f)^2] = \mathrm{Var}[X_m(f)], \end{aligned} \quad (5.12)$$

since $X_m(f)$ is centered. In particular, we get that

$$\tilde{\mathbb{E}}_f[X(f)] = \sum_{m=\tilde{N}+1}^{N-\tilde{N}} \mathrm{Var}[X_m(f)] = G_{\tilde{N}, N-\tilde{N}}(f,f) \sigma^2(\beta) \mathcal{V}_{\tilde{N}} = \mathrm{Var}[X(f)]. \quad (5.13)$$

We also have the following lemma, which controls $G_{\tilde{N}, N-\tilde{N}}(f,f)$, uniformly for $f \in \mathcal{M}_1^{\mathrm{disc}}(\sqrt{\tilde{N}})$.

Lemma 5.4. *There are universal constants $C, C' > 0$ such that, for any $1 \leq s < t$*

$$C \log\left(\frac{t}{s}\right) \leq \inf_{f \in \mathcal{M}_1^{\mathrm{disc}}(\sqrt{s})} G_{s,t}(f,f) \leq \sup_{f \in \mathcal{M}_1^{\mathrm{disc}}(\sqrt{s})} G_{s,t}(f,f) \leq C' \log\left(\frac{t}{s}\right).$$

Proof. The proof simply follows from the local central limit theorem. More precisely, there are universal constants c, c' such that, for any $m \in \mathbb{N}$ and $z \in \mathbb{Z}^2$ even (i.e. with $z_1 + z_2$ even) such that $|z| \leq 2\sqrt{m}$, we have $\frac{c}{m} \leq q_{2m}(z) \leq \frac{c'}{m}$. In particular, for $m > s$ we get that

$$\frac{c}{m} \leq \inf_{\substack{|x|, |x'| \leq \sqrt{s} \\ x-x' \text{ even}}} q_{2m}(x-x') \leq \sup_{\substack{|x|, |x'| \leq \sqrt{s} \\ x-x' \text{ even}}} q_{2m}(x-x') \leq \frac{c'}{m}, \quad (5.14)$$

from which one easily deduces that $\frac{c}{2m} \leq q_{2m}(f, f) \leq \frac{c'}{m}$ uniformly for $f \in \mathcal{M}_1^{\text{disc}}(\sqrt{s})$. Summing over $m \in \llbracket s+1, t \rrbracket$ gives the desired conclusion. \square

Combining the bounds in Lemmas 5.2 and 5.4, the formulas (5.11)-(5.13) then yield that

$$cC \log\left(\frac{N}{\tilde{N}} - 1\right) \frac{e^{\vartheta-\eta}}{\vartheta-\eta} \leq \inf_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{\tilde{N}})} \tilde{\mathbb{E}}_f[X(f)] \leq \sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{\tilde{N}})} \text{Var}[X(f)] \leq C'c' \log\left(\frac{N}{\tilde{N}}\right) \frac{e^{\vartheta-\eta}}{\vartheta-\eta},$$

which concludes the proofs of Lemmas 4.8 and 4.9. \square

Remark 5.5. Let us stress that since we have $\tilde{\mathbb{E}}_f[X(f)] = \text{Var}[X(f)]$, going back to Lemma 4.7 we get

$$\mathbb{P}(A_N(f)) \leq \frac{4}{\text{Var}[X(f)]} = \frac{4}{G_{\tilde{N}, N-\tilde{N}}(f, f) \sigma^2(\beta_N) \mathcal{V}_{\tilde{N}}}.$$

The role of Lemmas 5.2 and 5.4 is simply to make this bound more explicit — in fact, only lower bounds on $\sigma^2(\beta_N) \mathcal{V}_{\tilde{N}}$ and $\inf_{f \in \mathcal{M}_1^{\text{disc}}} G_{\tilde{N}, N-\tilde{N}}(f, f)$ are actually needed here.

5.3. Second moment estimates: proof of Lemma 5.2 and Proposition 2.10. Before we start the proofs, let us introduce some further notation and useful estimate. We define

$$u(n) := \sum_{x \in \mathbb{Z}^2} q_n(x)^2 = q_{2n}(0) = \mathbf{P}(S_{2n} = 0), \quad (5.15)$$

so that $R_N = \sum_{n=1}^N u(n)$. We recall that by (2.5)

$$\pi R_N = \log N + \alpha + o(1), \quad \alpha := \gamma + 4 \log 2 - \pi \approx 0.208. \quad (5.16)$$

(In fact, we have that $0 \leq o(1) \leq \frac{\pi}{\tilde{N}}$.)

For $I = \{i_1, \dots, i_k\}$ with $i_1 < \dots < i_k$, we also set

$$u(I) := \prod_{j=1}^k u(i_j - i_{j-1}),$$

with by convention $i_0 = 0$, and $u(\emptyset) = 1$. Let us also introduce, for $n \geq 2$, i.i.d. random variables $T^{(n)}, T_1^{(n)}, T_2^{(n)}, \dots$ taking values in $\{1, \dots, n\}$ with

$$\mathbf{P}(T^{(n)} \leq j) = \frac{R_j}{R_n} \mathbf{1}_{\{1, \dots, n\}}(j).$$

In particular, we have $\mathbf{P}(T^{(n)} = j) = \frac{u(j)}{R_n} \mathbf{1}_{\{1, \dots, n\}}(j)$. Therefore, we can write

$$\sum_{I \subseteq \llbracket 1, n \rrbracket, |I|=k} u(I) = (R_n)^k \mathbf{P}(\tau_k^{(n)} \leq n) \quad \text{with} \quad \tau_k^{(n)} := T_1^{(n)} + \dots + T_k^{(n)}.$$

5.3.1. *Proof of Lemma 5.2.* To lighten notation, assume for simplicity that $\lfloor \log N \rfloor$ is used where needed. Summing over the spatial coordinates in the definition (5.8) of $\mathcal{V}_{\tilde{N}}$, we have that

$$\mathcal{V}_{\tilde{N}} = \sum_{k=0}^{\log N - 1} \sigma^2(\beta)^k \sum_{I \subseteq \llbracket 1, \tilde{N} \rrbracket, |I|=k} u(I) = \sum_{k=0}^{\log N - 1} (\sigma^2(\beta) R_{\tilde{N}})^k \mathbb{P}(\tau_k^{(\tilde{N})} \leq \tilde{N}).$$

Bounding $\mathbb{P}(\tau_k^{(\tilde{N})} \leq \tilde{N}) \leq 1$ and $\mathbb{P}(\tau_k^{(\tilde{N})} \leq \tilde{N}) \geq \mathbb{P}(\tau_{\log N}^{(\tilde{N})} \leq \tilde{N})$ and summing the geometric sum, we therefore get that

$$\frac{(\sigma^2(\beta) R_{\tilde{N}})^{\log N} - 1}{\sigma^2(\beta) R_{\tilde{N}} - 1} \mathbb{P}(\tau_{\log N}^{(\tilde{N})} \leq \tilde{N}) \leq \mathcal{V}_{\tilde{N}} \leq \frac{(\sigma^2(\beta) R_{\tilde{N}})^{\log N} - 1}{\sigma^2(\beta) R_{\tilde{N}} - 1}.$$

Notice that $\log \tilde{N} = \log N - \eta \geq \frac{1}{2} \log N$ for N large enough (in fact for $N \geq e^{2\eta}$), so we can bound $\log N \leq 2 \log \tilde{N}$ and then use [CSZ19b, Proposition 1.3] to get that

$$\mathbb{P}(\tau_{\log N}^{(\tilde{N})} \leq \tilde{N}) \geq \mathbb{P}(\tau_{2 \log \tilde{N}}^{(\tilde{N})} \leq \tilde{N}) \xrightarrow{N \rightarrow \infty} \mathfrak{p} > 0,$$

where $\mathfrak{p} = \mathbf{P}(Y_2 \leq 1)$ is a universal constant related to the Dickman subordinator $(Y_t)_{t \geq 0}$. A direct computation using the density of Y_2 [CSZ19b, Theorem 1.1] shows that $\mathfrak{p} = \frac{1}{2} e^{-2\gamma} \approx 0.158$. All together, to prove our goal (5.9), we only need to get upper and lower bounds on $\sigma^2(\beta) R_{\tilde{N}} - 1$ and $(\sigma^2(\beta) R_{\tilde{N}})^{\log N} - 1$ as $N \rightarrow \infty$, when we fix $\beta = \beta_N$ in the critical regime (2.10).

First of all, we can use (5.16) and the fact that $\tilde{N} = e^{-\eta} N$ to get that $R_{\tilde{N}} - R_N = -\frac{\eta}{\pi} + o(1)$ as $N \rightarrow \infty$. Hence, for $\beta = \beta_N$ satisfying (2.10), we have that

$$\sigma^2(\beta_N) R_{\tilde{N}} - 1 = \frac{R_{\tilde{N}}}{R_N - \frac{\vartheta + o(1)}{\pi}} - 1 = \frac{\frac{\vartheta + o(1)}{\pi} - \frac{\eta}{\pi} + o(1)}{R_N - \frac{\vartheta + o(1)}{\pi}} = (1 + o(1)) \frac{\frac{\vartheta - \eta}{\pi}}{R_N},$$

where for the last inequality we note that $o(1) = o(\vartheta - \eta)$ since $\vartheta - \eta \geq 1$. In particular, using also that $\sigma^2(\beta_N) R_N = 1 + o(1)$ again by (2.10), we get that as $N \rightarrow \infty$

$$\frac{\sigma^2(\beta_N)}{\sigma^2(\beta_N) R_{\tilde{N}} - 1} = (1 + o(1)) \frac{\pi}{\vartheta - \eta}.$$

On the other hand, using again $R_{\tilde{N}} = R_N - \frac{\eta}{\pi} + o(1)$ and (2.10), we get as $N \rightarrow \infty$

$$\sigma^2(\beta_N) R_{\tilde{N}} = \frac{R_{\tilde{N}}}{R_N - \frac{\vartheta + o(1)}{\pi}} = \frac{1 - \frac{\eta}{\pi R_N} + \frac{o(1)}{R_N}}{1 - \frac{\vartheta}{\pi R_N} + \frac{o(1)}{R_N}} = e^{\frac{\vartheta - \eta + o(1)}{\pi R_N}}.$$

Taking the $\log N$ power and recalling that $\pi R_N \sim \log N$ as $N \rightarrow \infty$, we get that

$$(\sigma^2(\beta_N) R_{\tilde{N}})^{\log N} = (1 + o(1)) e^{\vartheta - \eta}.$$

Gathering the previous estimates, we therefore get that as $N \rightarrow \infty$

$$(1 + o(1)) \frac{\frac{\pi}{2} e^{-2\gamma}}{\vartheta - \eta} (e^{\vartheta - \eta} - 1) \leq \sigma^2(\beta_N) \mathcal{V}_{\tilde{N}} \leq (1 + o(1)) \frac{\pi}{\vartheta - \eta} (e^{\vartheta - \eta} - 1),$$

which concludes the proof of Lemma 5.2, using also that $\frac{1}{2} e^{\vartheta - \eta} \leq e^{\vartheta - \eta} - 1 \leq e^{\vartheta - \eta}$ for $\vartheta - \eta \geq 1$. \square

5.3.2. *Proof of Proposition 2.10.* First of all, starting from the chaos expansion (4.9), we can decompose over the starting point of non-empty subsets A : we can write, in analogy with (5.2),

$$Z_N^{\beta, \omega}(f) - \mathbb{E}[Z_N^{\beta, \omega}(f)] = \sum_{x \in \mathbb{Z}^2} f(x) \sum_{m=1}^N \sum_{\substack{A \subseteq \llbracket m, N \rrbracket \\ \text{start}(A)=m}} q^{(x)}(A) \xi(A).$$

Notice that here we have subtracted the contribution of $A = \emptyset$, *i.e.* $\mathbb{E}[Z_N^{\beta, \omega}(f)]$. Now, note that if (m, y) is the first point of A we can again write $q^{(x)}(A) = q_m(y - x) q(A')$, with $A' = A - (m, y)$

the set A translated by its first point (with this point being removed). Hence, decomposing over (m, y) similarly as in (5.3), we get by orthogonality and translation invariance that

$$\mathbb{V}\text{ar}[Z_N^{\beta, \omega}(f)] = \sum_{m=1}^N \sum_{y \in \mathbb{Z}^2} q_m^{(f)}(y) q_m^{(f)}(y) \sigma^2(\beta) \mathbb{E}[Z_{N-m}^{\beta, \omega}(0)^2].$$

By Chapman–Kolmogorov, notice that $\sum_{y \in \mathbb{Z}^2} q_m^{(f)}(y)^2 = q_{2m}(f, f)$, as in (5.7). Using that $\mathbb{E}[Z_{N-m}^{\beta, \omega}(0)^2] \leq \mathbb{E}[Z_N^{\beta, \omega}(0)^2]$ and recalling the definition (5.10) of the weighted Green function (writing $G_N(f, f) = G_{0, N}(f, f)$ for simplicity), we therefore end up with the following upper bound on the variance:

$$\mathbb{V}\text{ar}[Z_N^{\beta, \omega}(f)] \leq G_N(f, f) \sigma^2(\beta) \mathbb{E}[Z_N^{\beta, \omega}(0)^2].$$

Now, for $f = \mathcal{U}_{\rho\sqrt{N}}$, we get that

$$G_N(\mathcal{U}_{\rho\sqrt{N}}, \mathcal{U}_{\rho\sqrt{N}}) = \frac{1}{|B(\rho\sqrt{N}) \cap \mathbb{Z}^2|^2} \sum_{m=1}^N \sum_{x, x' \in B(\rho\sqrt{N}) \cap \mathbb{Z}^2} q_{2m}(x - x') \leq \frac{C}{\rho^2},$$

where we first used that $\sum_{x \in \mathbb{Z}^2} q_{2m}(x - x') = 1$ and then the fact that the cardinality of $B(\rho\sqrt{N}) \cap \mathbb{Z}^2$ is of order $\rho^2 N$.

All together, we only need to get an upper bound on $\sigma^2(\beta) \mathbb{E}[Z_N^{\beta, \omega}(0)^2]$, which is equal to

$$\sigma^2(\beta) \sum_{k=0}^N \sigma^2(\beta)^k \sum_{I \subseteq \llbracket 1, N \rrbracket, |I|=k} u(I) = \sigma^2(\beta) \sum_{k=0}^N (\sigma^2(\beta) R_N)^k \mathbb{P}(\tau_k^{(N)} \leq N). \quad (5.17)$$

We now bound the probability appearing in the sum thanks to Chernoff's bound: for any $\lambda > 0$,

$$\mathbb{P}(\tau_k^{(N)} \leq N) \leq e^{N\lambda} \mathbb{E} \left[\exp(-\lambda T^{(N)}) \right]^k. \quad (5.18)$$

To estimate the Laplace transform of $T^{(N)}$, we use the following Tauberian theorem, from [BGT89, Thm. 3.9.1].

Lemma 5.6. *For a sequence $(u(n))_{n \in \mathbb{N}}$ of positive numbers, define the quantities*

$$R(m) := \sum_{n=1}^m u(n) \quad \text{and} \quad \hat{R}(\lambda) := \sum_{n=1}^{+\infty} e^{-\lambda n} u(n).$$

If there exist constants $a, b > 0$ such that $aR(m) = \log m + b + o(1)$ as $m \rightarrow \infty$, then

$$a\hat{R}(\lambda) = \log\left(\frac{1}{\lambda}\right) + b - \gamma + \tilde{o}(1), \quad (5.19)$$

as $\lambda \rightarrow 0$, where γ is the Euler–Mascheroni constant.

Let $\varepsilon \in (0, 1)$ and let us set $\lambda = \lambda(\beta)$ such that

$$\hat{R}(\lambda) := \frac{1}{\sigma^2(\beta)} - \frac{\varepsilon}{\pi}.$$

With this choice of λ , we have

$$\mathbb{E} \left[\exp(-\lambda T^{(N)}) \right] = \frac{1}{R_N} \sum_{n=1}^N e^{-\lambda n} u(n) \leq \frac{\hat{R}(\lambda)}{R_N} = \frac{1 - \frac{\varepsilon}{\pi} \sigma^2(\beta)}{\sigma^2(\beta) R_N}.$$

Using (5.18) and plugging this into (5.17), we get

$$\sigma^2(\beta) \mathbb{E}[Z_N^{\beta, \omega}(0)^2] \leq e^{N\lambda} \sigma^2(\beta) \sum_{k=0}^{\infty} \left(1 - \frac{\varepsilon}{\pi} \sigma^2(\beta)\right)^k \leq \frac{\pi}{\varepsilon} e^{N\lambda}. \quad (5.20)$$

It remains to estimate $N\lambda$. Note that the assumptions of Lemma 5.6 are verified with $a = \pi$ and $b = \alpha$, see (5.16). As a corollary, by (5.19), we get $\log \lambda = -\frac{\pi}{\sigma^2(\beta)} + \varepsilon + \alpha - \gamma + o(1)$ as $\beta \downarrow 0$. Recalling also (2.14), we get

$$N\lambda = e^\varepsilon e^{\alpha-\gamma+o(1)} N e^{-\frac{\pi}{\sigma^2(\beta)}} \leq e^{2\varepsilon} e^{\vartheta(N,\beta)-\gamma} \quad \text{as } \beta \downarrow 0, N \rightarrow \infty. \quad (5.21)$$

Similarly, if we restrict $\beta \in (0, 1)$, by (5.19) we can bound $\log \lambda \leq -\frac{\pi}{\sigma^2(\beta)} + \varepsilon + \alpha - \gamma + c$ for a suitable $c \in (0, \infty)$, therefore again by (2.14) we get, for a suitable $C \in (0, \infty)$,

$$N\lambda \leq e^\varepsilon e^{\alpha-\gamma+c} N e^{-\frac{\pi}{\sigma^2(\beta)}} \leq C e^{\vartheta(N,\beta)} \quad \text{for all } \beta \in (0, 1), N \in \mathbb{N}. \quad (5.22)$$

Plugging (5.22) into (5.20), we get the uniform bound (2.28). Similarly, plugging (5.21) into (5.20), we obtain

$$\sigma^2(\beta) \mathbb{E}[Z_N^{\beta,\omega}(0)^2] \leq \frac{\pi}{\varepsilon} \exp(e^{2\varepsilon} e^{\vartheta(N,\beta)-\gamma}) \leq \exp(e^{3\varepsilon} e^{\vartheta(N,\beta)-\gamma}),$$

where the second inequality holds eventually, if we assume that $\vartheta(N, \beta) \rightarrow \infty$. Since $\varepsilon > 0$ is arbitrary, this completes the proof of (2.29), hence of Proposition 2.10. \square

6. CONTROL OF THE SIZE-BIASED VARIANCE

In this section, we control the size-biased variance, *i.e.* we prove Proposition 4.10. *Throughout this section, we fix $1 \leq \eta < \vartheta < \infty$ with $\vartheta - \eta \geq 1$, for $N \in \mathbb{N}$ we set $\tilde{N} = e^{-\eta}N$ and we consider $\beta = \beta_N(\vartheta)$ in the critical regime (2.6), or equivalently (2.10).*

First of all, recalling (5.2), we write $\tilde{\text{Var}}_f[X(f)]$ as the sum of size-biased covariances, that we split into two parts, that we call *diagonal* and *off-diagonal* terms:

$$\tilde{\text{Var}}_f[X(f)] = \sum_{\substack{m_1, m_2 = \tilde{N}+1 \\ |m_1 - m_2| \leq \tilde{N}}}^{N-\tilde{N}} \tilde{\text{Cov}}_f[X_{m_1}(f), X_{m_2}(f)] + \sum_{\substack{m_1, m_2 = \tilde{N}+1 \\ |m_1 - m_2| > \tilde{N}}}^{N-\tilde{N}} \tilde{\text{Cov}}_f[X_{m_1}(f), X_{m_2}(f)].$$

The proof reduces to proving the following estimates.

Lemma 6.1 (Diagonal terms). *There is a universal constant $C > 0$ such that*

$$\limsup_{N \rightarrow \infty} \sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{\tilde{N}})} \sum_{\substack{m_1, m_2 = \tilde{N}+1 \\ |m_1 - m_2| \leq \tilde{N}}}^{N-\tilde{N}} \tilde{\text{Cov}}_f[X_{m_1}(f), X_{m_2}(f)] \leq C \left(\frac{1}{\vartheta - \eta} e^{\vartheta - \eta} \right)^2. \quad (6.1)$$

Lemma 6.2 (Off-diagonal terms). *There is a universal constant $C > 0$ such that, for any $m_1, m_2 \in [\tilde{N} + 1, N - \tilde{N}]$ with $m_2 - m_1 > \tilde{N}$, we have*

$$\sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{\tilde{N}})} \tilde{\text{Cov}}_f[X_{m_1}(f), X_{m_2}(f)] \leq \frac{C}{(m_2)^2} \left(\frac{1}{\vartheta - \eta} e^{\vartheta - \eta} \right)^2.$$

As a consequence, the off-diagonal term verifies, uniformly in $f \in \mathcal{M}_1^{\text{disc}}(\sqrt{\tilde{N}})$,

$$\sum_{\substack{m_1, m_2 = \tilde{N}+1 \\ |m_1 - m_2| > \tilde{N}}}^{N-\tilde{N}} \tilde{\text{Cov}}_f[X_{m_1}(f), X_{m_2}(f)] \leq 2C \left(\frac{1}{\vartheta - \eta} e^{\vartheta - \eta} \right)^2 \sum_{\tilde{N} < m_1 < m_2 < N} \frac{1}{(m_2)^2} \leq 2C \left(\frac{1}{\vartheta - \eta} e^{\vartheta - \eta} \right)^2 \eta,$$

where for the last inequality we have used that there are at most m_2 terms in the sum over m_1 so that the sum is bounded by $\sum_{m_2=\tilde{N}+1}^{N-1} \frac{1}{m_2} \leq \log\left(\frac{N}{\tilde{N}}\right) = \eta$. Combining these two results concludes the proof of Proposition 4.10. \square

It therefore remains to prove Lemma 6.1 and Lemma 6.2. We first deal with the off-diagonal term, *i.e.* Lemma 6.2, since it is a bit less technical than the diagonal term.

6.1. Off-diagonal terms: proof of Lemma 6.2. Let $m_2 > m_1 > \tilde{N}$ with $m_2 - m_1 > \tilde{N}$. Then, recalling the expression (5.2) for $X_m(f)$ and expanding the covariance, we can write explicitly:

$$\tilde{\text{Cov}}_f[X_{m_1}(f), X_{m_2}(f)] = \sum_{A_1 \in \mathcal{I}_{m_1}, A_2 \in \mathcal{I}_{m_2}} q^{(f)}(A_1)q^{(f)}(A_2) \tilde{\text{Cov}}_f[\xi(A_1), \xi(A_2)].$$

Now, since the sets A_1, A_2 are disjoint (because the strips $[[m_1, m_1 + \tilde{N}]]$ and $[[m_2, m_2 + \tilde{N}]]$ are disjoint), we get that

$$\begin{aligned} \tilde{\text{Cov}}_f[\xi(A_1), \xi(A_2)] &= \tilde{\mathbb{E}}_f[\xi(A_1 \cup A_2)] - \tilde{\mathbb{E}}_f[\xi(A_1)]\tilde{\mathbb{E}}_f[\xi(A_2)] \\ &= \sigma^2(\beta)^{|A_1|+|A_2|} (q^{(f)}(A_1 \cup A_2) - q^{(f)}(A_1)q^{(f)}(A_2)), \end{aligned}$$

using also that

$$\tilde{\mathbb{E}}_f[\xi(A)] = \mathbb{E}[Z_N^{\beta, \omega}(f)\xi(A)] = \sigma^2(\beta)^{|A|} q^{(f)}(A), \quad (6.2)$$

recalling (4.9). All together, we have that $\tilde{\text{Cov}}_f[X_{m_1}(f), X_{m_2}(f)]$ is equal to

$$\sum_{A_1 \in \mathcal{I}_{m_1}, A_2 \in \mathcal{I}_{m_2}} \sigma^2(\beta)^{|A_1|+|A_2|} q^{(f)}(A_1)q^{(f)}(A_2) (q^{(f)}(A_1 \cup A_2) - q^{(f)}(A_1)q^{(f)}(A_2)).$$

Denoting (ℓ_1, z_1) the last point of A_1 and (m_2, y_2) the first point of A_2 , we can write

$$q^{(f)}(A_1 \cup A_2) = q^{(f)}(A_1)q_{m_2-\ell_1}(y_2 - z_1)q(A'_2) \quad \text{and} \quad q^{(f)}(A_2) = q_{m_2}^{(f)}(y_2)q(A'_2),$$

with $A'_2 = A_2 - (m_2, y_2)$ the set A_2 translated by its first point (with this point being removed). Hence, we have

$$\begin{aligned} q^{(f)}(A_1)q^{(f)}(A_2)(q^{(f)}(A_1 \cup A_2) - q^{(f)}(A_1)q^{(f)}(A_2)) \\ = q^{(f)}(A_1)^2 q_{m_2}^{(f)}(y_2) (q_{m_2-\ell_1}(y_2 - z_1) - q_{m_2}^{(f)}(y_2)) q(A'_2)^2. \end{aligned}$$

Summing over A'_2 , since $\sum_{A'_2} \sigma^2(\beta)^{|A_2|} q(A'_2)^2 = \sigma^2(\beta) \mathcal{V}_{\tilde{N}}$ by definition of (4.9), we obtain that $\tilde{\text{Cov}}_f[X_{m_1}(f), X_{m_2}(f)]$ is equal to

$$\begin{aligned} \sum_{A_1 \in \mathcal{I}_{m_1}} \sigma^2(\beta)^{|A_1|} q^{(f)}(A_1)^2 \left(\sum_{y_2 \in \mathbb{Z}^2} q_{m_2}^{(f)}(y_2) (q_{m_2-\ell_1}(y_2 - z_1) - q_{m_2}^{(f)}(y_2)) \right) \sigma^2(\beta) \mathcal{V}_{\tilde{N}} \\ = \sum_{A_1 \in \mathcal{I}_{m_1}} \sigma^2(\beta)^{|A_1|} q^{(f)}(A_1)^2 (q_{2m_2-\ell_1}^{(f)}(z_1) - q_{2m_2}(f, f)) \sigma^2(\beta) \mathcal{V}_{\tilde{N}}, \end{aligned} \quad (6.3)$$

where we used Chapman-Kolmogorov (see also (5.7)) and we also recall that (ℓ_1, z_1) is the last point of A_1 .

Note that $\ell_1 \leq m_1 + \tilde{N} < m_2$, by definition of \mathcal{I}_m . Now, let us show that uniformly for $(\ell_1, z_1) \in \mathbb{N} \times \mathbb{Z}^2$ with $\tilde{N} \leq m_1 \leq \ell_1 \leq m_1 + \tilde{N} \leq m_2$, uniformly for $f \in \mathcal{M}_1^{\text{disc}}(\sqrt{\tilde{N}})$, we have

$$q_{2m_2-\ell_1}^{(f)}(z_1) - q_{2m_2}(f, f) \leq C \frac{m_1}{(m_2)^2}. \quad (6.4)$$

First of all, since $q_{2n}(x)$ is maximized at $x = 0$, we have that

$$q_{2n}^{(f)}(z) = \sum_{x \in \mathbb{Z}^2} f(x) q_{2n}(z - x) \leq \sum_{x \in \mathbb{Z}^2} f(x) q_{2n}(0) = q_{2n}(0).$$

Hence, we simply estimate

$$q_{2m_2-\ell_1}^{(f)}(z_1) - q_{2m_2}(f, f) \leq (q_{2m_2-\ell_1}(0) - q_{2m_2}(0)) + (q_{2m_2}(0) - q_{2m_2}(f, f)).$$

where we assume for simplicity that ℓ_1 is even (the odd case is similar). We now control both terms separately.

- For the first term, since $q_{2n}(0) = \frac{1}{\pi} \frac{1}{n} + O(\frac{1}{n^2})$ as $n \rightarrow \infty$ by the local CLT (see e.g. [LL10, Thm. 2.1.1], in particular (2.5)), for $\ell_1 \leq m_2$ (so that $2m_2 - \ell_1 \geq m_2$) we have

$$q_{2m_2 - \ell_1}(0) - q_{2m_2}(0) = \frac{1}{\pi} \left(\frac{1}{m_2 - \ell_1/2} - \frac{1}{m_2} \right) + O\left(\frac{1}{(m_2)^2} \right) \leq c \frac{\ell_1}{m_2^2} \leq 2c \frac{m_1}{m_2^2},$$

where the last inequality comes from the fact that $\ell_1 \leq m_1 + \tilde{N} \leq 2m_1$ since $m_1 \geq \tilde{N}$.

- For the second term, recalling the definition (5.7) of $q_{2m_2}(f, f)$, we start by writing, for $f \in \mathcal{M}_1^{\text{disc}}(\sqrt{\tilde{N}})$

$$\begin{aligned} |q_{2m_2}(0) - q_{2m_2}(f, f)| &\leq \sum_{|x|, |x'| \leq \sqrt{\tilde{N}}} f(x)f(x') |q_{2m_2}(0) - q_{2m_2}(x - x')| \\ &\leq \sup_{|x| \leq 2\sqrt{\tilde{N}}} |q_{2m_2}(0) - q_{2m_2}(x)|. \end{aligned}$$

Then, again by the local CLT, we get that

$$|q_{2m_2}(0) - q_{2m_2}(x)| \leq \frac{1}{2\pi m_2} |1 - e^{-|x|^2/m_2}| + O\left(\frac{1}{(m_2)^2} \right) \leq c \frac{1 + |x|^2}{(m_2)^2} \leq 2c \frac{m_1}{(m_2)^2},$$

uniformly for $|x| \leq 2\sqrt{\tilde{N}} \leq 2\sqrt{m_1}$. This concludes the proof of (6.4).

Plugging (6.4) back into (6.3) (notice that all the terms are non-negative), we get that

$$\tilde{\text{Cov}}_f[X_{m_1}(f), X_{m_2}(f)] \leq C \frac{m_1}{(m_2)^2} \sigma^2(\beta) \mathcal{V}_{\tilde{N}} \sum_{A_1 \in \mathcal{I}_{m_1}} \sigma^2(\beta)^{|A_1|} q^{(f)}(A_1)^2.$$

Let us notice that the last sum is exactly $\tilde{\mathbb{E}}_f[X_{m_1}(f)]$, recalling (5.2) together with (6.2). Thus, plugging the expression (5.12) for $\tilde{\mathbb{E}}_f[X_m(f)]$ (recall (5.6)), we get that

$$\tilde{\text{Cov}}_f[X_{m_1}(f), X_{m_2}(f)] \leq C \frac{m_1}{(m_2)^2} q_{2m_1}(f, f) (\sigma^2(\beta) \mathcal{V}_{\tilde{N}})^2 \leq C' \frac{1}{(m_2)^2} (\sigma^2(\beta) \mathcal{V}_{\tilde{N}})^2, \quad (6.5)$$

where we have also used (5.14) to get that $q_{2m_1}(f, f) \leq \frac{c'}{m_1}$ uniformly in $f \in \mathcal{M}_1^{\text{disc}}(\sqrt{\tilde{N}})$. This concludes the proof of Lemma 6.2 thanks to Lemma 5.2. \square

Remark 6.3. Echoing Remark 5.5, notice that the bound (6.5) is again very general and does not rely on the specific value of $\sigma^2(\beta) \mathcal{V}_{\tilde{N}}$. In fact, combining (6.5) with (5.13) and Lemma 5.4, the contribution of the off-diagonal part of $\tilde{\text{Var}}_f[X(f)]$ in the bound for $\tilde{\mathbb{P}}_f(A_N(f))$ given in Lemma 4.7 is

$$\frac{1}{\tilde{\mathbb{E}}_f[X(f)]^2} \sum_{\substack{m_1, m_2 = \tilde{N}+1 \\ |m_1 - m_2| > \tilde{N}}}^{N - \tilde{N}} \tilde{\text{Cov}}_f[X_{m_1}(f), X_{m_2}(f)] \leq \frac{C}{(\log \frac{N}{\tilde{N}})^2} \sum_{\tilde{N} \leq m_1 < m_2 \leq N} \frac{1}{(m_2)^2} \leq \frac{C'}{\log \frac{N}{\tilde{N}}}.$$

In particular, this is small *irrespective of the value of* $\mathcal{V}_{\tilde{N}}$ (uniformly in $f \in \mathcal{M}_1^{\text{disc}}(\sqrt{\tilde{N}})$), provided that N/\tilde{N} is large.

6.2. Diagonal term: proof of Lemma 6.1. Let us introduce intervals of length $2\tilde{N}$:

$$I_j := [(j-1)\tilde{N} + 1, (j+1)\tilde{N}] \quad \text{for } j = 2, 3, \dots,$$

in order to bound the diagonal term as follows:

$$\sum_{\substack{m_1, m_2 = \tilde{N}+1 \\ |m_1 - m_2| \leq \tilde{N}}}^{N - \tilde{N}} \tilde{\text{Cov}}_f[X_{m_1}(f), X_{m_2}(f)] \leq \sum_{j=2}^{\frac{N}{\tilde{N}}-1} \left\{ \sum_{m_1, m_2 \in I_j} \tilde{\mathbb{E}}_f[X_{m_1}(f)X_{m_2}(f)] \right\}. \quad (6.6)$$

We will focus on the terms in brackets, showing that there exists a universal constant $C > 0$ such that, for N large enough, the following bound holds:

$$\forall j \geq 2: \quad \sum_{m_1, m_2 \in I_j} \tilde{\mathbb{E}}_f [X_{m_1}(f) X_{m_2}(f)] \leq \varepsilon_{j,N}(f) + \frac{C}{j^2} \left(\frac{1}{\vartheta - \eta} e^{\vartheta - \eta} \right)^2, \quad (6.7)$$

where the terms $\varepsilon_{j,N}(f)$ satisfy

$$\limsup_{N \rightarrow \infty} \sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{N})} \sum_{j=2}^{\frac{N}{\tilde{N}} - 1} \varepsilon_{j,N}(f) = 0. \quad (6.8)$$

These relations, when plugged into (6.6), yield (6.1), thus completing the proof of Lemma 6.1.

It remains to prove (6.7) and (6.8). We recall that

$$\tilde{\mathbb{E}}_f [X_{m_1}(f) X_{m_2}(f)] = \mathbb{E} [X_{m_1}(f) X_{m_2}(f) Z_N^{\beta, \omega}(f)].$$

Using the representation (5.2) of $X_m(f)$ and the decomposition (4.9) of $Z_N^{\beta, \omega}(f)$, we get that

$$\begin{aligned} & \sum_{m_1, m_2 \in I_j} \tilde{\mathbb{E}}_f [X_{m_1}(f) X_{m_2}(f)] \\ &= \sum_{\substack{A_1, A_2 \in \bigcup_{m \in I_j} \mathcal{I}_m \\ A_3 \subseteq [1, N] \times \mathbb{Z}^2}} \sigma(\beta)^{|A_1| + |A_2| + |A_3|} q^{(f)}(A_1) q^{(f)}(A_2) q^{(f)}(A_3) \mathbb{E} [\xi(A_1) \xi(A_2) \xi(A_3)]. \end{aligned} \quad (6.9)$$

Since the $\xi_{n,x}$'s are centered and independent, for $\mathbb{E} [\xi(A_1) \xi(A_2) \xi(A_3)]$ to be non-zero *each point in $A_1 \cup A_2 \cup A_3$ must belong to at least two sets among A_1, A_2, A_3* . This means that, with $A_1 \Delta A_2 := (A_1 \setminus A_2) \cup (A_2 \setminus A_1)$ the symmetric difference of A_1 and A_2 , we must have

$$A_1 \Delta A_2 \subseteq A_3 \subseteq A_1 \cup A_2, \quad \text{i.e. } A_3 = (A_1 \Delta A_2) \cup D \quad \text{for some } D \subseteq A_1 \cap A_2.$$

The terms with $A_3 \supsetneq A_1 \Delta A_2$, *i.e.* $D \neq \emptyset$, correspond to *triple intersections* (points which belong to all three sets A_1, A_2, A_3).

We define $\varepsilon_{j,N}(f)$ to be the contribution to (6.9) given by triple intersections, *i.e.* the restriction of the sum to $A_3 \supsetneq A_1 \Delta A_2$. Since $I_j \subseteq [\tilde{N} + 1, N]$ for $2 \leq j \leq \frac{N}{\tilde{N}} - 1$, we can bound

$$\begin{aligned} & \sum_{j=2}^{\frac{N}{\tilde{N}} - 1} \varepsilon_{j,N}(f) \\ & \leq \sum_{\substack{A_1, A_2, A_3 \subseteq [\tilde{N} + 1, N] \times \mathbb{Z}^2 \\ A_1, A_2 \neq \emptyset, A_3 \supsetneq A_1 \Delta A_2}} \sigma(\beta)^{|A_1| + |A_2| + |A_3|} q^{(f)}(A_1) q^{(f)}(A_2) q^{(f)}(A_3) |\mathbb{E} [\xi(A_1) \xi(A_2) \xi(A_3)]|. \end{aligned} \quad (6.10)$$

We now prove that (6.8) holds, that is *triple intersections give a negligible contribution*. To this purpose, we exploit [CSZ20, Proposition 4.3] to prove the following.

Lemma 6.4 (No triple intersections). *Fix $0 \leq \eta < \vartheta < \infty$. For $N \in \mathbb{N}$, set $\tilde{N} = e^{-\eta} N$ and consider $\beta = \beta_N(\vartheta)$ in the critical regime (2.6). Then (6.8) holds.*

Proof. We will compare (6.10) with the centered third moment $\mathbb{E} [(Z_{N-\tilde{N}}^{\beta, \omega}(\hat{f}) - \mathbb{E} [Z_{N-\tilde{N}}^{\beta, \omega}(\hat{f})])^3]$ of a partition function with time horizon $N - \tilde{N}$ and with initial condition \hat{f} given by

$$\hat{f}(z) = \hat{f}_{\tilde{N}}(z) := \sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{\tilde{N}})} q_N^{(f)}(z). \quad (6.11)$$

Indeed, recalling (4.9), we can upper bound $\mathbb{E} [(Z_{N-\tilde{N}}^{\beta, \omega}(\hat{f}) - \mathbb{E} [Z_{N-\tilde{N}}^{\beta, \omega}(\hat{f})])^3]$ by

$$\sum_{\substack{A'_1, A'_2, A'_3 \subseteq [1, N - \tilde{N}] \times \mathbb{Z}^2 \\ A'_1, A'_2, A'_3 \neq \emptyset}} \sigma(\beta)^{|A'_1| + |A'_2| + |A'_3|} q^{(\hat{f})}(A'_1) q^{(\hat{f})}(A'_2) q^{(\hat{f})}(A'_3) |\mathbb{E} [\xi(A'_1) \xi(A'_2) \xi(A'_3)]|. \quad (6.12)$$

We can cast (6.10) in this form applying Chapman-Kolmogorov at time \tilde{N} , that is, writing

$$q^{(f)}(A_i) = \sum_{z \in \mathbb{Z}^2} q_{\tilde{N}}^{(f)}(z) q^{(z)}(A'_i) \leq q^{(\hat{f})}(A'_i) \quad \text{with } A'_i := A_i - (\tilde{N}, 0), \quad i = 1, 2, 3.$$

This means that the right hand side of (6.10) can be bounded, uniformly over $f \in \mathcal{M}_1^{\text{disc}}(\sqrt{\tilde{N}})$, by (6.12) restricted to the terms with triple intersections $A_3 \supseteq A_1 \triangle A_2$.

The latter contribution, denoted by $M_{0,1-e^{-\eta}}^{N,\Gamma}(\varphi, 1)$, was studied in [CSZ19b] (see eq. (4.4) and Proposition 4.3) and shown to vanish as $N \rightarrow \infty$, under the assumption that $\hat{f}(x) \leq \frac{C}{N} \varphi(\frac{x}{\sqrt{N}})$ for some continuous and compactly supported function $\varphi : \mathbb{R}^2 \rightarrow [0, \infty)$. Our function \hat{f} from (6.11) satisfies $f_{\tilde{N}}(x) \leq \frac{C}{N} \varphi(\frac{x}{\sqrt{N}})$ with $\varphi(x) = e^{-c|x|^2}$ (for some $c = c_\eta > 0$, by the local CLT): note that φ is bounded, but *not* compactly supported. However, the relevant property which is actually used in the proof of [CSZ19b, Proposition 4.3] is that $\sum_{x \in \mathbb{Z}^2} \frac{1}{N} \varphi(\frac{x}{\sqrt{N}})$ remains *bounded* as $N \rightarrow \infty$ (see the proof of Lemma 7.1 and the first display on page 427 in [CSZ19b]). This is clearly satisfied for our choice of φ , which completes the proof. \square

We can henceforth focus on the sum (6.9) restricted to $A_3 = A_1 \triangle A_2$ (no triple intersections). Our goal is to *bound it by the last term in (6.7)*.

We make some manipulations on (6.9). Since subsets $A_i \in \mathcal{I}_m$ have width up to \tilde{N} , see (5.1), and since I_j are intervals of width $2\tilde{N}$, it follows that the sum over $A_i \in \bigcup_{m \in I_j} \mathcal{I}_m$ can be enlarged to $A_i \subseteq \tilde{I}_j \times \mathbb{Z}^2$ for the interval \tilde{I}_j of width $3\tilde{N}$ given by

$$\tilde{I}_j := I_j \cup I_{j+1} = \llbracket (j-1)\tilde{N} + 1, (j+2)\tilde{N} \rrbracket. \quad (6.13)$$

Since $|A_1| + |A_2| + |A_1 \triangle A_2| \leq 2(|A_1| + |A_2|)$, decomposing according to $k = |A_1|$ and $k' = |A_2|$ we can then bound (6.9) restricted to pairwise intersections $A_3 = A_1 \triangle A_2$ as follows:

$$\begin{aligned} \sum_{k, k'=1}^{\log N} \sum_{\substack{A_1, A_2 \subseteq \tilde{I}_j \times \mathbb{Z}^2 \\ |A_1|=k, |A_2|=k'}} \sigma^2(\beta)^{k+k'} q^{(f)}(A_1) q^{(f)}(A_2) q^{(f)}(A_1 \triangle A_2) \\ \leq \sum_{k, k'=1}^{\log N} (\sigma^2(\beta) R_{3\tilde{N}})^{k+k'} \mathcal{M}_{\tilde{I}_j}^{(f)}(k, k'), \end{aligned} \quad (6.14)$$

where for an interval I we define the normalized quantity (note that for $I = \tilde{I}_j$ we have $|I| = 3\tilde{N}$)

$$\mathcal{M}_I^{(f)}(k, k') := \sum_{\substack{A_1, A_2 \subseteq I \times \mathbb{Z}^2 \\ |A_1|=k, |A_2|=k'}} \frac{q^{(f)}(A_1) q^{(f)}(A_2) q^{(f)}(A_1 \triangle A_2)}{(R_{|I|})^{k+k'}}. \quad (6.15)$$

To derive (6.14), we also used that $\sigma^2(\beta) R_{3\tilde{N}} \geq 1$ for large N , as we now check. Recall that we take $\beta = \beta_N(\vartheta)$ in the critical regime (2.6). Applying (2.5), since $\tilde{N} = e^{-\eta} N$, we have $R_{3\tilde{N}}/R_N = 1 + (\log 3 - \eta + o(1))/\log N$ which yields, as $N \rightarrow \infty$,

$$\sigma^2(\beta) R_{3\tilde{N}} = \left(1 + \frac{\vartheta + o(1)}{\log N}\right) \left(1 + \frac{\log 3 - \eta + o(1)}{\log N}\right) = 1 + \frac{\vartheta - \eta + \log 3 + o(1)}{\log N}, \quad (6.16)$$

which is larger than 1 for large N , since $\vartheta \geq \eta$ by assumption.

We now use the following claim, that we prove below.

Claim 6.5. *There is a constant $C > 0$ such that, for \tilde{N} large enough, we have*

$$\forall j \geq 2: \quad \sup_{k, k' \geq 1} \sup_f \mathcal{M}_{\tilde{I}_j}^{(f)}(k, k') \leq \frac{C}{j^2 (R_{3\tilde{N}})^2}, \quad (6.17)$$

where the second sup ranges over all probability densities on \mathbb{Z}^2 .

With Claim 6.5 at hand, since $\sigma^2(\beta)R_{3\tilde{N}} \leq 1 + \frac{\vartheta-\eta+2}{\log N} \leq \exp(\frac{\vartheta-\eta+2}{\log N})$ for large N , see (6.16), we obtain that, uniformly in $f \in \mathcal{M}_1(\sqrt{N})$, the right hand side of (6.14) is bounded by

$$\frac{C}{j^2(R_{3\tilde{N}})^2} \left(\sum_{k=1}^{\log N} e^{k \frac{\vartheta-\eta+2}{\log N}} \right)^2 \leq \frac{C}{j^2(R_{3\tilde{N}})^2} \left(\frac{e^{\vartheta-\eta+2}}{\frac{\vartheta-\eta+2}{\log N}} \right)^2 \leq \frac{C'}{j^2} \left(\frac{e^{\vartheta-\eta}}{\vartheta-\eta} \right)^2,$$

where in the first inequality we summed the geometric series (bounding $e^x - 1 \geq x$ in the denominator) and in the second inequality we exploited (2.5) and bounded $\vartheta - \eta + 2 \geq \vartheta - \eta$. This yields our goal (6.7) and concludes the proof of Lemma 6.1. \square

6.2.1. *Proof of Claim 6.5.* The quantity $\mathcal{M}_I^{(f)}(k, k')$ that we need to estimate is close to (the contribution of pairwise intersections to) the third moment of the partition function, similarly to the proof of Lemma 6.4. We cannot apply results from [CSZ20] out of the box, because of the *local constraints* given by the fixed values of k, k' , but we can still adapt (and simplify) the arguments in [CSZ20] to our context.

Recalling (6.15), we split $\mathcal{M}_I^{(f)}(k, k')$ in two parts, say

$$\mathcal{M}_I^{(f)}(k, k') = \mathcal{M}_{I,=}^{(f)}(k, k') + \mathcal{M}_{I,\neq}^{(f)}(k, k'),$$

corresponding to the contributions of the terms with $A_1 = A_2$ and $A_1 \neq A_2$, respectively. We will show that both contributions satisfy the bound (6.17).

We start with $\mathcal{M}_{I,=}^{(f)}(k, k')$, so we restrict (6.15) to $A_1 = A_2$ (this is only possible for $k = k'$). Listing the elements of $A_1 = A_2 = \{(m_i, z_i)\}_{1 \leq i \leq k}$ by increasing time, we can write

$$q^{(f)}(A_1) q^{(f)}(A_2) q^{(f)}(A_1 \triangle A_2) = q^{(f)}(A_1)^2 = q^{(f)}(m_1, z_1)^2 \prod_{i=2}^k q(m_i - m_{i-1}, z_i - z_{i-1})^2. \quad (6.18)$$

Renaming $(a, x) = (m_1, z_1)$ and $(b, y) = (m_k, z_k)$ the first and last point of A_1 , and summing over the inner points (m_i, z_i) for $2 \leq i \leq k-1$, the product in (6.18) yields the space-time convolution $Q^{*(k-1)}(b-a, y-x)$, where we define the probability mass function on $\mathbb{N} \times \mathbb{Z}^2$

$$Q(m, z) := \frac{q(m, z)^2}{R_{3\tilde{N}}} \mathbf{1}_{\{1 \leq m \leq 3\tilde{N}\}}. \quad (6.19)$$

(By convention, $Q^{*0}(m, z) = \mathbf{1}_{m=0} \mathbf{1}_{z=0}$.) Looking back at (6.15), we can write the contribution of $A_1 = A_2$ as

$$\mathcal{M}_{I,=}^{(f)}(k, k') = \mathbf{1}_{\{k=k'\}} \sum_{\substack{a \leq b \in I \\ x, y \in \mathbb{Z}^2}} \frac{q^{(f)}(a, x)^2}{R_{3\tilde{N}}} Q^{*(k-1)}(b-a, y-x). \quad (6.20)$$

Note that $\sum_{y \in \mathbb{Z}^2} Q^{*(k-1)}(b-a, y-x) = K^{*(k-1)}(b-a)$ where we define

$$K(m) = \sum_{z \in \mathbb{Z}^2} Q(m, z) := \frac{u(m)}{R_{3\tilde{N}}} \mathbf{1}_{\{1 \leq m \leq 3\tilde{N}\}}, \quad (6.21)$$

recalling also (5.15). (Again, by convention, $K^{*0}(m) = \mathbf{1}_{m=0}$.) We now use the following basic estimate: there exists a constant $\hat{c} > 1$ such that

$$q(m, z) \leq \sup_{y \in \mathbb{Z}^2} q(m, y) \leq \hat{c} u(m), \quad \text{hence by (4.7) also } q^{(f)}(m, z) \leq \hat{c} u(m), \quad (6.22)$$

which we apply to *one instance* of $q^{(f)}(a, x)$ in (6.20). Since $\sum_{x \in \mathbb{Z}^2} q^{(f)}(a, x) = 1$, we obtain

$$\sup_{k, k' \geq 1} \sup_f \mathcal{M}_{I,=}^{(f)}(k, k') = \sup_{k \geq 1} \sum_{a \leq b \in I} \hat{c} \frac{u(a)}{R_{3\tilde{N}}} K^{*(k-1)}(b-a) \leq \sum_{a \in I} \hat{c} \frac{u(a)}{R_{3\tilde{N}}},$$

where we used the fact that $\sum_{m \in \mathbb{N}} K^{*(k-1)}(m) = 1$. When we consider $I = \tilde{I}_j$ from (6.13), we can bound $u(a) \leq \frac{c}{a} \leq \frac{c}{(j-1)\tilde{N}}$ by (5.15) and the local CLT. Since $|\tilde{I}_j| = 3\tilde{N}$, we have shown that

$$\sup_{k, k' \geq 1} \sup_f \mathcal{M}_{I,=}^{(f)}(k, k') \leq \frac{3\hat{c}}{(j-1)R_{3\tilde{N}}},$$

which yields our goal (6.17) for $j \geq 2$.

We next consider the contribution $\mathcal{M}_{I, \neq}^{(f)}(k, k')$ to (6.15) given by terms $A_1 \neq A_2$, which is more involved. We perform a change of variables: setting $A_3 := A_1 \triangle A_2$, we define the *disjoint subsets*

$$C_{12} := A_1 \cap A_2, \quad C_{23} := A_2 \cap A_3 = A_2 \setminus A_1, \quad C_{13} := A_1 \cap A_3 = A_1 \setminus A_2,$$

so that we can write

$$q^{(f)}(A_1) q^{(f)}(A_2) q^{(f)}(A_1 \triangle A_2) = q^{(f)}(C_{12} \sqcup C_{13}) q^{(f)}(C_{12} \sqcup C_{23}) q^{(f)}(C_{13} \sqcup C_{23}). \quad (6.23)$$

We will derive an explicit expression for this product according to “interaction diagrams”, see Figure 1 for a graphical illustration. To each space-time point $(m, z) \in A_1 \cup A_2 = C_{13} \cup C_{23} \cup C_{12}$ we associate a label $d = ij \in \{12, 23, 13\}$ indicating the set to which it belongs, *i.e.*, $(m, z) \in C_{ij}$. This partitions the set $C_{13} \cup C_{23} \cup C_{12}$ into *stretches* of points with common label ij , describing the “interaction” between the random walk configurations A_i and A_j (since $C_{ij} = A_i \cap A_j$).

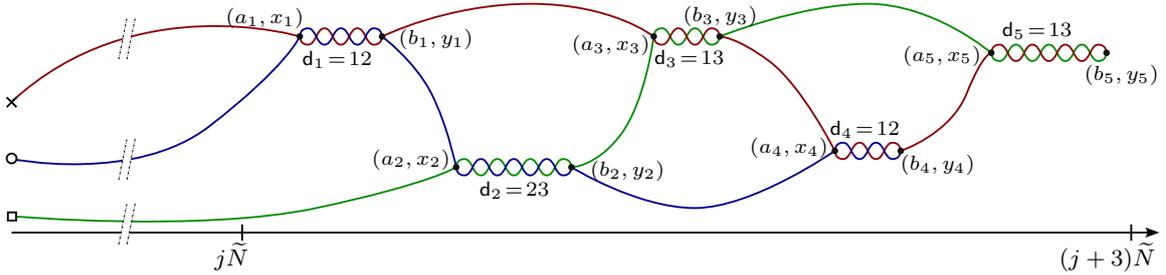


FIGURE 1. Illustration of an “interaction diagram”. Pairwise interactions are grouped in stretches of space-time points belonging to the same set C_{ij} , *i.e.* with the same label $d = ij$. A labeled diagram corresponds to a collection of stretches, where each stretch has a label d_p , a size (cardinality) k_p and ordered starting and ending points (a_p, x_p) , (b_p, y_p) . In the above diagram, there are $\ell = 5$ stretches.

To give a formal definition of the stretches, we order elements of $C_{13} \cup C_{23} \cup C_{12}$ by increasing time, obtaining a list $(m_i, z_i)_{1 \leq i \leq |C_{13} \cup C_{23} \cup C_{12}|}$. For the first stretch, we let $(a_1, x_1) = (m_1, z_1) = \min\{C_{13} \cup C_{23} \cup C_{12}\}$ be its first point, to which we associate label $d_1 \in \{12, 23, 13\}$ such that $(a_1, x_1) \in C_{d_1}$, and we then add elements to the first stretch as long as they are in C_{d_1} . The size of the first stretch is then $k_1 = \sup\{k : (m_i, z_i) \in C_{d_1} \forall i \leq k\}$, and $(b_1, y_1) = (m_{k_1}, z_{k_1})$ is its last point. The first stretch is then $\mathcal{S}_1 := \{(m_i, z_i), 1 \leq i \leq k_1\}$.

We then proceed iteratively to define the subsequent stretches. If the stretches $\mathcal{S}_1, \dots, \mathcal{S}_{p-1}$ with respective sizes k_1, \dots, k_{p-1} have been defined, the first element of the p -th stretch (if it exists) is then $(a_p, x_p) := \min\{(C_{13} \cup C_{23} \cup C_{12}) \setminus (\mathcal{S}_1 \cup \dots \cup \mathcal{S}_{p-1})\}$, which is in fact the element $(m_{k_1 + \dots + k_{p-1} + 1}, z_{k_1 + \dots + k_{p-1} + 1})$. The associated label is $d_p \in \{12, 23, 13\}$ such that $(a_p, x_p) \in C_{d_p}$, and note that $d_p \neq d_{p-1}$. We define the size k_p and the last element (b_p, y_p) of the stretch exactly as above. The p -th stretch is then $\mathcal{S}_p := \{(m_i, z_i) : k_1 + \dots + k_{p-1} + 1 \leq i \leq k_1 + \dots + k_p\}$.

We can now rewrite (6.23) as a product over stretches $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_\ell$, where $\ell \geq 2$ the total number of stretches (note that there are at least two stretches, since we consider the contribution

of $A_1 \neq A_2$). Recalling (6.23), we can write (see again Figure 1 for an illustration)

$$\begin{aligned} & q^{(f)}(A_1) q^{(f)}(A_2) q^{(f)}(A_1 \triangle A_2) \\ &= q^{(f)}(a_1, x_1)^2 \prod_{i=2}^{k_1} q(m_i - m_{i-1}, z_i - z_{i-1})^2 \\ &\quad \cdot q^{(f)}(a_2, x_2) q(a_2 - b_1, x_2 - y_1) \prod_{i=k_1+1}^{k_1+k_2} q(m_i - m_{i-1}, z_i - z_{i-1})^2 \\ &\quad \cdot \prod_{p=3}^{\ell} q(a_p - b_{p-2}, x_p - y_{p-2}) q(a_p - b_{p-1}, x_p - y_{p-1}) \prod_{i=k_1+\dots+k_{p-1}+1}^{k_1+\dots+k_p} q(m_i - m_{i-1}, z_i - z_{i-1})^2. \end{aligned}$$

We plug this expression into (6.15) restricted to $A_1 \neq A_2$, which defines $\mathcal{M}_{I, \neq}^{(f)}(k, k')$.

We now follow similar steps as for the case $A_1 = A_2$, see (6.18) and the following lines. First we perform a partial sum inside each stretch: if we fix the starting and ending points (a_p, x_p) , (b_p, y_p) of the p -th stretch, recalling the mass function Q defined in (6.19), the sum over internal space-time points (m_i, z_i) for $i_1 + \dots + k_{p-1} + 1 \leq i \leq k_1 + \dots + k_p - 1$ yields the space-time convolution $Q^{*(k_p-1)}(b_p - a_p, y_p - x_p)$ (recall the definition (6.19)). All together, we can rewrite (6.15) as a sum over diagrams, codified by the number $\ell \geq 2$ of stretches and by the label \mathbf{d}_p , size k_p , starting and ending points (a_p, x_p) , (b_p, y_p) of each stretch. The conditions $|A_1| = |C_{12}| + |C_{13}| = k$, $|A_2| = |C_{12}| + |C_{23}| = k'$ can be expressed as a constraint on the sizes k_1, \dots, k_ℓ depending on the labels:

$$\mathcal{C}_{k, k'}(\mathbf{d}_1, \dots, \mathbf{d}_\ell) := \left\{ k_1 \geq 1, \dots, k_\ell \geq 1 : \sum_{\substack{p=1, \dots, \ell: \\ \mathbf{d}_p \in \{12, 13\}}} k_p = k, \sum_{\substack{p=1, \dots, \ell: \\ \mathbf{d}_p \in \{12, 23\}}} k_p = k' \right\}. \quad (6.24)$$

This leads to the following explicit *identity*:

$$\begin{aligned} \mathcal{M}_{I, \neq}^{(f)}(k, k') &= \sum_{\ell=2}^{\infty} \sum_{\substack{\mathbf{d}_1, \dots, \mathbf{d}_\ell \in \{12, 23, 13\} \\ \mathbf{d}_p \neq \mathbf{d}_{p-1} \forall p=2, \dots, \ell}} \sum_{k_1, \dots, k_\ell \in \mathcal{C}_{k, k'}(\mathbf{d}_1, \dots, \mathbf{d}_\ell)} \sum_{\substack{a_1 \leq b_1 < \dots < a_\ell \leq b_\ell \in I \\ x_1, y_1, \dots, x_\ell, y_\ell \in \mathbb{Z}^2}} \\ &\quad \cdot \frac{q^{(f)}(a_1, x_1)^2}{R_{3\tilde{N}}} Q^{*(k_1-1)}(b_1 - a_1, y_1 - x_1) \\ &\quad \cdot \frac{q^{(f)}(a_2, x_2) q(a_2 - b_1, x_2 - y_1)}{R_{3\tilde{N}}} Q^{*(k_2-1)}(b_2 - a_2, y_2 - x_2) \\ &\quad \cdot \prod_{p=3}^{\ell} \frac{q(a_p - b_{p-2}, x_p - y_{p-2}) q(a_p - b_{p-1}, x_p - y_{p-1})}{R_{3\tilde{N}}} Q^{*(k_p-1)}(b_p - a_p, y_p - x_p). \end{aligned}$$

We next apply the estimate (6.22) to the kernels $q(a_p - b_{p-2}, x_p - y_{p-2})$, as well as to $q^{(f)}(a_2, x_2)$ and to *one instance* of $q^{(f)}(a_1, x_1)$. This allows us to sum over all space variables iteratively, starting from $y_\ell, x_\ell, y_{\ell-1}, x_{\ell-1}, \dots$ until y_1, x_1 . To this purpose, recalling (6.21), we have $\sum_{y_p \in \mathbb{Z}^2} Q^{*(k_p-1)}(b_p - a_p, y_p - x_p) = K^{*(k_p-1)}(b_p - a_p)$. Using also $\sum_{x_p \in \mathbb{Z}^2} q(a_p - b_{p-1}, x_p - y_{p-1}) = 1$ and $\sum_{x_1 \in \mathbb{Z}^2} q^{(f)}(a_1, x_1) = 1$, we then obtain

$$\begin{aligned} \mathcal{M}_I^{(f)}(k, k') &\leq \sum_{\ell=2}^{\infty} (\hat{c})^\ell \sum_{\substack{\mathbf{d}_1, \dots, \mathbf{d}_\ell \in \{12, 23, 13\} \\ \mathbf{d}_p \neq \mathbf{d}_{p-1} \forall p=2, \dots, \ell}} \sum_{k_1, \dots, k_\ell \in \mathcal{C}_{k, k'}(\mathbf{d}_1, \dots, \mathbf{d}_\ell)} \sum_{a_1 \leq b_1 < \dots < a_\ell \leq b_\ell \in I} \\ &\quad \cdot \frac{u(a_1)}{R_{3\tilde{N}}} K^{*(k_1-1)}(b_1 - a_1) \frac{u(a_2)}{R_{3\tilde{N}}} K^{*(k_2-1)}(b_2 - a_2) \cdot \prod_{p=3}^{\ell} K(a_p - b_{p-2}) K^{*(k_p-1)}(b_p - a_p). \end{aligned} \quad (6.25)$$

Note that there is no longer any dependence on f .

We now sum over the time variables $b_\ell, b_{\ell-1}$ which only appear in the last two stretches (these are free ends of the diagram, see Figure 1). As a consequence, the kernels K^{*k_ℓ} and $K^{*k_{\ell-1}}$ are erased from (6.25), since $\sum_{b_p} K^{*(k_p-1)}(b_p - a_p) \leq \sum_{m \in \mathbb{N}_0} K^{*(k_p-1)}(m) = 1$. This is in fact a crucial step, since we can now sum over the stretch sizes $k_\ell, k_{\ell-1}$ removing the constraint $\mathcal{C}_{k,k'}(\mathbf{d}_1, \dots, \mathbf{d}_\ell)$. Indeed, *there is at most one pair $(k_\ell, k_{\ell-1})$ for which the constraint is fulfilled*, see (6.24), hence $\sum_{k_\ell} \sum_{k_{\ell-1}} \mathbf{1}_{\{(k_1, \dots, k_\ell) \in \mathcal{C}_{k,k'}(\mathbf{d}_1, \dots, \mathbf{d}_\ell)\}} \leq 1$ for any $\mathbf{d}_1, \dots, \mathbf{d}_\ell$ and $k_1, \dots, k_{\ell-2}$.

If $\ell \geq 3$, we then sum freely over k_p for $1 \leq p \leq \ell - 3$, replacing $K^{*(k_p-1)}(b_p - a_p)$ by $U(b_p - a_p)$ with

$$U(m) := \sum_{k \geq 0} K^{*k}(m).$$

Estimating $u(a_1), u(a_2) \leq \frac{c}{(j-1)\tilde{N}}$ uniformly for $a_1, a_2 \geq (j-1)\tilde{N}$, see (6.13), and bounding the number of labels $(\mathbf{d}_1, \dots, \mathbf{d}_\ell)$ by $3 \cdot 2^{\ell-1}$, we finally get

$$\mathcal{M}_{\tilde{I}_j}^{(f)}(k, k') \leq \frac{3^3 c^2}{2(j-1)^2 (R_{3\tilde{N}})^2} \sum_{\ell=2}^{\infty} (2\hat{c})^\ell J_{\tilde{N}, \ell}$$

where we define $J_{\tilde{N}, \ell}$ as follows: for $\ell \in \{2, 3\}$ we set

$$J_{\tilde{N}, 2} := \sum_{a_1 < a_2 \in \tilde{I}_j} \frac{1}{(3\tilde{N})^2}, \quad J_{\tilde{N}, 3} := \sum_{a_1 \leq b_1 < a_2 < a_3 \in \tilde{I}_j} \frac{1}{3\tilde{N}} U(b_1 - a_1) \frac{1}{3\tilde{N}} K(a_3 - b_1), \quad (6.26)$$

while for $\ell \geq 4$ we set

$$J_{\tilde{N}, \ell} := \sum_{\substack{a_1 \leq b_1 < \dots < a_{\ell-2} \leq b_{\ell-2} \\ < a_{\ell-1} < a_\ell \in \tilde{I}_j}} \frac{1}{3\tilde{N}} U(b_1 - a_1) \frac{1}{3\tilde{N}} U(b_2 - a_2) \prod_{p=3}^{\ell-2} K(a_p - b_{p-2}) U(b_p - a_p) \cdot K(a_{\ell-1} - b_{\ell-3}) \cdot K(a_\ell - b_{\ell-2}). \quad (6.27)$$

Since by assumption we have $j \geq 2$, our goal (6.17) follows by the next claim, which shows that $J_{\tilde{N}, \ell}$ decays super-exponentially in ℓ , uniformly in \tilde{N} . This readily concludes the proof of Claim 6.5. \square

Claim 6.6. *For every $\varepsilon > 0$, there exists a constant $C_\varepsilon < +\infty$ such that*

$$\sup_{\tilde{N} \in \mathbb{N}} J_{\tilde{N}, \ell} \leq C_\varepsilon \varepsilon^\ell \quad \forall \ell \geq 2.$$

Remark 6.7 (Renewal interpretation). The quantity J_ℓ from (6.26)-(6.27) enjoys a probabilistic interpretation. Note that it only depends on the length of the interval \tilde{I}_j , so we can replace \tilde{I}_j by $\llbracket 1, 3\tilde{N} \rrbracket$. Let τ, τ' be independent renewal processes, started from τ_0, τ'_0 uniformly sampled in $\llbracket 1, 3\tilde{N} \rrbracket$ and with step probability mass function $K(m)$ from (6.21). Denote by $\mathcal{L}_{3\tilde{N}}(\tau, \tau')$ the number of alternating stretches of τ, τ' in the interval $\llbracket 1, 3\tilde{N} \rrbracket$, then we can write

$$J_{\tilde{N}, \ell} = \mathbb{P}(\mathcal{L}_{3\tilde{N}}(\tau, \tau') \geq \ell, A_{\tilde{N}, \ell}).$$

where the event $A_{\tilde{N}, \ell}$ is defined as follows: denoting by $\sigma_\ell = \sigma_\ell(\tau, \tau')$ the starting point of the ℓ -th alternating stretch (if it exists), we set

$$A_{\tilde{N}, \ell} = \{\tau_0 < \tau'_0, \tau \cap \tau' \cap \llbracket 1, \sigma_\ell - 1 \rrbracket = \emptyset\}.$$

Indeed, the right hand side of (6.26)-(6.27) gives precisely the probability that there are at least ℓ alternating stretches of τ, τ' , with $\tau_0 < \tau'_0$ and no common point before σ_ℓ (see Figure 2).

In particular, it follows by Claim 6.6 that, on the event $A_{\tilde{N}, \ell}$, the number of alternating stretches $J_{\tilde{N}, \ell}$ has finite exponential moments: $\mathbb{E}[e^{\lambda \mathcal{L}_{3\tilde{N}}(\tau, \tau')} \mathbf{1}_{A_{\tilde{N}, \ell}}] < +\infty$ for all $\lambda \in (0, \infty)$, which is equivalent to $\sum_{\ell=2}^{\infty} e^{\lambda \ell} J_\ell < +\infty$.

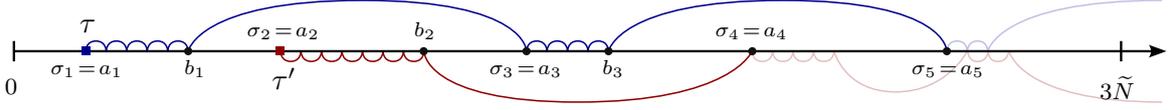


FIGURE 2. Illustration of the renewal interpretation of the formula (6.27) for J_ℓ . The first points of τ, τ' are chosen uniformly in $\llbracket 1, 3\tilde{N} \rrbracket$ and they the two renewals have inter-arrival distribution $K(m)$ defined above. The stretches alternate between τ and τ' and have starting and ending point denoted by a_p (or σ_p) and b_p , in reference to the interaction diagrams (see Figure 1). The number of alternating stretches is denoted $\mathcal{L}_{3\tilde{N}}(\tau, \tau')$: in the above picture we have $\mathcal{L}_{3\tilde{N}}(\tau, \tau') \geq \ell = 5$ and the two renewals $\tau \cap \tau'$ do not intersect before the beginning σ_5 of the 5th stretch (but they might intersect afterwards).

6.2.2. *Proof of Claim 6.6.* The proof is similar to what is done in [CSZ20, Section 5.3], but we give the details, because we provide a substantially shorter and simplified proof.

Let us fix $\ell \geq 2$. We can combine the definitions (6.26)-(6.27) in the single formula

$$J_{\tilde{N}, \ell} = \sum_{\substack{0 < a_1 \leq b_1 < \dots < a_{\ell-2} \\ \leq b_{\ell-2} < a_{\ell-1} < a_\ell \leq 3\tilde{N}}} \frac{1}{(3\tilde{N})^2} \prod_{i=1}^{\ell-2} U(b_i - a_i) \prod_{i=3}^{\ell} K(a_i - b_{i-2}).$$

Now, by [CSZ19a, Theorem 1.4] (and recalling (5.15)-(5.16) for controlling $u(m)$ and $R_{\tilde{N}}$), we have that there is a constant $C > 1$ such that

$$U(m) \leq C \frac{\log(3\tilde{N})}{3\tilde{N}} \cdot G_0\left(\frac{m+1}{3\tilde{N}}\right), \quad K(m) \leq \frac{C}{\log(3\tilde{N})} \frac{1}{m},$$

where $G_0(t) := \int_0^\infty \frac{1}{\Gamma(s+1)} s t^{s-1} e^{-\gamma s} ds$ for $t \in (0, 1]$ is the renewal function of the so-called Dickman subordinator. (By taking \tilde{N} large we could make the constant C arbitrarily close to 1.)

Plugging this in the above formula (and noticing that all the terms “ $\log(3\tilde{N})$ ” cancel out)

$$\begin{aligned} J_{\tilde{N}, \ell} &\leq C^{2(\ell-2)} \sum_{\substack{0 < a_1 \leq b_1 < \dots < a_{\ell-2} \\ \leq b_{\ell-2} < a_{\ell-1} < a_\ell \leq 3\tilde{N}}} \frac{1}{(3\tilde{N})^2} \prod_{i=1}^{\ell-2} \frac{1}{3\tilde{N}} G_0\left(\frac{b_i - a_i + 1}{3\tilde{N}}\right) \prod_{i=3}^{\ell} \frac{1}{3\tilde{N}} \frac{3\tilde{N}}{a_i - b_{i-2}} \\ &\leq (C')^\ell \int \dots \int_{\substack{0 < s_1 < t_1 < \dots < s_{\ell-2} \\ \leq t_{\ell-2} < s_{\ell-1} < s_\ell < 1}} \prod_{i=1}^{\ell-2} G_0(t_i - s_i) \prod_{i=3}^{\ell} \frac{1}{s_i - t_{i-2}} ds dt, \end{aligned}$$

the last inequality following from a Riemann sum bound. With a change of variable $u_i = t_i - s_i$ for $1 \leq i \leq \ell - 2$ and $v_i = s_i - t_{i-1}$ for $1 \leq i \leq \ell - 1$ (with $t_0 = 0$), as well as $v_\ell = s_\ell - s_{\ell-1}$, we get

$$J_{\tilde{N}, \ell} \leq (C')^\ell \int \dots \int_{\substack{u_i \in (0,1), v_i \in (0,1) \\ u_1 + \dots + u_{\ell-2} + v_1 + \dots + v_\ell < 1}} \prod_{i=1}^{\ell-2} G_0(u_i) \cdot \prod_{i=3}^{\ell-1} \frac{1}{v_i + u_{i-1} + v_{i-1}} \cdot \frac{1}{v_\ell + v_{\ell-1}} du dv.$$

Then, bounding $v_i + u_{i-1} + v_{i-1} \geq v_i + v_{i-1}$, introducing a multiplier $\lambda > 0$ and using that $\prod_{i=1}^{\ell-2} e^{\lambda u_i} \leq e^\lambda$, we get

$$J_{\tilde{N}, \ell} \leq (C')^\ell e^\lambda \left(\int_0^1 G_0(u) e^{-\lambda u} du \right)^{\ell-2} \int_{(0,1)^{\ell-1}} \prod_{i=3}^{\ell} \frac{1}{v_i + v_{i-1}} dv.$$

By [CSZ20, Lemma 5.2], there exists $c < \infty$ such that for all $\lambda \geq 1$

$$\int_0^1 G_0(u) e^{-\lambda u} du \leq \frac{c}{2 + \log \lambda},$$

so we end up with

$$J_{\tilde{N},\ell} \leq (C')^2 e^\lambda \left(\frac{cC'}{2 + \log \lambda} \right)^{\ell-2} \int_{(0,1)^{\ell-1}} \prod_{i=3}^{\ell} \frac{1}{v_i + v_{i-1}} d\mathbf{v},$$

and it remains to control the last integral.

For this, define $\varphi^{(0)}(v) \equiv 1$ and by iteration $\varphi^{(k)}(v) = \int_{(0,1)} \frac{1}{v+u} \varphi^{(k-1)}(u) du$, so that the integral is equal to $\int_0^1 \varphi^{(\ell-2)}(v) dv$. We now show by induction that for any $k \geq 1$

$$\forall v \in (0, 1) \quad \varphi^{(k)}(v) \leq \frac{\pi^k}{\sqrt{v}}.$$

The base case $k = 0$ is trivial, since $\varphi^{(0)}(v) = 1 \leq \frac{1}{\sqrt{v}}$ for $v \in (0, 1)$. For the inductive step, we assume that $\varphi^{(k-1)}(v) \leq \frac{\pi^{k-1}}{\sqrt{v}}$ for all $v \in (0, 1)$. Then, we have

$$\begin{aligned} \varphi^{(k)}(v) &= \int_0^1 \frac{1}{v+u} \varphi^{(k-1)}(u) du \leq \pi^{k-1} \int_0^1 \frac{1}{\sqrt{u}(v+u)} du \\ (\text{setting } u = t^2) &= 2\pi^{k-1} \int_0^1 \frac{1}{v+t^2} dt = \frac{2\pi^{k-1}}{\sqrt{v}} \arctan\left(\frac{1}{\sqrt{v}}\right) \leq \frac{\pi^k}{\sqrt{v}}. \end{aligned}$$

Therefore, we obtain that

$$\int_{(0,1)^{\ell-1}} \prod_{i=3}^{\ell} \frac{1}{v_i + v_{i-1}} d\mathbf{v} = \int_0^1 \varphi^{(\ell-2)}(v) dv \leq \int_0^1 \frac{\pi^{\ell-2}}{\sqrt{v}} dv \leq 2\pi^{\ell-2}.$$

All together, we conclude that

$$J_{\tilde{N},\ell} \leq 2(C')^2 e^\lambda \left(\frac{cC'\pi}{2 + \log \lambda} \right)^{\ell-2}$$

Taking $\lambda = \lambda_\varepsilon$ large enough concludes the proof of Claim 6.6. \square

7. PROOF OF THE OTHER MAIN RESULTS

In this section, we give the proofs of Theorems 1.1, 1.5, 1.7, 2.3 and 2.8. We start with Theorem 2.8, which is a direct consequence of Theorem 2.2.

Proof of Theorem 2.8. The lower bound in (2.26) is proved in Appendix B: we follow [BL17, §4], exploiting super-additivity and concentration of measure arguments for $\log Z_N^{\beta,\omega}$. We focus here on the upper bound in (2.26), which we deduce from Theorem 2.2.

We claim that we can truncate $Z_N^{\beta,\omega}$ at 1 in the definition (2.25) of the free energy and write

$$\mathbf{F}(\beta) = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E}[\log(Z_N^{\beta,\omega} \wedge 1)]. \quad (7.1)$$

Indeed, the proof is simple: since $Z_N^{\beta,\omega} = (Z_N^{\beta,\omega} \wedge 1)(Z_N^{\beta,\omega} \vee 1)$, it suffices to show that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E}[\log(Z_N^{\beta,\omega} \vee 1)] = 0.$$

But this follows by the inequalities $1 \leq Z_N^{\beta,\omega} \vee 1 \leq 1 + Z_N^{\beta,\omega}$, which yields

$$0 \leq \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E}[\log(Z_N^{\beta,\omega} \vee 1)] \leq \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E}[1 + Z_N^{\beta,\omega}] = 0,$$

recalling also that $\mathbb{E}[Z_N^{\beta,\omega}] = 1$.

Recalling (2.14) and applying (2.13) to $Z_N^{\beta,\omega} = Z_N^{\beta,\omega}(0) = Z_N^{\beta,\omega}(\mathbf{1}_{\{0\}})$, we get

$$\mathbb{E}[Z_N^{\beta,\omega} \wedge 1] \leq \frac{1}{c_2} \exp(-c_2 e^{\vartheta(N,\beta)}) = \frac{1}{c_2} \exp(-c_2 e^{\alpha+o(1)} N e^{-\frac{\pi}{\sigma^2(\beta)}}).$$

Applying relation (7.1) together with $\mathbb{E}[\log(Z_N^{\beta,\omega} \wedge 1)] \leq \log \mathbb{E}[Z_N^{\beta,\omega} \wedge 1]$ (by Jensen's inequality), we obtain the upper bound on the free energy in (2.26) with $c = c_2 e^\alpha$. \square

We then prove Theorem 1.1 about the Stochastic Heat Flow (SHF), which follows from the corresponding result for directed polymers, Theorem 2.2.

Proof of Theorem 1.1. We first prove (1.3). Fix $\vartheta \in \mathbb{R}$ and let $\beta_N = \beta_N(\vartheta)$ for $N \in \mathbb{N}$ satisfy (2.6), or equivalently (2.10). We are going to exploit (2.13) for $\beta = \beta_N$, so that $\vartheta_N(N, \beta) \rightarrow \vartheta$ as $N \rightarrow \infty$, see (2.11). Recall the convergence (2.9) of the directed polymer partition function to the SHF, and note that the support of $f = \varphi^{(N)}$ is \sqrt{N} times the size of the support of φ . If we let $N \rightarrow \infty$ in (2.13) for $f = \varphi^{(N)}$, since $\vartheta_N \rightarrow \vartheta$ we obtain precisely (1.3) for $t = 1$:

$$\frac{1}{c_1} e^{-c_1 e^\vartheta} \leq \sup_{\varphi \in \mathcal{M}_1(e^{c_0 e^\vartheta})} \mathbb{E}[\mathcal{Z}_1^\vartheta(\varphi) \wedge 1] \leq \frac{1}{c_2} e^{-c_2 e^\vartheta}. \quad (7.2)$$

For general $t > 0$, we use the scaling covariance property $\mathcal{Z}_1^{\vartheta+\log t}(\varphi) \stackrel{d}{=} \mathcal{Z}_t^\vartheta(\varphi_{\sqrt{t}})$, see the second relation in (1.7): applying (7.2) with ϑ replaced by $\vartheta + \log t$ yields

$$\frac{1}{c_1} e^{-c_1 t e^\vartheta} \leq \sup_{\varphi \in \mathcal{M}_1(e^{c_0 t e^\vartheta})} \mathbb{E}[\mathcal{Z}_t^\vartheta(\varphi_{\sqrt{t}}) \wedge 1] \leq \frac{1}{c_2} e^{-c_2 t e^\vartheta}.$$

We finally note that $\psi = \varphi_{\sqrt{t}} \in \mathcal{M}_1(e^{c_0 t e^\vartheta} \sqrt{t})$ for $\varphi \in \mathcal{M}_1(e^{c_0 t e^\vartheta})$, which proves (1.3).

We next prove (1.4). We already remarked that the upper bound follows by the upper bound in (1.3) and Markov's inequality $\mathbb{P}(Z \geq \varepsilon) \leq (\varepsilon \wedge 1)^{-1} \mathbb{E}[Z \wedge 1]$, which yields $C_{2,\varepsilon} = (c_2 \varepsilon)^{-1}$. For the lower bound, it suffices to consider the uniform density $\varphi = \mathcal{U}_{\sqrt{t}}$ on the ball of radius \sqrt{t} , see (1.5). The Paley–Zygmund inequality gives, for $Z = \mathcal{Z}_t^\vartheta(\mathcal{U}_{\sqrt{t}})$ with $\mathbb{E}[Z] = 1$,

$$\mathbb{P}(Z \geq \varepsilon) \geq (1 - \varepsilon)^2 \frac{\mathbb{E}[Z]^2}{\mathbb{E}[Z^2]} = \frac{(1 - \varepsilon)^2}{1 + \text{Var}[Z]}.$$

Since $\text{Var}[\mathcal{Z}_t^\vartheta(\mathcal{U}_{\sqrt{t}})] \leq c_3 e^{c_3 t e^\vartheta}$ by (2.28) from Proposition 2.10 and (2.9), we see that the lower bound in (1.4) holds with $c_1 = c_3$ and $C_{1,\varepsilon} = \frac{(1-\varepsilon)^2}{1+c_3}$. \square

We next deduce Theorem 1.5 from Theorem 1.1 and Proposition 2.10.

Proof of Theorem 1.5. Recall the uniform density (1.5). Fix $c, \delta > 0$ (to be determined later) and set $\rho = e^{c t e^\vartheta}$, $\varepsilon := t e^{-\delta t e^\vartheta}$. By Markov's inequality, setting $\varepsilon' := \frac{\varepsilon}{\pi \rho^2 t} = \frac{1}{\pi} e^{-(\delta+2c)t e^\vartheta} \leq 1$,

$$\mathbb{P}\left(\mathcal{Z}_t^\vartheta(B(0, \rho\sqrt{t})) > \varepsilon\right) = \mathbb{P}\left(\mathcal{Z}_t^\vartheta(\mathcal{U}_{\rho\sqrt{t}}) > \varepsilon'\right) \leq \frac{\mathbb{E}[\mathcal{Z}_t^\vartheta(\mathcal{U}_{\rho\sqrt{t}}) \wedge 1]}{\varepsilon' \wedge 1} \leq \frac{\frac{1}{c_2} e^{-c_2 t e^\vartheta}}{\frac{1}{\pi} e^{-(\delta+2c)t e^\vartheta}},$$

where we applied the upper bound from (1.3) in Theorem 1.1 assuming $c \leq c_0$. The right hand side is $\frac{\pi}{c_2} e^{-(c_2-2c-\delta)t e^\vartheta} \leq \frac{\pi}{c_2} e^{-\delta t e^\vartheta}$ if we fix $c < \min\{c_0, \frac{c_2}{2}\}$ and $\delta \leq \frac{1}{2}(c_2 - 2c)$. This proves the first line in (1.8) provided we further take $\delta \leq \frac{c_2}{\pi}$.

For the second line, we exploit the following upper bound on the variance of the SHF:

$$\text{Var}\left[\mathcal{Z}_t^\vartheta(\mathcal{U}_{\rho\sqrt{t}})\right] \leq c_3 \frac{\exp(c_3 t e^\vartheta)}{\rho^2}. \quad (7.3)$$

This follows by (2.28) for $t = 1$, recall (2.9), while the general case $t > 0$ can be deduced by the scaling properties of the SHF, see the second relation in (1.7). Let us set $\chi := t e^{\delta t e^\vartheta}$. Note that $\chi' := \frac{\chi}{\pi \rho^2 t} = \frac{1}{\pi} e^{(\delta-2c)t e^\vartheta} \leq \frac{1}{\pi}$, provided that $\delta \leq 2c$. Since $\mathbb{E}[\mathcal{Z}_t^\vartheta(\mathcal{U}_{\rho\sqrt{t}})] = 1$, Chebyshev's inequality yields

$$\mathbb{P}\left(\mathcal{Z}_t^\vartheta(B(0, \rho\sqrt{t})) \leq \chi\right) = \mathbb{P}\left(\mathcal{Z}_t^\vartheta(\mathcal{U}_{\rho\sqrt{t}}) \leq \chi'\right) \leq \frac{\text{Var}[\mathcal{Z}_t^\vartheta(\mathcal{U}_{\rho\sqrt{t}})]}{(1 - \frac{1}{\pi})^2} \leq 3c_3 \frac{e^{c_3 t e^\vartheta}}{\rho^2}.$$

Plugging in $\rho = e^{ct e^\vartheta}$ with $c = c'' > \frac{c_3}{2}$, the right hand side is $3c_3 e^{-(2c''-c_3)te^\vartheta} \leq \frac{1}{\delta} e^{-\delta te^\vartheta}$ provided we fix $\delta \leq \min\{2c'' - c_3, \frac{1}{3c_3}\}$. \square

We finally prove Theorems 1.7 and 2.3.

Proof of Theorem 1.7. Recalling (1.9) we may write, by a change of variables,

$$\hat{\mathcal{Z}}_t^{\vartheta,c}(\varphi) = \mathcal{Z}_t^{\vartheta}(\hat{\varphi}) \quad \text{with} \quad \hat{\varphi}(x) = R^{-2} \varphi(R^{-1}x), \quad R = e^{ct e^\vartheta} \sqrt{t}. \quad (7.4)$$

Without loss of generality we assume that $\varphi \in \mathcal{M}_1(1)$, see (1.2), hence $\hat{\varphi} \in \mathcal{M}_1(R)$. For $c < c_0$ we can then apply Theorem 1.1 to get $\mathbb{E}[\mathcal{Z}_t^{\vartheta}(\hat{\varphi}) \wedge 1] \rightarrow 0$ as $t \rightarrow \infty$ and/or $\vartheta \rightarrow \infty$, hence $\hat{\mathcal{Z}}_t^{\vartheta,c}(\varphi) \rightarrow 0$ in distribution, which proves the first line of (1.10) with $c' = c_0$.

Next, we observe that, since $\mathbb{E}[\hat{\mathcal{Z}}_t^{\vartheta,c}(\varphi)] = 1$, we can write by (7.4)

$$\mathbb{E}[(\hat{\mathcal{Z}}_t^{\vartheta,c}(\varphi) - 1)^2] = \text{Var}[\hat{\mathcal{Z}}_t^{\vartheta,c}(\varphi)] = \text{Var}[\mathcal{Z}_t^{\vartheta}(\hat{\varphi})] \leq \pi^2 \|\varphi\|_\infty^2 \text{Var}[\mathcal{Z}_t^{\vartheta}(\mathcal{U}_R)],$$

where we simply bounded $\hat{\varphi}(x) \leq \pi \|\varphi\|_\infty \mathcal{U}_R(x)$, see (1.5). Applying (7.3) we then get

$$\mathbb{E}[(\hat{\mathcal{Z}}_t^{\vartheta,c}(\varphi) - 1)^2] \leq c_3 \pi^2 \|\varphi\|_\infty^2 \frac{\exp(c_3 t e^\vartheta)}{(e^{ct e^\vartheta})^2},$$

hence the second line of (1.10) holds with $c'' = 2c_3$. \square

Proof of Theorem 2.3. The proof is similar to that of Theorem 1.7. We set $t = 1$ for simplicity. Recalling (2.20) and (2.18), as well as (2.3) and (2.8), we have the identity (in distribution, since we neglect the time-reversed environment)

$$\int_{\mathbb{R}^2} \varphi(x) \hat{u}_N^{\beta,c}(1, x) dx \stackrel{d}{=} Z_N^{\beta,\omega}(\hat{f}_N) \quad \text{with} \quad \hat{f}_N = \varphi^{((\rho_N^{\beta,c})^2 N)}, \quad \rho_N^{\beta,c} = e^{c e^\vartheta(N,\beta)}.$$

For $\varphi \in \mathcal{M}_1(1)$, see (1.2), we have $\hat{f}_N \in \mathcal{M}_1^{\text{disc}}(\rho_N^{\beta,c} \sqrt{N}) = \mathcal{M}_1^{\text{disc}}(e^{c e^\vartheta(N,\beta)} \sqrt{N})$, see (2.12). Applying Theorem 2.2, for $c < c_0$ we get $Z_N^{\beta,\omega}(\hat{f}_N) \rightarrow 0$ in distribution as $N \rightarrow \infty$, which proves the first line of (2.22) with $c' = c_0$. We next bound, by (2.28),

$$\mathbb{E}\left[\left(\int_{\mathbb{R}^2} \varphi(x) \hat{u}_N^{\beta,c}(1, x) - 1\right)^2\right] = \text{Var}[Z_N^{\beta,\omega}(\hat{f}_N)] \leq C \text{Var}[Z_N^{\beta,\omega}(\mathcal{U}_{\rho_N^{\beta,c} \sqrt{N}}^{\text{disc}})] \leq C' \frac{\exp(c_3 e^\vartheta(N,\beta))}{(\rho_N^{\beta,c})^2},$$

which yields the second line of (2.22) with $c'' = 2c_3$. \square

APPENDIX A. THE COARSE-GRAINING PROCEDURE

In this section, we prove Proposition 3.4, which we recall is a *finite-volume criterion* showing how having a small fractional moment at a given time scale triggers an exponential decay of the partition function at larger time scales. We recall the definition (2.12) of the family $\mathcal{M}_1^{\text{disc}}(r)$, where we replace for convenience $|\cdot|$ with $|\cdot|_\infty$.

Proof of Proposition 3.4. Recall that we assume that $L \in \mathbb{N}$ and $\beta \in (0, 1)$ are such that

$$\sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{L})} \mathbb{E}[Z_L^{\beta,\omega}(f)^{1/2}] \leq \frac{1}{113}. \quad (\text{A.1})$$

We will prove the result (3.6) only when N is an integer multiple of L , *i.e.* $N = mL$ for some $m \in \mathbb{N}$; the general case $N \geq L$ follows easily by monotonicity in N . We also assume for simplicity that \sqrt{L} is an integer.

For any integers $s < t$, for any probability measure μ on \mathbb{Z}^2 and any $B \subset \mathbb{Z}^2$, let us introduce the notation

$$Z_{s,t}^{\beta,\omega}(\mu; B) := \mathbf{E}_\mu \left[\exp \left(\sum_{n=s+1}^t (\beta \omega(n, S_n) - \lambda(\beta)) \right) \mathbf{1}_{\{S_t \in B\}} \right],$$

which is the partition function of a polymer with initial distribution μ at time s and constrained to end in B at time t . We also denote $Z_{s,t}^{\beta,\omega}(x,y)$ when μ is a Dirac mass at x and B is reduced to the set $\{y\}$.

Then, for some ‘‘skeleton’’ $\mathcal{Y} = (y_i)_{i \geq 1} \in (\mathbb{Z}^2)^\mathbb{N}$, we define a \sqrt{L} -scale coarse-grained partition function starting from $f \in \mathcal{M}_1^{\text{disc}}$ and with skeleton \mathcal{Y} , by setting for $m \in \mathbb{N}$

$$Z_{mL}^{\beta,\omega}(f; \mathcal{Y}) = \sum_{x_0 \in B(0)} \mu(x_0) \sum_{x_1 \in B(y_1)} \cdots \sum_{x_m \in B(y_m)} \prod_{j=1}^m Z_{(j-1)L, jL}^{\beta,\omega}(x_{j-1}, x_j),$$

where for simplicity we denoted $B(y) = B_{\sqrt{L}}(y) := 2y\sqrt{L} + \llbracket -\sqrt{L}, \sqrt{L} \rrbracket^2$ the (half open) L^∞ ball centered at $2y\sqrt{L}$ of radius \sqrt{L} , in such a way that $(B(y))_{y \in \mathbb{Z}^2}$ is a partition of \mathbb{Z}^2 . Note also that we have used the Markov property to write the partition function constrained to visit the x_i 's as a product of point-to-point partition functions.

Using the standard inequality $(\sum_i z_i)^{1/2} \leq \sum_i z_i^{1/2}$ for non-negative (z_i) , we then get that for any $m \in \mathbb{N}$,

$$Z_{mL}^{\beta,\omega}(f)^{1/2} = \left(\sum_{(y_1, \dots, y_m) \in (\mathbb{Z}^2)^m} Z_{mL}^{\beta,\omega}(f; \mathcal{Y}) \right)^{1/2} \leq \sum_{(y_1, \dots, y_m) \in (\mathbb{Z}^2)^m} Z_{mL}^{\beta,\omega}(f; \mathcal{Y})^{1/2}, \quad (\text{A.2})$$

so that we are reduced to estimating a fractional moment along a skeleton \mathcal{Y} .

Now, let us stress that we have some coarse-grained product structure for $Z_{kL}^{\beta,\omega}(f; \mathcal{Y})$. Indeed, we can write

$$Z_{(k+1)L}^{\beta,\omega}(f; \mathcal{Y}) = Z_{kL}^{\beta,\omega}(f; \mathcal{Y}) Z_{kL, (k+1)L}^{\beta,\omega}(\mu_{k,f,\mathcal{Y}}^{\beta,\omega}; B(y_{k+1})),$$

where $\mu_{k,f,\mathcal{Y}}^{\beta,\omega}$ is the ‘‘ \mathcal{Y} -skeleton polymer’’ probability distribution, supported on $B(y_k)$, given by

$$\mu_{k,f,\mathcal{Y}}^{\beta,\omega}(x) := \frac{1}{Z_{kL}^{\beta,\omega}(f; \mathcal{Y})} \sum_{x_0 \in B(0)} \mu(x_0) \sum_{x_1 \in B(y_1)} \cdots \sum_{x_{k-1} \in B(y_{k-1})} \prod_{j=1}^k Z_{(j-1)L, jL}^{\beta,\omega}(x_{j-1}, x_j) \mathbf{1}_{\{x_k=x\}}.$$

Therefore, taking the conditional expectation with respect to $\mathcal{F}_{kL} = \sigma(\omega(n, z) : n \leq kL, z \in \mathbb{Z}^2)$ and using that $\mu_{k,f,\mathcal{Y}}^{\beta,\omega}$ is \mathcal{F}_{kL} -measurable, we get that

$$\mathbb{E} \left[Z_{(k+1)L}^{\beta,\omega}(f; \mathcal{Y})^{1/2} \mid \mathcal{F}_{kL} \right] \leq Z_{kL}^{\beta,\omega}(f; \mathcal{Y})^{1/2} \times \sup_{\mu: \text{supp}(\mu) \subset B(y_k)} \mathbb{E} \left[Z_{kL, (k+1)L}^{\beta,\omega}(\mu; B(y_{k+1}))^{1/2} \right],$$

where in the supremum μ is a probability distribution. Therefore, if we define

$$\mathcal{Q}(y) := \sup_{\mu: \text{supp}(\mu) \subset B(0)} \mathbb{E} \left[Z_{0,L}^{\beta,\omega}(\mu; B(y))^{1/2} \right],$$

then by translation invariance we get by iteration that

$$\sup_{f \in \mathcal{M}_1^{\text{disc}}(\sqrt{L})} \mathbb{E} \left[Z_{mL}^{\beta,\omega}(f; \mathcal{Y})^{1/2} \right] \leq \prod_{i=1}^m \mathcal{Q}(y_i - y_{i-1}).$$

Therefore, plugged into (A.2) we get that

$$\mathbb{E} \left[(Z_{mL}^{\beta,\omega})^{1/2} \right] \leq \sum_{(y_1, \dots, y_m) \in (\mathbb{Z}^2)^m} \prod_{i=1}^m \mathcal{Q}(y_i - y_{i-1}) = \left(\sum_{y \in \mathbb{Z}^2} \mathcal{Q}(y) \right)^m.$$

It thus only remains to show that under (A.1) we have that $\sum_{y \in \mathbb{Z}^2} \mathcal{Q}(y) \leq e^{-1}$.

First of all, we always have that $\mathcal{Q}(y) \leq \frac{1}{113}$, thanks to Equation (A.1). On the other hand, simply applying Jensen's inequality, we have that

$$\mathbb{E} \left[Z_{0,L}^{\beta,\omega}(\mu; B(y))^{1/2} \right] \leq \sqrt{\sum_{x \in B(0)} \mu(x) \mathbf{P}_x(S_L \in B(y))} \leq \sqrt{\mathbf{P}(S_L \in 2y\sqrt{L} + \llbracket -2\sqrt{L}, 2\sqrt{L} \rrbracket^2)},$$

where we have widened the ball around $2y\sqrt{L}$ by \sqrt{L} to account for the worst case scenario for the starting point $x \in B(0) = \llbracket -\sqrt{L}, \sqrt{L} \rrbracket^2$. Now, notice that $(\pm S_n^{(1)} \pm S_n^{(2)})_{n \geq 0}$ are standard simple random walks in dimension 1, so that

$$\mathbf{P}\left(S_L \in 2y\sqrt{L} + \llbracket -2\sqrt{L}, 2\sqrt{L} \rrbracket^2\right) \leq \mathbf{P}\left(\text{SRW}_L \geq (2|y|_1 - 4)\sqrt{L}\right) \leq e^{-2(|y|_1 - 2)^2},$$

where the last inequality is standard.

Therefore, for any integer threshold $K \geq 1$, we obtain that

$$\sum_{y \in \mathbb{Z}^2} \mathcal{Q}(y) \leq \sum_{|y|_1 \leq K} \frac{1}{113} + \sum_{|y|_1 > K} \sqrt{e^{-2(|y|_1 - 2)^2}} = (2K^2 + 2K + 1) \cdot \frac{1}{113} + \sum_{r > K} 4re^{-(r-2)^2}.$$

Now, it turns out that for $K = 4$ the first term is $\frac{41}{113} \approx 0.3628$ and the second is ≈ 0.0025 , with the sum of the two being smaller than $0.366 < e^{-1}$. This concludes the proof. \square

APPENDIX B. LOWER BOUND ON THE FREE ENERGY

Let us prove the lower bound in (2.26) from Theorem 2.8, using the same strategy as in [BL17]. The idea is to start from the super-additivity of $\mathbb{E}[\log Z_N^{\beta, \omega}]$, which gives that

$$\mathbf{F}(\beta) = \sup_{N \geq 1} \frac{1}{N} \mathbb{E}[\log Z_N^{\beta, \omega}],$$

see e.g. [Com17, Theorem 2.1].

We will apply this inequality for some specific $N_c = N_c(\beta)$ such that $\sigma^2(\beta)R_{N_c} = 1$ (in other words such that $\vartheta(\beta, N) := \pi R_N - \frac{\pi}{\sigma^2(\beta)} = 0$ in (2.11)), which therefore gives that

$$\mathbf{F}(\beta) \geq \frac{1}{N_c(\beta)} \mathbb{E}[\log Z_{N_c}^{\beta, \omega}] \geq c e^{-\frac{\pi}{\sigma^2(\beta)}} \mathbb{E}[\log Z_{N_c}^{\beta, \omega}],$$

using that $\pi R_{N_c} = \log N_c + \alpha + o(1)$ as $N_c \rightarrow \infty$ (or $\beta \downarrow 0$), see (2.5). What remains to prove is therefore the following lemma, which estimates $\mathbb{E}[\log Z_{N_c}^{\beta, \omega}]$.

Lemma B.1. *Let $N \geq 1$ and let $\beta = \beta_c(N)$ be such that $\sigma^2(\beta_c)R_N = 1$. Then, there is some constant $C > 0$ such that, for all $N \geq 2$, we have*

$$\mathbb{E}[\log Z_N^{\beta_c(N), \omega}] \geq -C(\log N)^4.$$

With this lemma at hand, and since $\beta_c(N_c(\beta)) = \beta$, this readily gives that $\mathbb{E}[\log Z_N^{\beta_c(N), \omega}] \geq -C'\sigma^2(\beta)^{-4}$, using also that $\log N_c(\beta) = \frac{\pi}{\sigma^2(\beta)} + \alpha + o(1)$. This concludes the proof of the lower bound in Theorem 2.8.

Remark B.2. The bound in Lemma B.1 is of course not optimal. In particular, we expect that* $\mathbb{E}[\log Z_N^{\beta_c(N), \omega}] \sim -c \log \log N$. Combined with super-additivity, this would give the lower bound $-(cst.) \log\left(\frac{1}{\sigma^2(\beta)}\right) e^{-\pi/\sigma^2(\beta)}$ for the free energy. In fact, we believe that our upper bound in Theorem 2.8 is not completely sharp since we obtain it from an averaged starting point. We would therefore expect the following behavior for the free energy:

$$\mathbf{F}(\beta) \sim -c \log\left(\frac{1}{\sigma^2(\beta)}\right) e^{-\pi/\sigma^2(\beta)} \quad \text{as } \beta \downarrow 0.$$

Proof of Lemma B.1. The proof relies on concentration inequalities to estimate the left tail of $\log Z_N^\beta$. We use the following concentration inequality from [CTT17, Prop. 3.4].

Proposition B.3. *Assume that the environment is bounded, i.e. $|\omega| \leq K$, and let f be a convex function. Then, there exists some constant $c > 0$ such that for any a, M and $t > 0$, we have*

$$\mathbb{P}(f(\omega) \geq a; |\nabla f| \leq M) \mathbb{P}(f(\omega) \leq a - t) \leq 2e^{-c \frac{t^2}{K^2 M^2}}.$$

*In fact, one conjectures that $\lambda_N^{-1}(\log Z_N^{\beta_c(N), \omega} - \lambda_N^2)$ converges in distribution if $\lambda_N = \log \log N$; we expect asymptotic normality in the lower quasi-critical regime $\vartheta_N \rightarrow -\infty$, $\vartheta_N = o(\log N)$, with $\lambda_N = \log\left(\frac{\log N}{|\vartheta_N|}\right)$.

We will apply this result to $\log Z_N^{\beta,\omega}$, which is a convex function in ω , whose norm of the gradient is given by

$$|\nabla \log Z_N^{\beta,\omega}|^2 = \sum_{n=1}^N \sum_{|x| \leq n} \left(\frac{\partial}{\partial \omega_{n,x}} \log Z_N^{\beta,\omega} \right)^2.$$

Our first lemma controls the first factor in Proposition B.3.

Lemma B.4. *Assume that $\sigma^2(\beta)R_N \leq e^{\frac{\vartheta}{\log N}}$ for some $\vartheta \in \mathbb{R}$. Then, there is a constant $C = C(\vartheta) > 0$ such that*

$$\mathbb{P}\left(\log Z_N^{\beta,\omega} \geq -1; |\nabla \log Z_N^{\beta,\omega}|^2 \leq C(\log N)^3\right) \geq \frac{1}{C \log N}.$$

Then, in the case $\vartheta = 0$, applying Proposition B.3 with $a = -1$ and $M = \sqrt{C}(\log N)^{3/2}$, we get that for a bounded environment $|\omega| \leq K$,

$$\mathbb{P}(\log Z_N^{\beta,\omega} \leq -1 - t) \leq 2C \log N e^{-\frac{c}{C} \frac{t^2}{K^2(\log N)^3}}.$$

We can in fact reduce to a bounded environment with a large constant $K = (\log N)^{3/2}$: define $\tilde{\omega}_{n,x} = \omega_{n,x} \mathbf{1}_{\{|\omega_{n,x}| \leq (\log N)^{3/2}\}}$, and note that

$$\mathbb{P}(\exists n \in \llbracket 1, N \rrbracket, |x| \leq n \text{ such that } \tilde{\omega}_{n,x} \neq \omega_{n,x}) \leq 9N^3 \mathbb{P}(|\omega| \geq (\log N)^{3/2}) \leq 9N^3 e^{-c_0(\log N)^{3/2}}.$$

Therefore,

$$\mathbb{P}(\log Z_N^{\beta,\omega} \leq -1 - t) \leq \mathbb{P}(\log Z_N^{\beta,\tilde{\omega}} \leq -1 - t) + 9N^3 e^{-c_0(\log N)^{3/2}}.$$

Note that setting $\tilde{\lambda}(\beta) = \log \mathbb{E}[e^{\beta \tilde{\omega}}]$ and $\tilde{\sigma}^2(\beta) = e^{\tilde{\lambda}(2\beta) - 2\tilde{\lambda}(\beta)}$, we can check that for $\beta \in (0, 1)$ we have $\tilde{\lambda}(\beta) = \lambda(\beta) + O(e^{-c(\log N)^{3/2}})$ and $\tilde{\sigma}^2(\beta) = \sigma^2(\beta) + O(e^{-c(\log N)^{3/2}})$. In particular we can harmlessly replace $\lambda(\beta)$ by $\tilde{\lambda}(\beta)$ in $Z_N^{\beta,\tilde{\omega}}$, to which we can then apply Lemma B.4, say with $\vartheta = 1$ instead of $\vartheta = 0$. Applying Proposition B.3 with $K = (\log N)^{3/2}$, $a = -1$, $M = \sqrt{C}(\log N)^{3/2}$, we end up with

$$\mathbb{P}(\log Z_N^{\beta,\omega} \leq -1 - t) \leq 2C \log N e^{-\frac{c}{C} \frac{t^2}{(\log N)^6}} + 9N^3 e^{-c_0(\log N)^{3/2}}.$$

Then, using that $-\mathbb{E}[\log Z_N^{\beta,\omega}] \leq 1 + \int_1^\infty \mathbb{P}(-\log Z_N^{\beta,\omega} \geq u) du$, we can split the integral into two parts. The first part is

$$\int_1^{N^2} \mathbb{P}(\log Z_N^{\beta,\omega} \leq -1 - u) du \leq C'(\log N)^4 + 9N^5 e^{-c_0(\log N)^{3/2}}$$

where we have used the upper bound on the left tail of $\log Z_N^{\beta,\omega}$ found above. For the remaining part, we use a very rough bound: writing $\sum_{n=1}^N \beta \omega_{n,S_n} \geq \beta N \min_{n \in \llbracket 1, N \rrbracket, |x| \leq N} \{\omega_{n,x}\}$, we get that for $u \geq 2\lambda(\beta)N$

$$\begin{aligned} \mathbb{P}(\log Z_N^{\beta,\omega} \leq -u) &\leq \mathbb{P}\left(\beta N \min_{n \in \llbracket 1, N \rrbracket, |x| \leq N} \{\omega_{n,x}\} - \lambda(\beta)N \leq -u\right) \\ &\leq \mathbb{P}\left(\min_{n \in \llbracket 1, N \rrbracket, |x| \leq N} \{\omega_{n,x}\} \leq -\frac{1}{2} \frac{u}{\beta N}\right) \leq 9N^3 e^{-c_0 \frac{u}{2\beta N}}. \end{aligned}$$

Thus, the second part of the integral $\int_{N^2}^\infty \mathbb{P}(\log Z_N^{\beta,\omega} \leq -1 - u) du$ is bounded by $c\beta N^4 e^{-c_0 N/2\beta}$, which is negligible compared to the first term. This concludes the proof of Lemma B.1. \square

Proof of Lemma B.4. First of all, let us write

$$\begin{aligned} \mathbb{P}(\log Z_N^{\beta,\omega} \geq -1; |\nabla \log Z_N^{\beta,\omega}|^2 \leq C(\log N)^3) \\ = \mathbb{P}(Z_N^{\beta,\omega} \geq e^{-1}) - \mathbb{P}(Z_N^{\beta,\omega} \geq e^{-1}; |\log Z_N^{\beta,\omega}|^2 > C(\log N)^3). \end{aligned}$$

For the first term, we use Paley–Zygmund inequality to get that

$$\mathbb{P}(Z_N^{\beta,\omega} \geq e^{-1}) \geq (1 - e^{-1})^2 \frac{1}{\mathbb{E}[(Z_N^{\beta,\omega})^2]} \geq \frac{c}{\log N},$$

where we have used that, in the critical window, $\mathbb{E}[(Z_N^{\beta,\omega})^2] \leq c \log N$ for some constant $c = c(\vartheta)$. For the second term, a straightforward calculation gives that

$$|\nabla \log Z_N^{\beta,\omega}|^2 = \frac{\beta^2}{(Z_N^{\beta,\omega})^2} \mathbf{E}^{\otimes 2} \left[\sum_{n=1}^N \mathbf{1}_{\{S_n = \tilde{S}_n\}} e^{\sum_{n=1}^N \beta(\omega_n, S_n + \omega_n, \tilde{S}_n) - 2\lambda(\beta)} \right].$$

Bounding $\frac{1}{(Z_N^{\beta,\omega})^2} \leq e^2$ on the event $Z_N^{\beta,\omega} \geq e^{-1}$, we get that, applying also Markov's inequality

$$\begin{aligned} \mathbb{P}\left(Z_N^{\beta,\omega} \geq e^{-1}; |\nabla \log Z_N^{\beta,\omega}|^2 > C(\log N)^3\right) \\ \leq \frac{e^2}{C(\log N)^3} \mathbf{E}^{\otimes 2} \left[\beta^2 \sum_{n=1}^N \mathbf{1}_{\{S_n = \tilde{S}_n\}} e^{\lambda_2(\beta) \sum_{n=1}^N \mathbf{1}_{\{S_n = \tilde{S}_n\}}} \right], \end{aligned}$$

with $\lambda_2(\beta) = \lambda(2\beta) - 2\lambda(\beta)$. Then, we can use that, at criticality, we have the following bound, that we prove below

Claim B.5. *Assume that $\sigma^2(\beta)R_N \leq e^{\frac{\vartheta}{\log N}}$ for some $\vartheta \in \mathbb{R}_+$. Then there is a constant $C' = C'(\vartheta)$ such that*

$$\mathbf{E}^{\otimes 2} \left[\beta^2 \sum_{n=1}^N \mathbf{1}_{\{S_n = \tilde{S}_n\}} e^{\lambda_2(\beta) \sum_{n=1}^N \mathbf{1}_{\{S_n = \tilde{S}_n\}}} \right] \leq C' (\log N)^2.$$

All together, this gives that

$$\mathbb{P}\left(\log Z_N^{\beta,\omega} \geq -1; |\nabla \log Z_N^{\beta,\omega}|^2 \geq C(\log N)^3\right) \geq \frac{c}{\log N} - \frac{e^2 C'}{C \log N} \geq \frac{c}{2 \log N},$$

provided that we had fixed C large enough. \square

Proof of Claim B.5. Recalling that $\sigma^2(\beta) = e^{\lambda_2(\beta)} - 1$, we can perform the following chaos expansion:

$$\begin{aligned} \mathbf{E}^{\otimes 2} \left[\beta^2 \sum_{n=1}^N \mathbf{1}_{\{S_n = \tilde{S}_n\}} (1 + \sigma^2(\beta))^{\sum_{n=1}^N \mathbf{1}_{\{S_n = \tilde{S}_n\}}} \right] \\ = \beta^2 \sum_{k=0}^{\infty} \sigma^2(\beta)^k \sum_{1 \leq n_1 < \dots < n_k \leq N} \sum_{n=1}^N \mathbf{P}^{\otimes 2}(S_{n_i} = S_{n_i} \forall i \in \{1, \dots, k\}, S_n = \tilde{S}_n). \end{aligned} \tag{B.1}$$

Now, we consider two contributions. First, if $n \in \{n_1, \dots, n_k\}$, this gives a term

$$\beta^2 \sum_{k=0}^{\infty} k \sigma^2(\beta)^k \sum_{1 \leq n_1 < \dots < n_k \leq N} \prod_{i=1}^k u(n_i - n_{i-1}),$$

where k is simply a combinatorial factor due to the choice of index $i \in \{1, \dots, k\}$ such that $n = n_i$. Second, if $n \notin \{n_1, \dots, n_k\}$, this gives a term

$$\beta^2 \sum_{k=0}^{\infty} (k+1) \sigma^2(\beta)^k \sum_{1 \leq n_1 < \dots < n_{k+1} \leq N} \prod_{i=1}^k u(n_i - n_{i-1}),$$

where the combinatorial factor is due to the choice of interval (n_{i-1}, n_i) in which n falls. All together, after a change of index for the second term, the left-hand side in (B.1) is equal to

$$\beta^2 (1 + \sigma^2(\beta)^{-1}) \sum_{k=0}^{\infty} k \sigma^2(\beta)^k \sum_{1 \leq n_1 < \dots < n_k \leq N} \prod_{i=1}^k u(n_i - n_{i-1}).$$

Noticing that $\beta^2(1 + \sigma^2(\beta)^{-1})$ is bounded by a constant, we therefore focus on sum. We use the following upper bound, see [CSZ19a, Lemma 5.4]: there is a constant $c > 0$ such that, for every $k \geq 1$

$$\frac{1}{(R_N)^k} \sum_{1 \leq n_1 < \dots < n_k \leq N} \prod_{i=1}^k u(n_i - n_{i-1}) \leq e^{-c \frac{k}{\log N} \log^+(\frac{k}{\log N})}.$$

With this bound at hand, we get that

$$\begin{aligned} \sum_{k=0}^{\infty} k \sigma^2(\beta)^k \sum_{1 \leq n_1 < \dots < n_k \leq N} \prod_{i=1}^k u(n_i - n_{i-1}) &\leq \sum_{k=0}^{\infty} k (\sigma^2(\beta) R_N)^k e^{-c \frac{k}{\log N} \log^+(\frac{k}{\log N})} \\ &\leq (\log N)^2 \times \frac{1}{\log N} \sum_{k=0}^{\infty} \frac{k}{\log N} e^{\vartheta \frac{k}{\log N} - c \frac{k}{\log N} \log^+(\frac{k}{\log N})}, \end{aligned}$$

where we have also used that $\sigma^2(\beta) R_N \leq e^{\vartheta/\log N}$. The last term converges to $\int_0^{\infty} t e^{\vartheta t - ct \log_+(t)} dt$ by a Riemann approximation, so in particular it is bounded by some constant (that depends on ϑ). This concludes the proof. \square

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UNIVERSITÉ SORBONNE PARIS NORD, LABORATOIRE D'ANALYSE, GÉOMÉTRIE ET APPLICATIONS, CNRS UMR 7539, 99 AV. J-B CLÉMENT, 93436 VILLETANEUSE, FRANCE AND INSTITUT UNIVERSITAIRE DE FRANCE

Email address: `quentin.berger@math.univ-paris13.fr`

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA, VIA COZZI 55, 20125 MILANO, ITALY

Email address: `francesco.caravenna@unimib.it`

DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA, VIA COZZI 55, 20125 MILANO, ITALY

Email address: `nicola.turchi@unimib.it`