STRONG DISORDER FOR STOCHASTIC HEAT FLOW AND 2D DIRECTED POLYMERS

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ABSTRACT. The critical 2D Stochastic Heat Flow (SHF) is a universal process of random measures, arising as the scaling limit of two-dimensional directed polymer partition functions (or solutions to the stochastic heat equation with mollified noise) under a critical renormalisation of disorder strength. We investigate the SHF in the *strong disorder* (or *super-critical*) regime, proving that it vanishes locally with an optimal doubly-exponential decay rate in the disorder intensity. We also establish a strengthened version of this result for 2D directed polymers, from which we deduce sharp bounds on the free energy. Our proof is based on estimates for truncated and fractional moments, exploiting refined change of measure and coarse-graining techniques.

1. INTRODUCTION AND MAIN RESULT

The critical 2D Stochastic Heat Flow with disorder strength ϑ, denoted SHF(ϑ), is a stochastic process $(\mathscr{Z}^{\vartheta}_{s,t}(\mathrm{d}x, \mathrm{d}y))_{0 \le s \le t < \infty}$ of random measures on $\mathbb{R}^2 \times \mathbb{R}^2$. It was introduced in [CSZ23a] as the universal limit of 2D directed polymer partition functions, which we recall below, under a critical rescaling of disorder strength. It also arises as the scaling limit of solutions of the 2D stochastic heat equation with mollified noise, see [Tsa24], where an axiomatic definition is also provided. The fact that space dimension two is *critical* for the stochastic heat equation and directed polymers makes the SHF especially interesting, as a rare example of *non-Gaussian scaling limit in the critical dimension and at the critical point.* We refer to the lecture notes [CSZ24] for an extended discussion, as well as additional background and connections to singular SPDEs.

1.1. **A quick overview of the SHF literature.** Many features of the SHF have been investigated, in particular its moments. The second moment was first studied in [BC98] in the context of solutions to the 2D stochastic heat equation, exploiting a connection with the delta-Bose gas from [ABD95]; refined results, also in the setting of directed polymers, were later obtained in [CSZ19a]. The third moment was established in [CSZ19b] before all integer moments were derived in [GQT21]; see also [Che24] for further connections with the delta-Bose gas.

The asymptotic analysis of moments is challenging, due to their intricate structure, but important progress has been obtained recently in [GN25], where a sharp lower bound on their growth rate was established through a novel connection between moments of the SHF and the Gaussian Free Field, as well as in [LZ24], where small-scale asymptotics were derived, extending the approach developed by [CZ23] in the sub-critical regime.

Concerning the properties of the SHF as a random measure, estimates on its singularity and regularity were obtained in [CSZ25]. It was also proved in [CSZ23b] that the SHF is not a *Gaussian Multiplicative Chaos (GMC)* on $\mathbb{R}^2$ via comparison of moments. Very recently, the SHF was shown in [CT25] to enjoy a *conditional GMC structure* on path space, which yields as corollaries the strict positivity/full support property, also obtained independently in [Nak25b], and a qualitative asymptotic behavior for strong disorder, discussed below.

Date: 4th August 2025.

2020 *Mathematics Subject Classification.* Primary: 82B44; Secondary: 60K35, 82D60.

Key words and phrases. Change of Measure, Coarse-Graining, Directed Polymer in Random Environment, Disordered Systems, Size Bias, Stochastic Heat Equation, Stochastic Heat Flow.

Other features of the SHF include a Chapman-Kolmogorov property and the construction of associated polymer measures [CM24], continuity in time and the already mentioned characterization [Tsa24]. Let us also mention the black noise property [GT25] and an enhanced noise sensitivity property for directed polymer partition functions [CD25], which both yield independence between SHF and white noise. Recent progress on a martingale characterization of the SHF has also been obtained [Nak25a, Che25].

Nearly all of these results refer to the SHF for a *fixed disorder strength* $\vartheta \in \mathbb{R}$, which corresponds to the *critical* regime. Some results are also available in the *weak-disorder* limit $\vartheta \to -\infty$, corresponding to the *sub-critical* regime: see, e.g., the Gaussian fluctuations in [CCR25, Theorem 1.2] and the asymptotic log-normality in [CSZ24, Theorem 1.2]. Corresponding results, and many others, have been obtained in the sub-critical regime for directed polymers and the stochastic heat equation, for which we refer again to [CSZ24].

By contrast, much less is known on the SHF in the *strong disorder* limit $\vartheta \to \infty$, which corresponds to the *super-critical* regime. In this paper, we investigate precisely this regime. Our main results are *sharp novel estimates for the SHF*, see Theorems 1.1 and 1.5, which quantify the rate at which its *mass escapes to infinity*. These can be deduced from analogous results for the partition functions of 2D directed polymers, see Theorems 1.6 and 1.8, which take even stronger forms. From this we also obtain *refined estimates on the free energy*, see Theorem 1.11, which improve on the best available bounds in the literature.

Our proofs are based on estimating *truncated and fractional moments*, see Remark 1.2, exploiting *change of measure* and *coarse-graining*. We present in Sections 2 and 3 refined versions of these classical techniques, combined with *change of scale arguments* of independent interest. The main technical novelty of our approach is the identification of a *new proxy for the partition function* of 2D directed polymers, which is obtained from a suitable *coarse-grained chaos expansion* (see Section 3.3). Besides the technical aspects, we believe that our strategy is of wide applicability and we expect it to be useful in other contexts as well.

1.2. **Strong disorder for SHF.** We focus on the one-time marginal of the SHF:

$$\mathscr{Z}_t^{\vartheta}(\mathrm{d}x) := \mathscr{Z}_{0,t}^{\vartheta}(\mathrm{d}x, \mathbb{R}^2)$$

which is a random measure on $\mathbb{R}^2$ for each fixed $t > 0$ and $\vartheta \in \mathbb{R}$. Its first moment $\mathbb{E}[\mathscr{Z}_t^{\vartheta}(\mathrm{d}x)] = \mathrm{d}x$ is simply the Lebesgue measure on $\mathbb{R}^2$. In particular, using the functional notation

$$\mathscr{Z}_t^{\vartheta}(\varphi) := \int_{\mathbb{R}^2} \varphi(x)\, \mathscr{Z}_t^{\vartheta}(\mathrm{d}x)\,,$$

we have $\mathbb{E}[\mathscr{Z}_t^{\vartheta}(\varphi)] = 1$ for any probability density φ on $\mathbb{R}^2$ (we call φ the *initial condition*).

It turns out that the second moment diverges for strong disorder: for any probability density φ

$$\lim_{\vartheta \to \infty} \mathbb{E}[\mathscr{Z}_t^{\vartheta}(\varphi)^2] = \infty\,.$$

This implies that higher moments diverge quickly: namely, one can easily deduce by size-biasing and Jensen's inequality (see [CSZ25, Remark 1.14]) that

$$\forall h > 2: \qquad \frac{\mathbb{E}[\mathscr{Z}_t^{\vartheta}(\varphi)^h]}{\mathbb{E}[\mathscr{Z}_t^{\vartheta}(\varphi)^2]^{\frac{h}{2}}} \xrightarrow[\vartheta \to \infty]{} \infty$$

which expresses a so-called *intermittent behavior*.

In this setting, it is natural to conjecture that $\mathscr{Z}_t^{\vartheta}(\varphi)$ actually vanishes for strong disorder:

$$\mathscr{Z}_t^{\vartheta}(\varphi) \xrightarrow[\vartheta \to \infty]{} 0 \quad \text{in probability}\,. \tag{1.1}$$

We prove in this paper a *quantitative version* of this convergence, where we allow for varying initial conditions φ with possibly *diverging support* as $\vartheta \to \infty$, and we establish *optimal bounds*, displaying a doubly-exponential decay rate in the disorder strength ϑ.

Let us denote by $\mathcal{M}_1(r)$ the set of probability densities with support in the ball of radius r:

$$\mathcal{M}_1(r) = \Big\{ \varphi : \mathbb{R}^2 \to [0,\infty) \quad \text{s.t.} \quad \int_{\mathbb{R}^2} \varphi(x) = 1, \quad \varphi(x) = 0 \text{ for } |x| > r \Big\} . \tag{1.2}$$

We can now state our first main result.

Theorem 1.1 (Strong disorder for SHF)**.** *For any $t > 0$ and any $\varphi : \mathbb{R}^2 \to \mathbb{R}$ compactly supported, we have $\mathscr{Z}_t^\vartheta(\varphi) \to 0$ in probability as $\vartheta \to \infty$. This can be quantified as follows: there exist universal constants $c_0, c_1, c_2 \in (0,\infty)$ such that, for any $t \geq 0$ and $\vartheta \in \mathbb{R}$,*

$$\frac{1}{c_1} \, \mathrm{e}^{-c_1 \, t \, \mathrm{e}^{\vartheta}} \leq \sup_{\varphi \in \mathcal{M}_1\left(\sqrt{(\mathrm{e}^{c_0 \, t \, \mathrm{e}^{\vartheta}})\, t}\right)} \mathbb{E}\big[\mathscr{Z}_t^\vartheta(\varphi) \wedge 1\big] \leq \frac{1}{c_2} \, \mathrm{e}^{-c_2 \, t \, \mathrm{e}^{\vartheta}} . \tag{1.3}$$

Note that the convergence (1.1) follows by the upper bound in (1.3) and Markov's inequality, since for a random variable $Z \geq 0$ and $\varepsilon > 0$ we can bound $\mathbb{P}(Z \geq \varepsilon) \leq (\varepsilon \wedge 1)^{-1} \, \mathbb{E}[Z \wedge 1]$.

We prove Theorem 1.1 in Section 2, deducing it from a corresponding result for 2D directed polymers, see Theorems 1.6 and 1.8.

Remark 1.2. Independently of our work, the result (1.1) was very recently obtained in [CT25], deducing it from a *conditional GMC structure* enjoyed by the SHF. Our proof exploits different techniques, based on change of measure and coarse-graining (see Sections 2 and 3), which additionally yield the quantitative estimate (1.3) and its consequences, to be discussed below.

Remark 1.3 (Truncated vs. fractional moments). We quantify the convergence $\mathscr{Z}_t^\vartheta(\varphi) \to 0$ via the *truncated mean* $\mathbb{E}[\mathscr{Z}_t^\vartheta(\varphi) \wedge 1]$, which yields interesting information on the *total variation distance* between the law of the disorder and its *size-biased version* by $\mathscr{Z}_t^\vartheta(\varphi)$ (see Remark 3.2). Similar bounds hold for *fractional moments* $\mathbb{E}[\mathscr{Z}_t^\vartheta(\varphi)^\gamma]$ with $\gamma \in (0,1)$ (see Lemma 2.2 below). We will use both truncated mean and fractional moments, since each quantity has its own advantages and limitations; see Section 3 for a discussion.

Remark 1.4 (Scaling covariance, strong disorder and large time). The dependence of the bounds in (1.3) on the parameters t and ϑ agrees with the *scaling covariance property of the SHF* established in [CSZ23a, Theorem 1.2], which states that for any t, ϑ, φ and for any $a > 0$ we have the equality in distribution

$$\mathscr{Z}_{at}^\vartheta(\varphi_{\sqrt{a}}) \overset{d}{=} \mathscr{Z}_t^{\vartheta + \log a}(\varphi) \qquad \text{where we set} \;\; \varphi_{\sqrt{a}}(x) := \tfrac{1}{a} \, \varphi\big(\tfrac{x}{\sqrt{a}}\big) . \tag{1.4}$$

Interestingly, this property allows us to connect the regimes of strong disorder $\vartheta \to \infty$ and large time $t \to \infty$: replacing t by t/a and setting $a = \mathrm{e}^{-\vartheta}$, resp. $a = t$, we obtain

$$\mathscr{Z}_t^\vartheta(\varphi_{\sqrt{\mathrm{e}^{-\vartheta}}}) \overset{d}{=} \mathscr{Z}_{t \, \mathrm{e}^\vartheta}^0(\varphi) , \qquad \mathscr{Z}_t^\vartheta(\varphi_{\sqrt{t}}) \overset{d}{=} \mathscr{Z}_1^{\vartheta + \log t}(\varphi) . \tag{1.5}$$

Since it was proved in [CSZ25, Theorem 1.4] that $\mathscr{Z}_t^\vartheta(\varphi) \to 0$ in probability as $t \to \infty$ for fixed ϑ, we could deduce from the first relation in (1.5) that $\mathscr{Z}_t^\vartheta(\varphi_{\sqrt{\mathrm{e}^{-\vartheta}}}) \to 0$ as $\vartheta \to \infty$. We stress, however, that this property is *much weaker* than (1.1), and even more than (1.3), because shrinking the support of the initial condition φ helps convergence to zero*.

Since we now know that property (1.1) holds, we can deduce from the second relation in (1.5) that $\mathscr{Z}_t^\vartheta(\varphi_{\sqrt{t}}) \to 0$ in probability as $t \to \infty$ for fixed ϑ, which strengthens [CSZ25, Theorem 1.4]. However, the bounds in (1.3) yield a much stronger result, which we next discuss.

Theorem 1.1 allows us to derive an exponential bound on the *total mass* $\mathscr{Z}_t^\vartheta(B(0,r))$ *assigned by the SHF to a ball* $B(0,r)$ *with large radius* $r \to \infty$. Interestingly, we are not limited to the regime $\vartheta \to \infty$ for fixed $t > 0$, but we can consider *any regime of $t > 0$, $\vartheta \in \mathbb{R}$ for which* $t \, \mathrm{e}^\vartheta \to \infty$, which also includes $t \to \infty$ for fixed $\vartheta \in \mathbb{R}$. The proof of the next result is given in

*For instance, by [CSZ25, Theorem 1.1], we have $\mathscr{Z}_t^\vartheta(\varphi_{\sqrt{a}}) \to 0$ in probability as $a \downarrow 0$ even *for fixed* t, ϑ.

Section 2. We write "with high probability" to mean that the probability of a given statement converges to 1.

Theorem 1.5 (Vanishing mass of SHF in large balls)**.** *There exist universal constants $c, \delta > 0$ such that*

$$\mathscr{Z}_t^\vartheta\Big(B\big(0, \sqrt{(\mathrm{e}^{c\, t\, \mathrm{e}^\vartheta})\, t}\,\big)\Big) \le t\, \mathrm{e}^{-\delta\, t\, \mathrm{e}^\vartheta} \quad \textit{with high probability as} \;\; t\, \mathrm{e}^\vartheta \to \infty\,.$$

1.3. **Strong disorder for 2D directed polymers.** The SHF(ϑ) was obtained in [CSZ23a] as the limit of 2D directed polymer partition functions in some appropriate *critical window* of disorder strength. As already mentioned, we are going to derive our Theorem 1.1 from a corresponding result for the directed polymer model, whose definition we now recall.

Let $S = (S_n)_{n\ge 0}$ be the simple (symmetric, nearest-neighbor) random walk on $\mathbb{Z}^2$, and denote $\mathbf{P}_x$ its law when $S_0 = x \in \mathbb{Z}^2$; let also $\mathbf{E}_x$ be the corresponding expectation. We simply write $\mathbf{P}, \mathbf{E}$ for $\mathbf{P}_0, \mathbf{E}_0$. Additionally, consider a collection $\omega = \big(\omega(n,x)\big)_{n\in\mathbb{N},\, x\in\mathbb{Z}^2}$ of i.i.d. random variables independent of S with law denoted by $\mathbb{P}$. Using an abuse of notation to denote by ω a generic random variable $\omega_{n,x}$, we assume that

$$\mathbb{E}[\omega] = 0\,, \qquad \mathbb{E}[\omega^2] = 1\,, \qquad \lambda(\beta) := \log \mathbb{E}[\mathrm{e}^{\beta\omega}] < +\infty \;\;\text{ for all } \beta \in \mathbb{R}\,. \tag{1.6}$$

For $N \in \mathbb{N}$ and $\beta > 0$, the point-to-plane (1+2-dimensional) directed polymer model is defined as the Gibbs measure with Hamiltonian $H_N^{\beta,\omega}(S) := \sum_{n=1}^N (\beta\omega(n, S_n) - \lambda(\beta))$. We are interested in the *point-to-plane partition function* started at $x \in \mathbb{Z}^2$, defined by

$$Z_N^{\beta,\omega}(x) := \mathbf{E}_x\Big[\mathrm{e}^{H_N^{\beta,\omega}(S)}\Big] \qquad \text{with} \qquad H_N^{\beta,\omega}(S) := \sum_{n=1}^N (\beta\omega(n, S_n) - \lambda(\beta))\,. \tag{1.7}$$

We may view $(Z_N^{\beta,\omega}(x))_{x\in\mathbb{Z}^2}$ as a random field: for a function $f \in \ell^1(\mathbb{Z}^2)$ we define

$$Z_N^{\beta,\omega}(f) := \sum_{x\in\mathbb{Z}^2} f(x)\, Z_N^{\beta,\omega}(x)\,, \tag{1.8}$$

which is the integral of f with respect to the random measure $\sum_{x\in\mathbb{Z}^2} Z_N^{\beta,\omega}(x)\,\delta_x$. The main result of [CSZ23a] shows that this measure, diffusively rescaled, converges to a unique limit, which they named Critical 2D Stochastic Heat Flow SHF(ϑ), provided the disorder strength β is rescaled in the so-called *critical window*, which we now define.

Recalling (1.6), we define for $\beta \ge 0$ and $N \in \mathbb{N}$

$$\sigma^2(\beta) := \mathbb{V}\mathrm{ar}\big(\mathrm{e}^{\beta\omega - \lambda(\beta)}\big) = \mathrm{e}^{\lambda(2\beta) - 2\lambda(\beta)} - 1\,, \qquad R_N := \sum_{n=1}^N \mathbf{P}(S_{2n} = 0)\,. \tag{1.9}$$

Note that $\sigma^2(\beta) \sim \beta^2$ as $\beta \downarrow 0$ and we can write, see [CSZ19a, Proposition 3.2],

$$R_N = \frac{1}{\pi}(\log N + \alpha_N) \qquad \text{with} \qquad \lim_{N\to\infty} \alpha_N \;=\; \alpha \;:=\; 4\log 2 + \gamma - \pi \;\simeq\; 0.208\,. \tag{1.10}$$

Then, the critical window corresponds to taking

$$\beta = \beta_N(\vartheta) \downarrow 0 \qquad \text{such that} \qquad \sigma^2(\beta) = \frac{1}{R_N}\Big(1 + \frac{\vartheta + o(1)}{\log N}\Big)\,, \tag{1.11}$$

where $\vartheta \in \mathbb{R}$ is a fixed parameter, called *disorder strength* in the critical regime.

We can now recall the main result of [CSZ23a]. Denote by $\mathbb{Z}^2_{\text{even}}$ the set of points $x = (x^1, x^2) \in \mathbb{Z}^2$ with $x^1 + x^2$ even (to comply with the random walk periodicity). Given an integrable function $\varphi : \mathbb{R}^2 \to \mathbb{R}$, define its rescaled version $\varphi^{(N)} : \mathbb{Z}^2_{\text{even}} \to \mathbb{R}$ by $\varphi^{(N)}(x) := \frac{1}{2/N}\int_{|y - \frac{x}{\sqrt{N}}|_1 \le \frac{1}{\sqrt{N}}} \varphi(y)\, \mathrm{d}y$ (where $|\cdot|_1$ denotes the ℓ^1 norm). We then have the convergence in distribution

$$\forall \vartheta \in \mathbb{R}\;: \qquad Z_{\lfloor Nt\rfloor}^{\beta_N,\omega}(\varphi^{(N)}) \xrightarrow[N\to\infty]{\mathrm{d}} \mathscr{Z}_t^\vartheta(\varphi) \qquad \text{for } \beta_N = \beta_N(\vartheta) \text{ as in (1.11)}\,. \tag{1.12}$$

The quasi-critical window. In this article, we go *beyond the critical window*, *i.e.* we allow for diverging $\vartheta = \vartheta_N \to \infty$ as $N \to \infty$. We first consider the regime $\vartheta = o(\log N)$, which we call *(upper) quasi-critical window* by analogy with [CCR25], where the regime $\vartheta \to -\infty$ with $|\vartheta| = o(\log N)$ was considered and called (lower) quasi critical.

In order to allow for $\vartheta \to \infty$, we need to refine the correspondence (1.11) between β, N and ϑ as follows[†]:

$$\sigma^2(\beta) = \frac{1}{R_N - \frac{\vartheta}{\pi}} = \frac{1}{R_N} \frac{1}{1 - \frac{\vartheta}{\pi R_N}} \,. \tag{1.13}$$

Let us stress that *this relation is equivalent to* (1.11) *for any fixed* $\vartheta \in \mathbb{R}$ (and even for $|\vartheta| = o(\sqrt{\log N})$, see the expansion (1.10) of R_N, but *not* for $|\vartheta| \geq (cst.)\sqrt{\log N}$).

We introduce, similarly to (1.2),

$$\mathcal{M}_1^{\mathrm{disc}}(r) = \Big\{ f : \mathbb{Z}^2 \to [0,1] \quad \text{s.t.} \quad \sum_{z \in \mathbb{Z}^2} f(z) = 1, \quad f(z) = 0 \ \forall |z| > r \Big\} \,.$$

We can now state our first result for 2D directed polymers, proved in Section 2.

Theorem 1.6 (Strong disorder for 2D directed polymers: quasi-critical regime)**.** *There exist universal constants* $c_0, c_1, c_2 \in (0, \infty)$ *such that the following holds: given any* $0 \leq \vartheta_N = o(\log N)$*, if we define* $\beta = \beta_N$ *such that* (1.13) *holds with* $\vartheta = \vartheta_N$*, then for* N *large enough we have*

$$\frac{1}{c_1}\, \mathrm{e}^{-c_1 \mathrm{e}^{\vartheta_N}} \leq \sup_{f \in \mathcal{M}_1^{\mathrm{disc}}\left(\sqrt{\mathrm{e}^{c_0 \mathrm{e}^{\vartheta_N}} N}\right)} \mathbb{E}[Z_N^{\beta_N,\omega}(f) \wedge 1] \leq \frac{1}{c_2}\, \mathrm{e}^{-c_2 \mathrm{e}^{\vartheta_N}} \,. \tag{1.14}$$

Remark 1.7 (Strong disorder and second moment). The lower bound in (1.14) will be deduced by a Paley-Zygmund-type inequality coupled with a second moment upper bound, see Proposition 1.13 below. (The same could be done for the lower bound in Theorem 1.1, at the level of the SHF.) This will also show that the bounds in (1.14) are sharp, up to the constants c_1, c_2, when f is uniform in the discrete ball of radius $\sqrt{N}$, see (1.20). For such initial conditions, we thus have $Z_N^{\beta_N,\omega}(f) \to 0$ *if and only if* $\mathbb{E}[Z_N^{\beta_N,\omega}(f)^2] \to \infty$, that is precisely when $\vartheta_N \to \infty$.

We refer to [JL25, Proposition 2.9] for a result in the same spirit concerning the *point to plane partition function* $Z_N^{\beta_N,\omega}(0)$, which corresponds to $f = \mathbf{1}_{\{0\}}$. In this case, it is known that $Z_N^{\beta_N,\omega}(0) \to 0$ as soon as $\sigma^2(\beta_N)\, R_N \to 1$, which includes the whole critical window $\vartheta_N \equiv \vartheta$ for any $\vartheta \in \mathbb{R}$, see [CSZ17, Theorem 2.8]. In particular, the upper bound in (1.14) is *not* sharp in this case, but we still have that $Z_N^{\beta_N,\omega}(0) \to 0$ *if and only if* $\mathbb{E}[Z_N^{\beta_N,\omega}(0)^2] \to \infty$.

Beyond the quasi-critical window. We can deduce Theorem 1.1 for SHF(ϑ) by plugging $\vartheta_N \equiv \vartheta$ into Theorem 1.6 for directed polymers, see Section 2, but Theorem 1.6 is *stronger* since it allows for $\vartheta \to \infty$ as long as $\vartheta = o(\log N)$. Next, we show that we can go much farther.

Let us rewrite relation (1.13) as follows:

$$\vartheta = \pi R_N - \frac{\pi}{\sigma^2(\beta)} \,. \tag{1.15}$$

If we plug this expression into (1.14), our next main result shows that *the upper bound still holds for any* $\beta > 0$ *small and with no restriction on* $N \in \mathbb{N}$.

Theorem 1.8 (Strong disorder for 2D directed polymers: uniform version)**.** *There are constants* $\hat{\beta}, c_0, c_2 \in (0, \infty)$ *such that, uniformly over* $N \in \mathbb{N}$ *and* $\beta \in (0, \hat{\beta})$*, we can bound*

$$\sup_{f \in \mathcal{M}_1^{\mathrm{disc}}\left(\sqrt{\mathrm{e}^{c_0 \mathrm{e}^{\vartheta(N,\beta)}} N}\right)} \mathbb{E}[Z_N^{\beta,\omega}(f) \wedge 1] \leq \frac{1}{c_2} \exp\Big(-c_2\, \mathrm{e}^{\vartheta(N,\beta)}\Big) \,, \tag{1.16}$$

[†]We do not include any $o(1)$ term in (1.13) because the parameter ϑ is not fixed, unlike in (1.11), hence it is allowed to vary with N and β. See also Remark 1.9 below for a discussion.

where we define $\vartheta(N,\beta) := \pi R_N - \frac{\pi}{\sigma^2(\beta)}$ as in (1.15) (see also (1.13)). Note that, in view of (1.10), we have

$$e^{\vartheta(N,\beta)} = e^{\alpha_N} N \, e^{-\frac{\pi}{\sigma^2(\beta)}} = e^{\alpha+o(1)} N \, e^{-\frac{\pi}{\sigma^2(\beta)}} . \tag{1.17}$$

The uniform upper bound (1.16) clearly implies the upper bound in (1.14). Moreover, it yields interesting information on the *free energy*, which we discuss next.

Remark 1.9 (On the critical regime). We may interpret the critical relation (1.13) in several ways. If we fix $\vartheta \in \mathbb{R}$, then we can vary either β or N as a function of the other parameter:

- We can choose $\beta = \beta_N = \beta_N(\vartheta) \downarrow 0$ as $N \to \infty$ so that (1.13) holds, this is the so-called *intermediate disorder* regime;
- We can also take $N = N_\beta = N_\beta(\vartheta) \to \infty$ as $\beta \downarrow 0$ so that (1.13) holds, this is natural for directed polymers, which are often studied as $N \to \infty$ for fixed $\beta > 0$.

Alternatively, given arbitrary $\beta > 0$ and $N \in \mathbb{N}$, we can *define $\vartheta = \vartheta(N,\beta)$ by* (1.13)*, or equivalently by* (1.15), which quantifies the disorder strength ϑ corresponding to β, N.

Remark 1.10 (On the 2D Stochastic Heat Equation). As we already mentioned, the critical 2D Stochastic Heat Flow also arises as the scaling limit of solutions of the 2D stochastic heat equation with mollified noise, see [Tsa24]. Adapting the techniques of the present paper, we can establish versions of Theorem 1.6 and Theorem 1.8 for the stochastic heat equation, which we will do in a forthcoming work.

1.4. **Free energy estimates.** In the space dimension $d = 2$ that we consider, the point-to-plane partition function $Z_N^{\beta,\omega} := Z_N^{\beta,\omega}(0)$ converges a.s. to 0 as $N \to \infty$ *for any fixed disorder strength* $\beta > 0$, and it does so exponentially fast, as shown in [Lac10]. Its exponential decay rate to 0 is called (up to a sign) the free energy (or pressure) and it is defined as

$$\mathtt{F}(\beta) := \lim_{N\to\infty} \frac{1}{N} \log Z_N^{\beta,\omega} = \lim_{N\to\infty} \frac{1}{N} \mathbb{E}[\log Z_N^{\beta,\omega}] \in (-\infty, 0] , \tag{1.18}$$

where the limit is known to exist a.s. and in $L^1(\mathbb{P})$, see e.g. [Com17, Thm. 2.1]. We point out that free energy is related to some localization properties of the polymer, see e.g. [CH06, CSY03].

It was shown in [Lac10] that $\mathtt{F}(\beta) < 0$ for any $\beta > 0$ with some explicit bounds; a few years later, [BL17] refined the bounds and showed that

$$\mathtt{F}(\beta) = -\exp\Big(-(1+o(1))\frac{\pi}{\beta^2}\Big) \qquad \text{as } \beta \downarrow 0 .$$

Our next result substantially improves these bounds: we identify the exact exponential decay rate as $\pi/\sigma^2(\beta)$, rather than simply[‡] π/β^2, and we “bring the $o(1)$ out of the exponential”.

Theorem 1.11 (Improved free energy bounds)**.** *There are constants $c, c' \in (0,\infty)$ such that, for any $\beta \in (0,1)$,*

$$-\frac{c'}{\sigma^2(\beta)^4} \exp\Big(-\frac{\pi}{\sigma^2(\beta)}\Big) \le \mathtt{F}(\beta) \le -c \exp\Big(-\frac{\pi}{\sigma^2(\beta)}\Big) , \tag{1.19}$$

where we recall that $\sigma^2(\beta) = e^{\lambda(2\beta)-2\lambda(\beta)} - 1$.

We prove the lower bound in (1.19) following closely the strategy of [BL17, §4], based on super-additivity and concentration arguments for $\log Z_N^{\beta,\omega}$; see Appendix B for the details. The upper bound in (1.19) is the main novelty: we deduce it from Theorem 1.8, more precisely from the upper bound in (1.16). We refer to Section 2 for the proof.

[‡]Note that $\lambda(\beta) = \frac{1}{2}\beta^2 + \frac{\kappa_3}{3!}\beta^3 + \frac{\kappa_4}{4!}\beta^4 + O(\beta^5)$ as $\beta \downarrow 0$, where κ_3, κ_4 are the third and fourth cumulants of the disorder distribution. It follows that we have $e^{-\pi/\beta^2} \sim (cst.)\, e^{-\pi/\sigma^2(\beta)}$ only when $\kappa_3 = 0$.

Remark 1.12. We do not expect either bound in (1.19) to be optimal, but the upper bound should be sharper than the lower bound (the prefactor $\sigma^2(\beta)^{-4}$ is due to a limitation of the current techniques). As discussed in Appendix B, the precise asymptotic behavior might be

$$\mathtt{F}(\beta) \sim -c \log\big(\tfrac{1}{\sigma^2(\beta)}\big)\, \mathrm{e}^{-\frac{\pi}{\sigma^2(\beta)}} \quad \text{as } \beta \downarrow 0\,.$$

1.5. **Second moment estimates.** Our last results concern a *second moment estimate* for the directed polymer partition function, which will be used in Section 2 to prove the lower bound in (1.14). We focus on the quasi-critical regime, *i.e.* we fix $\beta = \beta_N$ which satisfies (1.13) with $\vartheta = \vartheta_N = o(\log N)$. We consider the following setting:

- we assume that $\vartheta_N \to \infty$ (since in the critical window $\vartheta_N \to \vartheta \in \mathbb{R}$ the second moment is known to be bounded and can in fact be computed explicitly, see [CSZ19a]);
- we focus on the initial condition given by the uniform distribution on the discrete ball of radius $\sqrt{N}$, namely

$$\mathcal{U}^{\mathrm{disc}}_{\sqrt{N}}(x) := \frac{1}{|B(0,\sqrt{N}) \cap \mathbb{Z}^2|}\, \mathbf{1}_{B(0,\sqrt{N})\cap\mathbb{Z}^2}(x)\,, \tag{1.20}$$

We can now state our second moment estimate, to be proved in Section 4.4.

Proposition 1.13 (Quasi-critical second moment for directed polymers)**.** *Given $(\vartheta_N)_{N\geq 1}$ such that $\vartheta_N \to \infty$ and $\vartheta_N = o(\log N)$, define $\beta = \beta_N$ such that* (1.13) *holds with $\vartheta = \vartheta_N$. Then, as $N \to \infty$, we have*

$$\mathbb{E}\big[Z_N^{\beta_N,\omega}(\mathcal{U}^{\mathrm{disc}}_{\sqrt{N}})^2\big] \leq \exp\big((1+o(1))\, \mathrm{e}^{\vartheta_N - \gamma}\big) \tag{1.21}$$

where $\gamma := -\int_0^\infty \mathrm{e}^{-x} \log x \, \mathrm{d}x \simeq 0.577$ is the Euler–Mascheroni constant.

Remark 1.14. We believe the upper bound (1.21) to be sharp, *i.e.* one should also be able to prove that, in the same regime as in Proposition 1.13, we have

$$\mathbb{E}\big[Z_N^{\beta_N,\omega}(\mathcal{U}^{\mathrm{disc}}_{\sqrt{N}})^2\big] \geq \exp\big((1+o(1))\, \mathrm{e}^{\vartheta_N - \gamma}\big)\,.$$

Since this lower bound is not useful to us, we have decided to skip its proof. One could also try to improve these estimates by "bringing the $o(1)$ out of the exponential".

1.6. **Structure of the paper.** The rest of the paper is devoted mainly to proving Theorem 1.8: the strategy for its proof is presented in Section 2, which also contains the proofs of the results that derive from it (Theorem 1.1, Theorem 1.5, Theorem 1.6 and Theorem 1.11).

Section 3 contains the proof of the key proposition Proposition 2.1 used to prove Theorem 1.8. Section 4 and Section 5 contain the proofs of some estimates used in the previous sections. Some further technical results (which follow well-established paths) are postponed to the appendices.

1.7. **Notation.** For a point $x = (x_1, x_2)$ in $\mathbb{Z}^2$ we indicate with $|x| = \sqrt{x_1^2 + x_2^2}$ its Euclidean norm. Given two positive sequences $(a_N)_{N\in\mathbb{N}}$ and $(b_N)_{N\in\mathbb{N}}$, we write $a_N \sim b_N$ if $\lim_{N\to\infty} a_N/b_N = 1$ and $a_N \ll b_N$ if $\lim_{N\to\infty} a_N/b_N = 0$, $a_N \gg b_N$ if $\lim_{N\to\infty} a_N/b_N = +\infty$. For $N \in \mathbb{N}$ we write $[\![1, N]\!]$ for the set $\{1, \ldots, N\}$. When A is a set we indicate with $|A|$ its cardinality.

Acknowledgements. We are grateful to Rongfeng Sun and Nikos Zygouras for their comments on a preliminary version of the manuscript, which helped us improve the presentation. F.C. and N.T. acknowledge the support of INdAM/GNAMPA. Q.B. acknowledges the support of Institut Universitaire de France and ANR Local (ANR-22-CE40-0012-02).

2. Strategy of the proofs

In this section, we first present the strategy of the proof of Theorem 1.8, see Section 2.1. We then show how to deduce from Theorem 1.8 the other main results, namely Theorem 1.1, Theorem 1.5, Theorem 1.6 and Theorem 1.11, see Section 2.2.

2.1. Strategy of the proof of Theorem 1.8.

We prove Theorem 1.8 in three main steps.

(1) First, we formulate a key result, Proposition 2.1, which is sub-optimal (with respect to Theorem 1.8) in several senses: (i) the starting point is on the basic diffusive scale $\sqrt{N}$ without the factor $\mathrm{e}^{c_0\,\mathrm{e}^{\vartheta}}$ in (1.16); (ii) the parameter $\vartheta \in (1,\infty)$ is fixed so Proposition 2.1 is proved only in the critical window; (iii) the bound we obtain is polynomial in ϑ instead of doubly exponential.

(2) Second, we use *coarse-graining techniques* to extend the previous bound to any $\beta > 0$ small and $N \in \mathbb{N}$, meanwhile also obtaining an exponential decay in N, see Proposition 2.3. By now, this is a well-established method, which dates back to [Lac10] (in the context of directed polymers); technical details are postponed to Appendix A.

(3) Third, we use a change of scale argument to quantify the effect of enlarging the scale of the starting point: this leads to Proposition 2.6, which completes the proof of Theorem 1.8.

As a consequence of these steps, *Theorem 1.8 is reduced to the key Proposition 2.1, whose proof is given in the next Section 3.* Let us now give some details on the steps outlined above.

Step 1: Key (sub-optimal) result. We first state a weaker version of Theorem 1.8 formulated in the next key proposition. This is actually the core of the paper; we will present the key ideas of its proof in Section 3.

Proposition 2.1 (Key proposition)**.** *There exists a universal constant $C > 0$ such that for any given $\vartheta \in (1,\infty)$, if we define $\beta_N = \beta_N(\vartheta)$ such that (1.13) holds, we have*

$$\limsup_{N\to\infty} \sup_{f \in \mathcal{M}_1^{\mathrm{disc}}(\sqrt{N})} \mathbb{E}[Z_N^{\beta_N,\omega}(f) \wedge 1] \le \frac{C}{\vartheta} . \tag{2.1}$$

We view Proposition 2.1 as an estimate in the (upper) critical window, *i.e.* we will use it for ϑ large but fixed, in order to apply a "finite-volume criterion" in the next step. We deduce from (2.1) a corresponding bound on a fractional moment of $Z_N^{\beta_N,\omega}(f)$:

$$\limsup_{N\to\infty} \sup_{f \in \mathcal{M}_1^{\mathrm{disc}}(\sqrt{N})} \mathbb{E}\Big[Z_N^{\beta_N,\omega}(f)^{1/2}\Big] \le \frac{\sqrt{2C}}{\sqrt{\vartheta}} , \tag{2.2}$$

thanks to the following general result.

Lemma 2.2. *For any random variable $Z \ge 0$ with $\mathbb{E}[Z] = 1$ we have*

$$\mathbb{E}[Z \wedge 1] \le \mathbb{E}[Z^{1/2}] \le \sqrt{2}\,\mathbb{E}[Z \wedge 1]^{1/2} .$$

Proof. The first inequality simply uses that $x \le x^{1/2}$ for $x \in [0,1]$ to get that $\mathbb{E}[Z \wedge 1] \le \mathbb{E}[(Z \wedge 1)^{1/2}] \le \mathbb{E}[Z^{1/2}]$. For the second one, write

$$\mathbb{E}[Z^{1/2}] = \mathbb{E}[(Z\mathbf{1}_{\{Z\le 1\}})^{1/2}] + \mathbb{E}[Z^{1/2}\mathbf{1}_{\{Z>1\}}] \le \mathbb{E}[Z\mathbf{1}_{\{Z\le 1\}}]^{1/2} + \mathbb{E}[Z]^{1/2}\mathbb{P}(Z>1)^{1/2} ,$$

where we have used Jensen's inequality for the first term, and Cauchy–Schwarz inequality for the second one. Then, using the inequality $(a+b)^2 \le 2(a^2+b^2)$ and the fact that $\mathbb{E}[Z] = 1$, we get that $\mathbb{E}[Z^{1/2}]^2 \le 2(\mathbb{E}[Z\mathbf{1}_{\{Z\le 1\}}] + \mathbb{P}(Z>1)) = 2\mathbb{E}[Z \wedge 1]$. □

Step 2: Finite volume criterion via a coarse-graining procedure. Let us upgrade the result from the key Proposition 2.1, in the fractional moment version (2.2), extending it to arbitrary $N \in \mathbb{N}$ and $\beta > 0$ small (*i.e.* not necessarily in the critical window) and improving the bound with an exponential decay in N.

Recalling (1.15) and (1.10), we first define $N_\beta(\vartheta) \in \mathbb{N}$ for any $\beta > 0$, $\vartheta \in \mathbb{R}$ as follows:

$$N_\beta(\vartheta) := \mathrm{e}^{-\alpha_N}\, \mathrm{e}^{\vartheta}\, \mathrm{e}^{\frac{\pi}{\sigma^2(\beta)}} \sim \mathrm{e}^{-\alpha}\, \mathrm{e}^{\vartheta}\, \mathrm{e}^{\frac{\pi}{\sigma^2(\beta)}} \qquad \text{as } \beta \downarrow 0\,. \tag{2.3}$$

Proposition 2.3 (Improved bound). *There exist constants $\hat{\beta}, \hat{\vartheta} \in (0,\infty)$ such that the following holds: defining $\hat{N}_\beta := N_\beta(\hat{\vartheta})$ by (2.3), we have*

$$\forall \beta \in (0, \hat{\beta})\,,\ \forall N \in \mathbb{N}: \qquad \sup_{f \in \mathcal{M}_1^{\mathrm{disc}}\left(\sqrt{\hat{N}_\beta}\right)} \mathbb{E}\big[Z_N^{\beta,\omega}(f)^{1/2}\big] \le 3\, \mathrm{e}^{-N/\hat{N}_\beta}\,. \tag{2.4}$$

Proof. To prove this result, we first rewrite the relation (2.2) inverting the roles of β and N (see Remark 1.9): instead of letting $N \to \infty$ with $\beta = \beta_N(\vartheta)$, we let $\beta \downarrow 0$ with $N = N_\beta(\vartheta)$ defined in (2.3), this is just a rewriting of relation (1.13), recall (1.15). Therefore (2.2) can be rewritten as follows:

$$\limsup_{\beta \downarrow 0} \sup_{f \in \mathcal{M}_1^{\mathrm{disc}}(\sqrt{N_\beta(\vartheta)})} \mathbb{E}\big[Z_{N_\beta(\vartheta)}^{\beta,\omega}(f)^{1/2}\big] \le \frac{\sqrt{2C}}{\sqrt{\vartheta}}\,.$$

In particular, given any $\vartheta \ge 1$, we can fix a suitable $\tilde{\beta}(\vartheta) > 0$ small enough such that (say)

$$\forall \beta \in (0, \tilde{\beta}(\vartheta)): \qquad \sup_{f \in \mathcal{M}_1^{\mathrm{disc}}(\sqrt{N_\beta(\vartheta)})} \mathbb{E}\big[Z_{N_\beta(\vartheta)}^{\beta,\omega}(f)^{1/2}\big] \le \frac{2\sqrt{C}}{\sqrt{\vartheta}}\,. \tag{2.5}$$

We next improve this estimate allowing $N \in \mathbb{N}$ to be arbitrary. The core idea is the following *coarse-graining* result, which gives a finite-size volume criterion for the exponential decay of the partition function: it shows that if the fractional moment is small at some scale L, then it starts decreasing exponentially in N/L. This result is somehow classical in the literature, but we provide a self-contained proof in Appendix A below.

Proposition 2.4 (Coarse-graining). *If there exist $L \in \mathbb{N}$, $\beta > 0$ such that*

$$\sup_{f \in \mathcal{M}_1^{\mathrm{disc}}(\sqrt{L})} \mathbb{E}\big[Z_L^{\beta,\omega}(f)^{1/2}\big] \le \frac{1}{300}\,, \tag{2.6}$$

then for all $N \in \mathbb{N}$, we have

$$\sup_{f \in \mathcal{M}_1^{\mathrm{disc}}(\sqrt{L})} \mathbb{E}\big[Z_N^{\beta,\omega}(f)^{1/2}\big] \le 3\, \mathrm{e}^{-N/L}\,. \tag{2.7}$$

Remark 2.5. It is enough to prove (2.7) for $N \ge L$, since for $N < L$ we have $\mathbb{E}[Z_N^{\beta,\omega}(f)^{1/2}] \le \mathbb{E}[Z_N^{\beta,\omega}(f)]^{1/2} = 1$. Also, let us stress that the constant $\frac{1}{300}$ depends on the distribution of the random walk (we have simply taken a number that works for the simple random walk).

Recalling (2.5), we now fix $\hat{\vartheta} \ge 1$ such that $\frac{2\sqrt{C}}{\sqrt{\hat{\vartheta}}} \le \frac{1}{300}$, *i.e.* $\hat{\vartheta} = (600)^2\, C$. If we correspondingly define $\hat{\beta} := \tilde{\beta}(\hat{\vartheta}) > 0$ and $\hat{N}_\beta := N_\beta(\hat{\vartheta})$, then (2.5) yields

$$\forall \beta \in (0, \hat{\beta}): \qquad \sup_{f \in \mathcal{M}_1^{\mathrm{disc}}(\sqrt{\hat{N}_\beta})} \mathbb{E}\big[Z_{\hat{N}_\beta}^{\beta,\omega}(f)^{1/2}\big] \le \frac{1}{300}\,. \tag{2.8}$$

It only remains to apply Proposition 2.4 with $L = \hat{N}_\beta$ to complete the proof of Proposition 2.3. $\square$

Step 3. The change of scale argument. We finally show that the scale of the starting point in the bound (2.4) can be enlarged, to get the following result.

Proposition 2.6 (Large scale bound)**.** *In the same setting as Proposition 2.3, we have*

$$\forall \beta \in (0, \hat{\beta})\,,\ \forall N \in \mathbb{N}: \qquad \sup_{f \in \mathcal{M}_1^{\mathrm{disc}}\left(\sqrt{\mathrm{e}^{N/\hat{N}_\beta} N}\,\right)} \mathbb{E}[Z_N^{\beta,\omega}(f)^{1/2}] \le 7\,\mathrm{e}^{-\frac{1}{3}N/\hat{N}_\beta}\,. \tag{2.9}$$

This completes the proof of Theorem 1.8. Indeed, if we define $\vartheta = \vartheta(N, \beta)$ in agreement with (1.15), we have the identity $N = N_\beta(\vartheta(N, \beta))$ with $N_\beta(\cdot)$ defined in (2.3). Recalling that $\hat{N}_\beta = N_\beta(\hat{\vartheta})$, we obtain

$$\frac{N}{\hat{N}_\beta} = \frac{N_\beta(\vartheta(N,\beta))}{N_\beta(\hat{\vartheta})} = \mathrm{e}^{\vartheta(N,\beta) - \hat{\vartheta}}\,,$$

which plugged into (2.9) yields (1.16), by Lemma 2.2, with $c_0 = \mathrm{e}^{-\hat{\vartheta}}$ and $c_2 = \min\{\frac{1}{3}\mathrm{e}^{-\hat{\vartheta}}, \frac{1}{7}\}$.

Proof of Proposition 2.6. The key tool is the following general result, which allows to control fractional moments with starting points at two different scales.

Lemma 2.7 (Changing scales)**.** *For any $1 \le A \le B$ and any $\gamma \in (0,1)$, we have*

$$\sup_{f \in \mathcal{M}_1^{\mathrm{disc}}(\sqrt{B})} \mathbb{E}\Big[Z_N^{\beta,\omega}(f)^\gamma\Big] \le \left(4\,\frac{B}{A}\right)^{1-\gamma} \sup_{g \in \mathcal{M}_1^{\mathrm{disc}}(\sqrt{A})} \mathbb{E}\Big[Z_N^{\beta,\omega}(g)^\gamma\Big]\,.$$

Proof. We partition the L^∞ ball of radius $\sqrt{B}$ into $K := \left\lceil \frac{2\lfloor\sqrt{B}\rfloor + 1}{2\lfloor\sqrt{A}\rfloor+1} \right\rceil^2 \le 4\frac{B}{A}$ balls $(B_i)_{1 \le i \le K}$ of radius $\sqrt{A}$. For any probability density $f \in \mathcal{M}_1^{\mathrm{disc}}(\sqrt{B})$, we can decompose it as $f = \sum_{i=1}^K \alpha_i g_i$ where $\alpha_i := \int f\,\mathbf{1}_{B_i}$ and $g_i := \frac{1}{\alpha_i} f\,\mathbf{1}_{B_i}$ is just f conditioned on B_i. This way, we may write

$$Z_N^{\beta,\omega}(f) = \sum_{i=1}^K \alpha_i\, Z_N^{\beta,\omega}(g_i)\,. \tag{2.10}$$

For $\gamma \in (0,1)$, using the subadditive inequality $(\sum_i z_i)^\gamma \le \sum_i z_i^\gamma$ for $z_i \ge 0$, we obtain that

$$\mathbb{E}[Z_N^{\beta,\omega}(f)^\gamma] \le \sum_{i=1}^K \alpha_i^\gamma\, \mathbb{E}[Z_N^{\beta,\omega}(g_i)^\gamma] \le \sup_{g \in \mathcal{M}_1^{\mathrm{disc}}(\sqrt{A})} \mathbb{E}[Z_N^{\beta,\omega}(g)^\gamma] \sum_{i=1}^K \alpha_i^\gamma\,, \tag{2.11}$$

using also translation invariance. Now, using Hölder's inequality, we can bound $\sum_{i=1}^K \alpha_i^\gamma \le K^{1-\gamma}$, so recalling that $K \le 4B/A$ this concludes the proof. $\square$

Thanks to Lemma 2.7, we deduce from Proposition 2.3 that for all $\beta \in (0, \hat{\beta})$ and $N \in \mathbb{N}$

$$\sup_{f \in \mathcal{M}_1^{\mathrm{disc}}\left(\sqrt{\mathrm{e}^{N/\hat{N}_\beta} N}\,\right)} \mathbb{E}\Big[Z_N^{\beta,\omega}(f)^{1/2}\Big] \le 2\left(\frac{N}{\hat{N}_\beta}\,\mathrm{e}^{N/\hat{N}_\beta}\right)^{1/2} 3\,\mathrm{e}^{-N/\hat{N}_\beta} = 6\left(\frac{N}{\hat{N}_\beta}\right)^{1/2} \mathrm{e}^{-\frac{1}{2}N/\hat{N}_\beta}\,.$$

This complete the proof of (2.9), since $6\sqrt{x}\,\mathrm{e}^{-\frac{1}{2}x} \le 7\,\mathrm{e}^{-\frac{1}{3}x}$ for $x \ge 0$. $\square$

2.2. **Proof of the other main results.** We now give the proofs of Theorems 1.1, 1.5, 1.6 and 1.11. We start from Theorem 1.11, which is a direct consequence of Theorem 1.8.

Proof of Theorem 1.11. The lower bound in (1.19) is proved in Appendix B: we follow [BL17, §4], exploiting super-additivity and concentration of measure arguments for $\log Z_N^{\beta,\omega}$. We focus here on the upper bound in (1.19), which we deduce from Theorem 1.8.

We claim that we can truncate $Z_N^{\beta,\omega}$ at 1 in the definition (1.18) of the free energy and write

$$\mathtt{F}(\beta) = \lim_{N\to\infty} \frac{1}{N}\,\mathbb{E}[\log(Z_N^{\beta,\omega} \wedge 1)]\,. \tag{2.12}$$

Indeed, the proof is simple: since $Z_N^{\beta,\omega} = (Z_N^{\beta,\omega} \wedge 1)(Z_N^{\beta,\omega} \vee 1)$, it suffices to show that

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}[\log(Z_N^{\beta,\omega} \vee 1)] = 0\,,$$

which follows by the inequalities $1 \le Z_N^{\beta,\omega} \vee 1 \le 1 + Z_N^{\beta,\omega}$ because, since $\mathbb{E}[Z_N^{\beta,\omega}] = 1$,

$$0 \le \lim_{N\to\infty} \frac{1}{N}\,\mathbb{E}\big[\log(Z_N^{\beta,\omega} \vee 1)\big] \le \lim_{N\to\infty} \frac{1}{N}\log\mathbb{E}\big[1 + Z_N^{\beta,\omega}\big] = 0\,.$$

Recalling (1.17) and applying (1.16), we can write for $Z_N^{\beta,\omega} = Z_N^{\beta,\omega}(0) = Z_N^{\beta,\omega}(\mathbf{1}_{\{0\}})$

$$\mathbb{E}\big[Z_N^{\beta,\omega} \wedge 1\big] \le \frac{1}{c_2}\,\exp\Big(-c_2\,\mathrm{e}^{\vartheta(N,\beta)}\Big) = \frac{1}{c_2}\,\exp\Big(-c_2\,\mathrm{e}^{\alpha+o(1)}N\,\mathrm{e}^{-\frac{\pi}{\sigma^2(\beta)}}\Big)\,. \tag{2.13}$$

Applying relation (2.12) together with $\mathbb{E}\big[\log(Z_N^{\beta,\omega}\wedge 1)\big] \le \log\mathbb{E}\big[Z_N^{\beta,\omega}\wedge 1\big]$ (by Jensen's inequality), we obtain the upper bound on the free energy in (1.19) with $c = c_2\,\mathrm{e}^{\alpha}$. ☐

We next prove Theorem 1.6, deducing it from Theorem 1.8 and Proposition 1.13 (which is is proved in Section 4.4).

Proof of Theorem 1.6. The upper bound in (1.14) is a special case of (1.16), so it remains to prove the lower bound in (1.14). To this purpose, we consider the following inequality, in the spirit of Paley-Zygmund: for any random variable $Z \ge 0$

$$\mathbb{E}[Z\wedge 1] \ge \frac{\mathbb{E}[Z]^2}{1+\mathbb{E}[Z^2]}\,. \tag{2.14}$$

The proof is simple: starting from the identity $Z = (Z\wedge 1)(Z\vee 1)$, we get by Cauchy-Schwarz

$$\mathbb{E}[Z]^2 \le \mathbb{E}[(Z\wedge 1)^2]\,\mathbb{E}[(Z\vee 1)^2] = \mathbb{E}[Z^2\wedge 1]\,\mathbb{E}[Z^2\vee 1] \le \mathbb{E}[Z\wedge 1]\,(1+\mathbb{E}[Z^2])\,.$$

To prove the lower bound in (1.14), it suffices to apply (2.14) to $Z = Z_N^{\beta_N,\omega}(f)$ with $f = \mathcal{U}^{\mathrm{disc}}_{\sqrt{N}}$, noting that $\mathbb{E}[Z] = 1$ and plugging in the estimate (1.21) from Proposition 1.13. ☐

We then prove Theorem 1.1 about the Stochastic Heat Flow (SHF), which follows from the corresponding result for directed polymers, Theorem 1.6, that we have just proved.

Proof of Theorem 1.1. Consider (1.14) from Theorem 1.6 in the special case $\vartheta_N \equiv \vartheta \in \mathbb{R}$. If we fix $t > 0$, we can note that *relation* (1.13) *is invariant if we replace* (N,ϑ) *by* $(\lfloor Nt\rfloor, \vartheta + \log t)$, in view of (1.10). Then we can apply relation (1.14) replacing N by $\lfloor Nt\rfloor$ and e^{ϑ_N} by $\mathrm{e}^{\vartheta+\log t} = t\,\mathrm{e}^{\vartheta}$, which yields for N large enough

$$\frac{1}{c_1}\,\mathrm{e}^{-c_1\,t\,\mathrm{e}^{\vartheta}} \le \sup_{f\in\mathcal{M}_1^{\mathrm{disc}}\left(\sqrt{(\mathrm{e}^{c_0\,t\,\mathrm{e}^{\vartheta}})\,t\,N}\right)} \mathbb{E}\big[Z_{\lfloor Nt\rfloor}^{\beta_N,\omega}(f)\wedge 1\big] \le \frac{1}{c_2}\,\mathrm{e}^{-c_2\,t\,\mathrm{e}^{\vartheta}}\,. \tag{2.15}$$

Recalling (1.12), since the size of the support of $\varphi^{(N)}$ is $\sqrt{N}$ times the size of the support of φ, we deduce that (1.3) holds. ☐

We finally deduce Theorem 1.5 from Theorem 1.1.

Proof of Theorem 1.5. Let us fix $0 < c < \min\{c_0, c_2\}$ and set $\delta := \frac{1}{2}(c_2 - c) > 0$. Define the uniform density

$$\mathcal{U}_r(x) := \frac{1}{\pi\,r^2}\,\mathbf{1}_{B(0,r)}(x) \qquad \text{with} \qquad B(0,r) := \{x\in\mathbb{R}^2\colon |x| \le r\}\,. \tag{2.16}$$

If we define $\varepsilon := t\,\mathrm{e}^{-\delta\,t\,\mathrm{e}^{\vartheta}}$, setting $\varepsilon' := \frac{\varepsilon}{\pi\,(\mathrm{e}^{c\,t\,\mathrm{e}^{\vartheta}})\,t} = \frac{1}{\pi}\mathrm{e}^{-(c+\delta)\,t\,\mathrm{e}^{\vartheta}}$, Markov's inequality yields

$$\mathbb{P}\Big(\mathscr{Z}_t^{\vartheta}\big(B(0,\sqrt{(\mathrm{e}^{c\,t\,\mathrm{e}^{\vartheta}})\,t}\,)\big) > \varepsilon\Big) = \mathbb{P}\Big(\mathscr{Z}_t^{\vartheta}\big(\mathcal{U}_{\sqrt{(\mathrm{e}^{c\,t\,\mathrm{e}^{\vartheta}})\,t}}\big) > \varepsilon'\Big) \le \frac{\mathbb{E}\big[\mathscr{Z}_t^{\vartheta}(\mathcal{U}_{\sqrt{(\mathrm{e}^{c\,t\,\mathrm{e}^{\vartheta}})\,t}})\wedge 1\big]}{\varepsilon'\wedge 1}\,.$$

By the upper bound in (1.3), the right hand side is $O(\mathrm{e}^{-(c_2-c-\delta)\,t\,\mathrm{e}^{\vartheta}}) \to 0$ as $t\,\mathrm{e}^{\vartheta}\to\infty$. ☐

3. Proof of Proposition 2.1

We have shown in Section 2 how to reduce Theorem 1.8 to the weaker key Proposition 2.1. This section is devoted to the proof of Proposition 2.1: we explain the general strategy formulating some general propositions, to be proved in the next subsections. We first introduce the key notion of *size biased measure*.

Definition 3.1 (Size-biased measure)**.** Denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space on which the disorder ω is defined, and recall the point-to-plane partition function $Z_N^{\beta,\omega}(x)$ from (1.7).

For any $x \in \mathbb{Z}^2$, we define the *size-biased measure* with starting point x by

$$\widetilde{\mathbb{P}}_x(A) = \widetilde{\mathbb{P}}^{\beta}_{x,N}(A) := \mathbb{E}[\mathbf{1}_A \, Z_N^{\beta,\omega}(x)] \qquad \text{for any } A \in \mathcal{F}\,. \tag{3.1}$$

More generally, the *size-biased measure with initial condition* f (a mass function) is

$$\widetilde{\mathbb{P}}_f(\cdot) = \widetilde{\mathbb{P}}^{\beta}_{f,N}(\cdot) := \mathbb{E}[\mathbf{1}_{(\cdot)} \, Z_N^{\beta}(f)]\,. \tag{3.2}$$

Remark 3.2 (Size bias and total variation distance)*.* Fix any $Z \geq 0$ with $\mathbb{E}[Z] = 1$ and consider the size-biased measure $\mathrm{d}\tilde{\mathbb{P}} = Z\,\mathrm{d}\mathbb{P}$. For any event $A \in \mathcal{F}$ we have $\mathbb{E}[Z \wedge 1] \leq \mathbb{P}(A) + \tilde{\mathbb{P}}(A^c)$ (just bound $Z \wedge 1 \leq 1$ on A and $Z \wedge 1 \leq Z$ on A^c) and the inequality is sharp, since we can take $A = \{Z \geq 1\}$ to get equality. It follows that

$$\mathbb{E}[Z \wedge 1] = \sup_{A \in \mathcal{F}} \{\mathbb{P}(A) + \tilde{\mathbb{P}}(A^c)\} = 1 - d_{\mathrm{TV}}(\mathbb{P}, \tilde{\mathbb{P}})\,,$$

where d_{TV} is the total variation distance between probability measures. Therefore, *showing that* $\mathbb{E}[Z \wedge 1]$ *is small corresponds to showing that* $d_{\mathrm{TV}}(\mathbb{P}, \tilde{\mathbb{P}})$ *is close to* 1.

Remark 3.3 (Anomalous path detection)*.* In view of Remark 3.2, showing that $\mathbb{E}[Z_N^{\beta,\omega}(f) \wedge 1] \to 0$ amounts to a statistical problem of being able to find an appropriate event $A = A_N$ to discriminate between two environment distribution: $\mathbb{P}$, and $\widetilde{\mathbb{P}}^{\beta}_{f,N}$. Such "anomalous path detection problems" have been investigated (mostly in dimension 1), see for instance in [ACCHZ08, CZ18], or [ABBDL10] for a discussion on similar hypothesis testing problems.

3.1. Strategy of the proof of Proposition 2.1.

We need to give an upper bound on $\mathbb{E}[Z_N^{\beta,\omega}(f) \wedge 1]$ inside the critical window, *i.e.* for a fixed $\vartheta \in [1, \infty)$, and for diffusive initial conditions f, *i.e.* supported in a ball of radius $\sqrt{N}$.

We combine a *change of scale* argument, that we use to reduce the initial diffusive scale $\sqrt{N}$ to a smaller scale $\sqrt{\tilde{N}}$, with a *change of measure* argument, for which we will use and refine ideas developed in the case of point-to-plane partition functions (see e.g. the proof of Theorem 2.9 in [JL25]). These lead to the next result, proved in Section 3.2.

Proposition 3.4 (Change of scale and measure)**.** *For any $\tilde{N} \leq N$ and for any event A_N, recalling* (3.1)*, we can bound*

$$\sup_{f \in \mathcal{M}_1^{\mathrm{disc}}(\sqrt{N})} \mathbb{E}[Z_N^{\beta,\omega}(f) \wedge 1] \leq 2\,\frac{N}{\tilde{N}}\,\mathbb{P}(A_N) + 2 \sup_{|x| \leq \sqrt{\tilde{N}}} \widetilde{\mathbb{P}}_x(A_N^c)\,. \tag{3.3}$$

We will fix $\tilde{N} = \mathrm{e}^{-\eta} N$ for a suitable $1 \leq \eta < \vartheta$. For the bound (3.3) to be useful, we must find an event A_N for which both $\mathbb{P}(A_N)$ and $\widetilde{\mathbb{P}}_x(A_N^c)$ are small, *i.e.* A_N *is atypical under* $\mathbb{P}$ *but typical under the size-biased measures* $\widetilde{\mathbb{P}}_x$. Our next result states that this can be achieved.

Proposition 3.5 (Bounds for the event A_N)**.** *Fix any $1 \leq \eta < \vartheta < \infty$, consider $\beta > 0, N \in \mathbb{N}$ which verify* (1.13)*, and set $\tilde{N} = \mathrm{e}^{-\eta} N$. Then we can find events $A_N \in \mathcal{F}$ such that*

$$\limsup_{N \to \infty} \mathbb{P}(A_N) \leq C_1\,\frac{\vartheta - \eta}{\eta}\,\mathrm{e}^{-(\vartheta - \eta)}\,, \tag{3.4}$$

$$\limsup_{N \to \infty} \sup_{|x| \leq \sqrt{\tilde{N}}} \widetilde{\mathbb{P}}_x(A_N^c) \leq \frac{C_2}{\eta}\,, \tag{3.5}$$

where $C_1, C_2 > 0$ are universal constants.

The reason why the bounds (3.4) and (3.5) have the specified dependence on η, ϑ will be clear below (they are determined by the mean and variance of a suitable random variable X). For the moment, it suffices to note that plugging these bounds in (3.3) we obtain

$$\limsup_{N\to\infty} \sup_{f\in\mathcal{M}_1^{\mathrm{disc}}(\sqrt{N})} \mathbb{E}[Z_N^{\beta,\omega}(f)\wedge 1] \leq 2C_1 \frac{\vartheta-\eta}{\eta} \mathrm{e}^{2\eta-\vartheta} + \frac{C_2}{\eta}\,.$$

Then, if we choose $\eta = \vartheta/3$, this concludes the proof of Proposition 2.1.

We are left with proving Proposition 3.5. To find A_N which is atypical under $\mathbb{P}$, but which becomes typical under the size-biased measure $\widetilde{\mathbb{P}}_x$, one could in theory take $A_N = \{Z_N^{\beta_N}(x) > \varepsilon_N\}$ for some $\varepsilon_N \downarrow 0$ slowly enough: in that case one easily gets $\widetilde{\mathbb{P}}_x(A_n^c) \leq \varepsilon_N \to 0$ by definition, but the difficult part remains to show that $\mathbb{P}(A_N) = \mathbb{P}(Z_N^{\beta_N}(x) > \varepsilon_N) \to 0$ for a well-chosen $\varepsilon_N \downarrow 0$, so we are back to square one.

A manageable solution is to find a *simpler random variable X which acts as a proxy for* $Z_N^{\beta,\omega}(x)$, for which we are able to compute the expectation and variance under $\mathbb{P}, \widetilde{\mathbb{P}}_x$. We clarify this strategy in the following lemma.

Lemma 3.6 (Choice of the event A_N)**.** *Consider some random variable $X = X_N$ such that*

$$\mathbb{E}[X] = 0 \qquad \textit{and} \qquad \widetilde{\mathbb{E}}_{\inf}[X] := \inf_{|x|\leq\sqrt{\tilde{N}}} \widetilde{\mathbb{E}}_x[X] > 0$$

and define the event

$$A_N = \{X \geq \tfrac{1}{2}\,\widetilde{\mathbb{E}}_{\inf}[X]\}\,.$$

Then, we get that

$$\mathbb{P}(A_N) \leq 4\,\frac{\mathbb{V}\mathrm{ar}(X)}{\widetilde{\mathbb{E}}_{\inf}[X]^2}\,, \qquad \widetilde{\mathbb{P}}_x(A_N^c) \leq 4\,\frac{\widetilde{\mathbb{V}}\mathrm{ar}_x(X)}{\widetilde{\mathbb{E}}_{\inf}[X]^2}\,.$$

The proof of the lemma follows directly by Chebychev's inequality. It therefore only remains to find a functional X such that $\mathbb{V}\mathrm{ar}(X), \widetilde{\mathbb{V}}\mathrm{ar}_x(X) \ll \widetilde{\mathbb{E}}_{\inf}[X]^2$ uniformly for $|x| \leq \sqrt{\tilde{N}}$.

The choice of X is the most delicate point in the strategy and the main novelty of our proof. We discuss this issue in Section 3.3 arriving at the explicit choice of X defined in (3.18), which may be described as *the first (linear) term in a coarse-grained chaos expansion of the partition function over time intervals of length $\tilde{N}$*. This choice of X depends on the following parameter (recall $\tilde{N} = \mathrm{e}^{-\eta}N$ from Proposition 3.5):

$$M := \frac{N}{2\tilde{N}} = \frac{\mathrm{e}^{\eta}}{2}\,. \tag{3.6}$$

We finally state our main estimates on $\mathbb{V}\mathrm{ar}(X)$, $\widetilde{\mathbb{E}}_x[X]$, $\widetilde{\mathbb{V}}\mathrm{ar}_x(X)$ that will be proved in the next sections. The first two lemmas follow from second moment calculations and are proven in Section 4. The last estimate is more difficult and will be proven in Section 5.

Lemma 3.7 (Variance bound)**.** *We have $\mathbb{E}[X] = 0$ and there is a constant $C > 0$ such that, for N sufficiently large*

$$\mathbb{V}\mathrm{ar}[X] \leq C\,\frac{\log M}{\vartheta-\eta}\,\mathrm{e}^{\vartheta-\eta}\,.$$

Lemma 3.8 (Size-biased mean bound)**.** *There is a constant $C > 0$ such that, for N sufficiently large*

$$\widetilde{\mathbb{E}}_{\inf}[X] := \inf_{|x|\leq\sqrt{\tilde{N}}} \widetilde{\mathbb{E}}_x[X] \geq C\,\frac{\log M}{\vartheta-\eta}\,\mathrm{e}^{\vartheta-\eta}\,.$$

Proposition 3.9 (Size-biased variance bound)**.** *There is a constant $C' > 0$ such that, for N sufficiently large*

$$\sup_{|x|\le\sqrt{\tilde N}} \widetilde{\mathbb{V}}\mathrm{ar}_x[X] \le C' \log M \left(\frac{1}{\vartheta-\eta}\,\mathrm{e}^{\vartheta-\eta}\right)^2 .$$

Together with Lemma 3.6, these estimates readily show that, for N sufficiently large,

$$\mathbb{P}(A_N) \le C\,\frac{\vartheta-\eta}{\log M}\,\mathrm{e}^{-(\vartheta-\eta)}, \qquad \text{and} \qquad \sup_{|x|\le\sqrt{\tilde N}} \widetilde{\mathbb{P}}_x(A_N) \le C\,\frac{1}{\log M} .$$

Since $\log M = \eta - \log 2$, see (3.6), this gives the bounds announced in Proposition 3.5. (Of course, the reason why we stated these bounds precisely in the form (3.4) and (3.5) was dictated by the computation of the mean and variance of X.)

The remainder of this section is devoted to the proof of Proposition 3.4 (see Section 3.2) and to the choice of the proxy X (see Section 3.3). We conclude with some additional considerations on the size-biased measure (see Section 3.4).

3.2. **Change of scale and measure: proof of Proposition 3.4.** We start with a *change of scale* for the starting point. To this purpose, we show an analogue of Lemma 2.7, except for the truncated mean $\mathbb{E}[Z_N^{\beta,\omega}(f)\wedge 1]$ rather than the fractional moment $\mathbb{E}[Z_N^{\beta,\omega}(f)^\gamma]$.

Lemma 3.10 (Change of scale)**.** *For any $\tilde N \le N$, we have*

$$\sup_{f\in\mathcal{M}_1^{\mathrm{disc}}(\sqrt{N})} \mathbb{E}[Z_N^{\beta,\omega}(f)\wedge 1] \le 2 \sup_{f\in\mathcal{M}_1^{\mathrm{disc}}(\sqrt{\tilde N})} \mathbb{E}\Big[Z_N^{\beta,\omega}(f)\wedge \frac{N}{\tilde N}\Big] . \tag{3.7}$$

The proof is a direct consequence of the following general lemma, combined with the same decomposition as in (2.10).

Lemma 3.11. *Let $(\alpha_i)_{1\le i\le K}$ be non-negative numbers with $\sum_{i=1}^K \alpha_i = 1$, and let $(Z_i)_{1\le i\le K}$ be non-negative random variables. Then, if we set $Z = \sum_{i=1}^K \alpha_i Z_i$, we have*

$$\mathbb{E}[Z\wedge 1] \le 2 \max_{1\le i\le K} \mathbb{E}[Z_i \wedge K].$$

Proof. Define $A_i := \{Z_i > K\}$ and let $B = \bigcup_{i=1}^K A_i$. Then, bounding $Z\wedge 1 \le 1$ on the event B and $Z\wedge 1\le Z$ on B^c, we have that

$$\mathbb{E}[Z\wedge 1] \le \mathbb{P}(B) + \mathbb{E}[Z\,\mathbf{1}_{B^c}] \le \sum_{i=1}^K \mathbb{P}(A_i) + \sum_{i=1}^K \alpha_i\,\mathbb{E}[Z_i\,\mathbf{1}_{A_i^c}] ,$$

where we have used sub-additivity for the first term and $B^c = \bigcap_{i=1}^K A_i^c \subseteq A_i^c$ for all i. Now, since $A_i = \{Z_i > K\}$, we have that $\mathbb{P}(A_i) \le \frac{1}{K}\mathbb{E}[Z_i\wedge K]$ by Markov's inequality and $\mathbb{E}[Z_i\,\mathbf{1}_{A_i^c}] \le \mathbb{E}[Z_i\wedge K]$ by definition of A_i^c. Plugging this in the above gives that

$$\mathbb{E}[Z\wedge 1] \le \sum_{i=1}^K \frac{1}{K}\,\mathbb{E}[Z_i\wedge K] + \sum_{i=1}^K \alpha_i\,\mathbb{E}[Z_i\wedge K] ,$$

which concludes the proof. □

We next use a *change of measure* argument to estimate the right-hand side of (3.7). We state it both for the truncated mean $\mathbb{E}[Z_N^{\beta,\omega}(f)\wedge \frac{N}{\tilde N}]$ and for the fractional moment $\mathbb{E}[Z_N^{\beta,\omega}(f)^\gamma]$, since the proof we have is simplified with respect to what we found in the literature.

Lemma 3.12 (Change of measure)**.** *Let $Z \ge 0$ be a non-negative random variable. For any $K > 0$ and any event $A \in \mathcal{F}$, we have*

$$\mathbb{E}[Z\wedge K] \le K\,\mathbb{P}(A) + \mathbb{E}[Z\mathbf{1}_{A^c}] .$$

Also, if $\mathbb{E}[Z]=1$, *for any* $\gamma\in(0,1)$, *we have, for any event* $A\in\mathcal{F}$

$$\mathbb{E}[Z^\gamma]\le \mathbb{P}(A)^{1-\gamma}+\mathbb{E}[Z\mathbf{1}_{A^c}]^\gamma\,.$$

The above improves and simplifies [JL24, Lem. 2.2], which controls the moment of order 1/2; in fact, we simplify its proof and get a general fractional moment (note that [JL25, Lem. 3.2] also controls a fractional moment, but in a non-optimal way).

Proof. For the first inequality, we simply bound $Z\wedge K\le K$ on A and $Z\wedge 1\le Z$ on A^c: this gives the desired bound.

For the fractional moment, we write $\mathbb{E}[Z^\gamma]=\mathbb{E}[Z^\gamma\mathbf{1}_A]+\mathbb{E}[Z^\gamma\mathbf{1}_{A^c}]$. For the first term, use Hölder's inequality to get $\mathbb{E}[Z^\gamma\mathbf{1}_A]\le\mathbb{E}[Z]^{1-\gamma}\mathbb{P}(A)^{1-\gamma}=\mathbb{P}(A)^{1-\gamma}$. For the second term, we use Jensen's inequality to get $\mathbb{E}[Z^\gamma\mathbf{1}_{A^c}]\le\mathbb{E}[Z\mathbf{1}_{A^c}]^\gamma$. This concludes the proof. ☐

In view of (3.7), we apply Lemma 3.12 and we obtain

$$\sup_{f\in\mathcal{M}_1^{\mathrm{disc}}(\sqrt{N})}\mathbb{E}\big[Z_N^{\beta,\omega}(f)\wedge 1\big]\le 2\left(\frac{N}{\tilde N}\,\mathbb{P}(A_N)+\sup_{f\in\mathcal{M}_1^{\mathrm{disc}}(\sqrt{\tilde N})}\mathbb{E}\big[Z_N^{\beta,\omega}(f)\mathbf{1}_{A_N^c}\big]\right),\tag{3.8}$$

for any $\tilde N\le N$ and any event A_N. Hence, using that $Z_N^{\beta,\omega}(f)=\sum_x f(x)Z_N^{\beta,\omega}(x)$ and $\sum_x f(x)=1$, we obtain (3.3) which completes the proof of Proposition 3.4.

3.3. **Choosing of a good proxy for the partition function.** We next discuss the *choice of the proxy* $X=X_N$ for the partition function $Z_N^{\beta,\omega}(x)$. Let us introduce some useful notation. For $n,N\in\mathbb{N}$, $x\in\mathbb{Z}^2$, we denote by $q_n(x)$ the simple random walk transition probability, that is

$$q_n(x)=q(n,x):=\mathbf{P}(S_n=x)\,.\tag{3.9}$$

A first approach: chaos expansion and L^2 projections. Let us define

$$\xi_{n,x}=\xi_{n,x}^{(\beta)}:=\mathrm{e}^{\beta\omega(n,x)}-1$$

and notice that the $(\xi_{n,x})$ are i.i.d. with $\mathbb{E}[\xi_{n,x}]=0$ and $\mathbb{E}[(\xi_{n,x})^2]=\mathrm{e}^{\lambda(2\beta)-2\lambda(\beta)}-1=:\sigma^2(\beta)$. Then, we can rewrite the partition function in the product form: $Z_N^{\beta,\omega}(x)=\mathbf{E}_x[\prod_{i=1}^n(1+\xi_{n,S_n})]$. Expanding the product, we can write

$$Z_N^{\beta,\omega}(x)=1+\sum_{k=1}^N\sum_{1\le n_1<\cdots<n_k\le N}\sum_{x_1,\dots,x_k\in\mathbb{Z}^2}\prod_{i=1}^k q(n_i-n_{i-1},x_i-x_{i-1})\prod_{i=1}^k\xi_{n_i,x_i}\,,\tag{3.10}$$

with by convention $n_0=0,x_0=x$. Let us notice the terms $\prod_{i=1}^k\xi_{n_i,x_i}$ in the above expansion are orthogonal in L^2. Therefore, one can reinterpret the above as the L^2 decomposition of $Z_N^{\beta,\omega}(x)$ over the linear subspace of L^2 generated by the orthogonal variables

$$\xi(A)=\prod_{z\in A}\xi_z\qquad\text{for } A\subset\mathbb{N}\times\mathbb{Z}^2\,.$$

Then, the chaos expansion above can be rewritten as

$$Z_N^{\beta,\omega}(x)=\sum_{A\subset[\![1,N]\!]\times\mathbb{Z}^2}q^{(x)}(A)\,\xi(A)\,,\qquad\text{with }\ q^{(x)}(A):=\mathbf{P}_x(A\subset\{(i,S_i)\}_{i\ge1})\,,\tag{3.11}$$

with the term corresponding to $A=\varnothing$ being equal to 1, by convention; we also denote $q(A)=q^{(0)}(A)$ for simplicity.

A simple choice for a proxy X_n for $Z_N^{\beta,\omega}=Z_N^{\beta,\omega}(0)$ is to take the first term in the chaos expansion, namely

$$\sum_{n=1}^N\sum_{x\in\mathbb{Z}^2}q_n(x)\xi_{n,x}\,.\tag{3.12}$$

This corresponds to the L^2 projection of $Z_N^{\beta,\omega}$ on the linear subspace of L^2 generated by the $(\xi_{n,x})$. We refer for instance to [JL25, Section 6] where the functional (3.12) is used to show that the martingale critical point is equal to 0 as soon as $R_N \to \infty$. In fact, one needs a slightly finer strategy than simply use Chebychev's inequality to bound $\tilde{\mathbb{P}}(A_N^c)$, but let us not dwell on details here. The method can be pushed to show that $Z_N^{\beta,\omega} \to 0$ in probability as soon as $\sigma^2(\beta)R_N \to \infty$; in analogy with what is done in [Ber, §4.2.2] (see Remarque 4.8).

In [BL17], the authors consider a more involved functional, namely (a slightly modified version of) the k-th order term in the chaos expansion (3.10), that is

$$\sum_{A\subset [\![1,N]\!]\times\mathbb{Z}^2, |A|=k} q(A)\,\xi(A)\,. \tag{3.13}$$

They take $k = k_N \to \infty$ slowly (in fact $k_N = \log\log N$) to show that $Z_N^{\beta,\omega} \to 0$ in probability as soon as $\liminf \sigma^2(\beta)R_N > 1$; the result is in fact stronger and the authors prove a bound on the free energy.

A new approach: a coarse-grained version of the chaos expansion. One could think of taking an even more faithful approximation of $Z_N^{\beta,\omega}$ than (3.13). A close to optimal proxy would indeed be to keep in the chaos expansion (3.10) all orders $k \le \log N$, namely

$$\sum_{A\subset [\![1,N]\!]\times\mathbb{Z}^2, |A|\le\log N} q(A)\,\xi(A)\,, \tag{3.14}$$

since it bears a positive proportion of the variance of $Z_N^{\beta,\omega}$ at criticality, *i.e.* when $\sigma^2(\beta)R_N = 1$. However, this would make the analysis incredibly technical; the calculations in [BL17] are already difficult, so dealing with variance terms in (3.14) would quickly turn into a computation nightmare.

To overcome this, we introduce a new idea to build a proxy, by using a *coarse-grained version of the chaos expansion*. Recall that we introduced in Proposition 3.5 some intermediate scale $\tilde N = e^{-\eta}N$, for some $\eta \in [1,\vartheta)$. We then decompose the partition function into contributions of strips of width $\tilde N$.

Let us assume that $N/\tilde N$ is an even integer and write $N/\tilde N = 2M$. For $j \in \{1,\dots,2M\}$, we decompose the Hamiltonian H_N in (1.7) over time intervals of size $\tilde N$, writing $H_N^{\beta,\omega} = \sum_{j=1}^{2M} \mathcal{H}_j^{\beta,\omega}(S)$ where we define

$$\mathcal{H}_j^{\beta,\omega}(S) = \sum_{n\in I_j} (\beta\omega(n,S_n) - \lambda(\beta)) \qquad \text{with} \quad I_j := [\![(j-1)\tilde N + 1, j\tilde N]\!]\,, \tag{3.15}$$

so that we can write

$$Z_N^{\beta,\omega} = \mathbf{E}\Big[\prod_{j=1}^{2M} e^{\mathcal{H}_j^{\beta,\omega}(S)}\Big]\,.$$

Then, writing each term $e^{\mathcal{H}_j^{\beta,\omega}(S)} = 1 + \Xi_j(S)$, we can perform a *coarse-grained version* of the chaos expansion: we obtain

$$Z_N^{\beta,\omega} = 1 + \sum_{j=1}^{2M} \mathbf{E}[\,\Xi_j(S)\,] + \sum_{1\le j_1<j_2\le 2M} \mathbf{E}[\,\Xi_{j_1}(S)\,\Xi_{j_1}(S)\,] + \cdots\,, \tag{3.16}$$

where we did not write the full expansion to lighten notation. Our idea is to choose the proxy X_N as (a modification of) the first term of this coarse-grained expansion. This corresponds to taking the coarse-grained version of the linear approximation (3.12).

Let us note that the random variables $\mathbf{E}[\Xi_j(S)]$ defined above can be seen as the projection of $Z_N^{\beta,\omega} - 1$ onto the subspace spanned by the $\xi(A)$ with $A \subset I_j \times \mathbb{Z}^2$. Let us thus define, for

$j \in \{1, \dots, 2M\}$,

$$X_j := \sum_{A \subset I_j \times \mathbb{Z}^2, |A| \le \log N} q(A)\, \xi(A)\,, \tag{3.17}$$

where we have added the constraint that $|A| \le \log N$; this corresponds to a truncation but will keep the main contribution to the second moment, similarly to (3.14). We then define

$$X := \sum_{\ell=1}^{M} X_{2\ell} = \sum_{\ell=1}^{M} \Pi_{\le \log N}\Big(\mathbf{E}\big[e^{\mathcal{H}_{2\ell}^{\beta,\omega}(S)} \big] - 1 \Big)\,, \tag{3.18}$$

where we recall that $\mathcal{H}_j^{\beta,\omega}(S)$ from (3.15) is the Hamiltonian restricted to the interval I_j, and we have denoted by $\Pi_{\le \log N}$ the orthogonal projection onto the subspace spanned by chaos components of order up to $\log N$ (*i.e.* by $\xi(A)$ with $|A| \le \log N$).

Remark 3.13. We have restricted the sum to even indices j to simplify some calculations later on, namely when computing covariances of $X_{2\ell}, X_{2\ell'}$. Also note that the $X_{2\ell}$ only involves indices (n,x) with $n > \tilde{N}$: this will somehow allow us to forget about the initial condition.

3.4. **More on size-biasing.** We conclude this section with some considerations on the size-biased measure. Recalling Definition 3.1, we can rewrite $\widetilde{\mathbb{P}}_f$ using the following two ingredients:

- Let $\mathbf{P}_f = \sum_x f(x) \mathbf{P}_x$ denote the law of a simple random walk with initial distribution f;
- Given a trajectory $s = (s_1, \dots, s_N)$, let $\widetilde{\mathbb{P}}_N^{\beta,(s)}$ be the product measure

$$d\widetilde{\mathbb{P}}_N^{\beta,(s)} = \prod_{n=1}^{N} e^{\beta\omega(n,s_n) - \lambda(\beta)}\, d\mathbb{P}\,,$$

  understood as the distribution of ω tilted along the trajectory of s.

Then, we have the following interpretation of the tilted measure:

$$\widetilde{\mathbb{P}}_f(A) = \mathbf{E}_f\big[\widetilde{\mathbb{P}}_N^{\beta,(S)}(A) \big]\,. \tag{3.19}$$

In a few words, $\widetilde{\mathbb{P}}_{f,N}^{\beta}$ is constructed by drawing a random walk S under $\mathbf{P}_f$ and then *tilting* the environment along the trajectory of S. This last formulation (3.19) is an easy consequence of the Fubini–Tonelli theorem, writing $Z_N^{\beta,\omega}(f)$ as $\mathbf{E}_f[\prod_{n=1}^N e^{\beta\omega(n,S_n) - \lambda(\beta)}]$ and exchanging the expectations $\mathbf{E}_f$ and $\mathbb{E}$ in (3.2).

Remark 3.14 (Size bias, reprise). Another approach to bound $\widetilde{\mathbb{P}}_x(A_N^c)$ is to use the size-biased representation (3.19). Indeed, one can introduce some well-chosen (random walk) event $B \in \sigma\{S_n, n \le N\}$ and then write $\widetilde{\mathbb{P}}_x(A_N^c) \le \mathbf{E}_x[\widetilde{\mathbb{P}}_N^{\beta,(S)}(A_N^c)\mathbf{1}_B] + \mathbf{P}_x(B^c)$. This is what is usually done in this setting, see for instance [BL17, §3] or [JL25, §6.2]. The advantage of this idea is that, once one has reduced to work on the event B, it possibly makes it easier to control $\widetilde{\mathbb{E}}_N^{\beta,(S)}[X_N]$ and $\widetilde{\mathbb{V}\mathrm{ar}}_N^{\beta,(s)}(X_N)$, and thus $\widetilde{\mathbb{P}}_N^{\beta,(S)}(A_N^c)$. We will not need such strategy, since our choice for event A_N will already make the computation of $\widetilde{\mathbb{E}}_x[X_N]$, $\widetilde{\mathbb{V}\mathrm{ar}}_x(X_N)$ manageable.

4. Second moment estimates

In this section, our goal is to prove Lemma 3.7 and Lemma 3.8, which mostly rely on second moment estimates. Proposition 3.9 requires some technical third-moment estimates and is the bulk of the proof: we prove it afterwards, in Section 5. Before that, we provide some preliminary second moment estimates, and in particular we prove Proposition 1.13.

4.1. **Notation and preliminary estimates.** For $n \in \mathbb{N}$ and $f : \mathbb{Z}^2 \to \mathbb{R}_+$, let us define $\hat{Z}_N^{\beta}(f)$ to be the L^2 projection of $Z_n^{\beta,\omega}(f)$ onto the linear subspace of L^2 generated by the $\xi(A)$ with $1 \le |A| \le \log N$: more precisely,

$$\hat{Z}_n^{\beta}(f) := \sum_{x \in \mathbb{Z}^2} f(x) \sum_{A \subset [\![1,N]\!] \times \mathbb{Z}^2, 1 \le |A| \le \log N} q^{(x)}(A)\, \xi(A). \tag{4.1}$$

Recall here that, for $A = \{(n_1, x_1), \ldots, (n_k, x_k)\}$ with $k \ge 1$, we have

$$q^{(x)}(A) := \mathbf{P}_x(A \subset \{(i, S_i)\}_{i \ge 1}) = \prod_{i=1}^{|A|} q(n_i - n_{i-1}, x_i - x_{i-1}), \tag{4.2}$$

with by convention $n_0 = n < n_1$, $x_0 = x$; recall also that $q(A) = q^{(0)}(A)$.

We stress that $\mathbb{E}[\hat{Z}_n^{\beta}(f)] = 0$, and when computing the variance (or covariances) of X_j we will need to estimate covariances of $\hat{Z}_n^{\beta}(f), \hat{Z}_n^{\beta}(g)$. For this, let us introduce the weighted collision kernel

$$q_{2i}(f, g) = \sum_{x,y \in \mathbb{Z}^2} f(x) q_{2i}(x - y) g(y),$$

and the corresponding weighted Green function

$$G_n(f, g) = \sum_{i=1}^{n} q_{2i}(f, g).$$

Let us also define, for $m \ge 1$,

$$\mathcal{V}_n = \mathcal{V}_n(\beta, N) := \sum_{k=0}^{\log N - 1} \sigma^2(\beta)^k \sum_{A \subseteq [\![1,n]\!] \times \mathbb{Z}^2,\, |A| = k} q(A)^2. \tag{4.3}$$

which will appear repeatedly in the following (with $n = \tilde{N}$ or $n = \frac{1}{2}\tilde{N}$) as the contribution to the variance of X_j after a first collision; it corresponds to the variance of a point-to-plane partition function with chaos truncated at $\log N - 1$. We then have the following result.

Proposition 4.1. *For $f, g : \mathbb{Z}^2 \to \mathbb{R}_+$, we have that*

$$\mathbb{E}[\hat{Z}_n^{\beta,\omega}(f) \hat{Z}_n^{\beta,\omega}(g)] = \sigma^2(\beta) \sum_{i=1}^{n} q_{2i}(f, g) \mathcal{V}_{n-i}.$$

As a consequence, we have that

$$\sigma^2(\beta) \mathcal{V}_{\frac{n}{2}}\, G_{\frac{n}{2}}(f, g) \le \mathbb{E}[\hat{Z}_n^{\beta,\omega}(f) \hat{Z}_n^{\beta,\omega}(g)] \le \sigma^2(\beta) \mathcal{V}_n\, G_n(f, g).$$

Proof. We use the decomposition (4.1). By orthogonality of the $\xi(A)$, we get that

$$\mathbb{E}[\hat{Z}_n^{\beta}(f) \hat{Z}_n^{\beta}(g)] = \sum_{x,y \in \mathbb{Z}^2} f(x) g(y) \sum_{k=1}^{\log N} \sigma^2(\beta)^k \sum_{A \subset [\![1,n]\!] \times \mathbb{Z}^2, |A| = k} q^{(x)}(A) q^{(y)}(A).$$

Now, notice that $q^{(x)}(A) = q(n_1, x - x_1) q(A')$, where $A' = A - (n_1, x_1)$ is the set A translated by (n_1, x_1) and where the first point has been removed (so $|A'| = k - 1$). Using the definition (4.3) of $\mathcal{V}_{n-n_1}$, we therefore observe that

$$\mathbb{E}[\hat{Z}_n^{\beta}(f) \hat{Z}_n^{\beta}(g)] = \sum_{x,y \in \mathbb{Z}^2} f(x) g(y) \sum_{n_1=1}^{n} \sum_{x_1 \in \mathbb{Z}^2} \sigma^2(\beta) q(n_1, x_1 - x) q(n_1, x_1 - y) \mathcal{V}_{n-n_1}.$$

Now, summing over x_1, we get by Chapman–Kolmogorov that

$$\sum_{x_1 \in \mathbb{Z}^2} q(n_1, x_1 - x) q(n_1, x_1 - y) = q_{2n_1}(x - y).$$

This shows that

$$\mathbb{E}[\hat{Z}_n^\beta(f)\hat{Z}_n^\beta(g)] = \sigma^2(\beta) \sum_{x,y\in\mathbb{Z}^2} f(x)g(y) \sum_{n_1=1}^{n} q_{2n_1}(x-y)\mathcal{V}_{n-n_1}\,,$$

which concludes the proof of the first part of Proposition 4.1, by definition of $q_{2n_1}(f,g)$.

For the remaining bound, we simply use that $n \mapsto \mathcal{V}_n$ is non-decreasing, to get that

$$\sigma^2(\beta) \sum_{n_1=1}^{n/2} q_{2n_1}(f,g)\mathcal{V}_{\frac{n}{2}} \le \mathbb{E}[\hat{Z}_n^\beta(f)\hat{Z}_n^\beta(g)] \le \sigma^2(\beta) \sum_{n_1=1}^{n} q_{2n_1}(f,g)\mathcal{V}_n\,,$$

which concludes the proof, by definition of $G_n(f,g)$. □

We now conclude this section by giving the following estimate on $\mathcal{V}_{\frac{1}{2}\tilde{N}}, \mathcal{V}_{\tilde{N}}$. We postpone its proof to Section 4.4 below.

Lemma 4.2. *There exist constants $c, c' \in (0,1)$ such that, for any $0 < \eta < \vartheta \ll \sqrt{\log N}$ with $\vartheta - \eta \ge 1$, if $\sigma^2(\beta) = (R_N - \frac{\vartheta}{\pi})^{-1}$ and $\tilde{N} = \mathrm{e}^{-\eta}N$, we have*

$$\frac{c}{\vartheta-\eta}\,\mathrm{e}^{\vartheta-\eta} \le \sigma^2(\beta)\mathcal{V}_{\frac{1}{2}\tilde{N}} \le \sigma^2(\beta)\mathcal{V}_{\tilde{N}} \le \frac{c'}{\vartheta-\eta}\,\mathrm{e}^{\vartheta-\eta}\,.$$

In fact, once we have Proposition 4.1 and Lemma 4.2, the estimates of $\mathbb{V}\mathrm{ar}(X)$ and $\widetilde{\mathbb{E}}_x(X)$ become straightforward.

4.2. **Variance estimate: proof of Lemma 3.7.** First of all, by orthogonality, we have that

$$\mathbb{V}\mathrm{ar}(X) = \sum_{\ell=1}^{M} \mathbb{V}\mathrm{ar}(X_{2\ell})\,.$$

Now, for any j, notice that X_j is the orthogonal projection of $\mathbf{E}[\exp(\mathcal{H}_j^{\beta,\omega})]$ onto subsets with $1 \le |A| \le \log N$. By the Markov property, $\mathbf{E}[\exp(\mathcal{H}_j^{\beta,\omega})]$ has the same distribution as $Z_{\tilde{N}}^{\beta,\omega}(\mu_j)$ where $\mu_j(z) := q((j-1)\tilde{N}, z)$ is the law of the simple random walk at time $(j-1)\tilde{N}$. Hence, we get that X_j has the same distribution as $\hat{Z}_{\tilde{N}}^{\beta,\omega}(\mu_j)$, so it follows from Proposition 4.1 and Lemma 4.2 that

$$\mathbb{V}\mathrm{ar}[X_j] = \mathbb{E}[\hat{Z}_{\tilde{N}}^{\beta,\omega}(\mu_j)^2] \le \sigma^2(\beta)\mathcal{V}_{\tilde{N}} G_{\tilde{N}}(\mu_j,\mu_j) \le \frac{c'}{\vartheta-\eta}\,\mathrm{e}^{\vartheta-\eta}\,G_{\tilde{N}}(\mu_j,\mu_j)\,.$$

Now, note that by Chapman–Kolmogorov we have that

$$q_{2i}(\mu_j,\mu_j) = \sum_{x,y\in\mathbb{Z}^2} q((j-1)\tilde{N},x)q(2i,x-y)q((j-1)\tilde{N},y) = q(2((j-1)\tilde{N}+i),0)\,.$$

In particular, we get for any $j \ge 2$ and any $i \ge 0$ that $q_{2i}(\mu_j,\mu_j) \le \frac{c}{j\tilde{N}}$ so that summing over $1 \le i \le \tilde{N}$ we end up with

$$G_{\tilde{N}}(\mu_j,\mu_j) = \sum_{i=0}^{\tilde{N}} q_{2i}(\mu_j,\mu_j) \le \frac{c}{j}\,,$$

and it follows that $\mathbb{V}\mathrm{ar}(X_j) \le \frac{c'}{j}\,\frac{1}{\vartheta-\eta}\,\mathrm{e}^{\vartheta-\eta}$. We conclude that

$$\mathbb{V}\mathrm{ar}[X] = \sum_{j=1}^{M} \mathbb{V}\mathrm{ar}[X_{2j}] \le c\,\frac{\log M}{\vartheta-\eta}\,\mathrm{e}^{\vartheta-\eta}\,,$$

as announced. □

4.3. **Tilted expectation estimate: proof of Lemma 3.8.** First of all, we obviously have that $\widetilde{\mathbb{E}}_x[X] = \sum_{j=1}^{M} \widetilde{\mathbb{E}}_x[X_{2j}]$. Now, we have by definition of $\widetilde{\mathbb{E}}_x$ and then by orthogonality and translation invariance of ω,

$$\widetilde{\mathbb{E}}_x[X_j] = \mathbb{E}[X_j Z_N^{\beta,\omega}(x)] = \mathbb{E}[\hat{Z}_{\tilde{N}}^{\beta}(\mu_j^{(0)})\hat{Z}_{\tilde{N}}^{\beta}(\mu_j^{(x)})]\,,$$

where $\mu_j^{(x)}$ is the law at time $(j-1)\tilde{N}$ of the random walk started from x, that is $\mu_j^{(x)}(z) := q((j-1)\tilde{N}, z-x)$.

Therefore, by Proposition 4.1 and Lemma 4.2, we end up with

$$\widetilde{\mathbb{E}}_x[X_j] \geq \frac{c}{\vartheta - \eta}\, \mathrm{e}^{\vartheta-\eta}\, G_{\frac{1}{2}\tilde{N}}(\mu_j^{(0)}, \mu_j^{(x)})\,.$$

We can now give a lower bound for $G_{\frac{1}{2}\tilde{N}}(\mu_j^{(0)}, \mu_j^{(x)})$ for $j \geq 2$: indeed, recalling the definition of $\mu_j^{(0)}, \mu_j^{(x)}$, we get by Chapman–Kolmogorov that

$$G_{\frac{1}{2}\tilde{N}}(\mu_j^{(0)}, \mu_j^{(x)}) = \sum_{i=1}^{\tilde{N}/2} q(2(i+(j-1)\tilde{N}), x) \geq \sum_{i=1}^{\tilde{N}/2} \frac{c}{j\tilde{N}} = \frac{c}{2j}\,.$$

Here, we have used for the first inequality that there is a constant $c > 0$ such that $q(t,x) \geq \frac{c}{t}$ uniformly for $|x| \leq \sqrt{\tilde{N}}$ and $t \geq \tilde{N}$, by the local CLT. Going back to the main estimate, we therefore get the following lower bound: for any $j \geq 2$

$$\inf_{|x|\leq\sqrt{\tilde{N}}} \widetilde{\mathbb{E}}_x[X_j] \geq \frac{c'}{j}\, \frac{1}{\vartheta-\eta}\, \mathrm{e}^{\vartheta-\eta}\,.$$

Summing over $j = 2\ell$ for $1 \leq \ell \leq M$, we obtain the announced lower bound. □

4.4. **Second moment estimates: proof of Lemma 4.2 and Proposition 1.13.** Before we start the proofs, let us introduce some further notation and useful estimate. We define

$$u(n) := \sum_{x\in\mathbb{Z}^2} q_n(x)^2 = \mathbf{P}(S_{2n} = 0)\,, \tag{4.4}$$

so that $R_N = \sum_{n=1}^{N} u(n)$. We recall that by (1.10)

$$\pi R_N = \log N + \alpha + o(1), \qquad \alpha := \gamma + 4\log 2 - \pi \approx 0.208\,. \tag{4.5}$$

In fact, we have that $0 \leq o(1) \leq \frac{\pi}{N}$.

For $I = \{i_1, \ldots, i_k\}$ with $i_1 < \cdots < i_k$, we also set

$$u(I) := \prod_{j=1}^{k} u(i_j - i_{j-1})\,,$$

with by convention $i_0 = 0$, and $u(\emptyset) = 1$. Let us also introduce, for $n \geq 2$, i.i.d random variables $T^{(n)}, T_1^{(n)}, T_2^{(n)}, \ldots$ taking values in $\{1, \ldots, n\}$ with c.d.f given by

$$\mathrm{P}(T^{(n)} \leq j) = \frac{R_j}{R_n} \mathbf{1}_{\{1,\ldots,n\}}(j)\,.$$

so in particular $\mathrm{P}(T^{(n)} = j) = \frac{u(j)}{R_n}\mathbf{1}_{\{1,\ldots,n\}}(j)$. Therefore, letting $\tau_k^{(n)} := T_1^{(n)} + \cdots + T_k^{(n)}$, we can write

$$\sum_{I\subseteq[\![1,n]\!]\,,\,|I|=k} u(I) = (R_n)^k \mathrm{P}(\tau_k^{(n)} \leq n)\,. \tag{4.6}$$

4.4.1. *Proof of Lemma 4.2.* First of all, summing over the spatial coordinate in the definition (4.3) of $\mathcal{V}_n$, we have that

$$\mathcal{V}_n = \sum_{k=0}^{\log N-1} \sigma^2(\beta)^k \sum_{I\subseteq [\![1,n]\!]\,,\,|I|=k} u(I) = \sum_{k=0}^{\log N-1} \big(\sigma^2(\beta)R_n\big)^k \mathrm{P}\big(\tau_k^{(n)} \leq n\big)\,.$$

Bounding $\mathrm{P}(\tau_k^{(n)} \leq n) \leq 1$ and $\mathrm{P}(\tau_k^{(n)} \leq n) \geq \mathrm{P}(\tau_{\log N}^{(n)} \leq n)$ and summing the geometric sum, we therefore get that

$$\frac{(\sigma^2(\beta)R_n)^{\log N}-1}{\sigma^2(\beta)R_n - 1}\,\mathrm{P}\big(\tau_{\log N}^{(n)} \leq n\big) \leq \mathcal{V}_n \leq \frac{(\sigma^2(\beta)R_n)^{\log N}-1}{\sigma^2(\beta)R_n - 1}\,.$$

Notice that for the lower bound with $n=\frac12\tilde N = \frac12 \mathrm{e}^{-\eta}N$, we have that $\log N = \log n + \eta + \log 2$, so in particular $\log N \leq 2\log n$. We can then use [CSZ19b, Proposition 1.3] to get that $\mathrm{P}(\tau_{2\log n}^{(n)} \leq n)$ converges to a positive constant. All together, we only need to get upper and lower bounds on $\sigma^2(\beta)R_n - 1$ and $(\sigma^2(\beta)R_n)^{\log N}-1$ with $n=\tilde N$ and $n=\frac12 \tilde N$.

First of all, we can use (4.5) and the fact that $\tilde N = \mathrm{e}^{-\eta}N$ to get that $R_N - R_{\tilde N} = \frac{\eta}{\pi} + o(1)$. Hence, on the one hand we have that

$$\sigma^2(\beta)R_{\tilde N} - 1 = \frac{R_{\tilde N}}{R_N - \frac{\vartheta}{\pi}} - 1 \geq \frac{\frac{\vartheta}{\pi} - \frac{\eta}{\pi} + o(1)}{R_N - \frac{\vartheta}{\pi}} \geq c\,\frac{\vartheta-\eta}{R_N}\,, \tag{4.7}$$

using also that $\frac{\vartheta}{\pi} \leq \frac12 R_N$ and $\vartheta - \eta \geq 1$. Similarly, noticing that $\frac12 \tilde N = \mathrm{e}^{-\tilde\eta}N$ with $\tilde\eta := \eta + \log 2$, we also get that

$$\sigma^2(\beta)R_{\frac12 \tilde N} - 1 \leq c'\,\frac{\vartheta-\tilde\eta}{R_N} \leq c'\,\frac{\vartheta-\eta}{R_N}\,.$$

In particular, using also that $\sigma^2(\beta)R_N = 1 + o(1)$, we get that

$$\frac{c}{\vartheta-\eta} \leq \frac{\sigma^2(\beta)}{\sigma^2(\beta)R_{\frac12\tilde N} - 1} \qquad\text{and}\qquad \frac{\sigma^2(\beta)}{\sigma^2(\beta)R_{\tilde N} - 1} \leq \frac{c'}{\vartheta-\eta}\,.$$

On the other hand, using also that $\frac{\vartheta}{\pi R_N} = o(1)$, we get for N large

$$\sigma^2(\beta)R_{\tilde N} \leq \frac{1 - \frac{\eta}{\pi R_N} + \frac{o(1)}{R_N}}{1 - \frac{\vartheta}{\pi R_N}} \leq \mathrm{e}^{-\frac{\eta}{\pi R_N} + \frac{o(1)}{R_N}}\,\mathrm{e}^{\frac{\vartheta}{\pi R_N} + 2(\frac{\vartheta}{\pi R_N})^2}\,. \tag{4.8}$$

Taking the $\log N$ power, and noting that $(\frac{\vartheta}{\pi R_N})^2 = o(\frac{1}{\log N})$, we get that

$$(\sigma^2(\beta)R_{\tilde N})^{\log N} \leq (1+o(1))\mathrm{e}^{\frac{\vartheta-\eta}{\pi R_N}\log N} \leq (1+o(1))\mathrm{e}^{\vartheta-\eta}\,,$$

using again (4.5) for the last inequality. Similarly, we get that $(\sigma^2(\beta)R_{\frac12\tilde N})^{\log N} \geq (1+o(1))\mathrm{e}^{\vartheta-\tilde\eta}$ so that, using that $\mathrm{e}^{\vartheta-\tilde\eta} = \frac12 \mathrm{e}^{\vartheta-\eta} \geq \mathrm{e}/2$, we also get that

$$(\sigma^2(\beta)R_{\frac12\tilde N})^{\log N} - 1 \geq c\,\mathrm{e}^{\vartheta-\eta}\,.$$

These estimates conclude the proof of Lemma 4.2. ☐

4.4.2. *Proof of Proposition 1.13.* First of all, notice that as for Proposition 4.1, we have that

$$\mathbb{E}[Z_N^{\beta,\omega}(f)^2] = \sigma^2(\beta)\sum_{i=1}^N q_{2i}(f,f)\mathbb{E}[Z_{N-i}^{\beta,\omega}(0)^2] \leq \sigma^2(\beta)G_N(f,f)\mathbb{E}[Z_N^{\beta,\omega}(0)^2]\,. \tag{4.9}$$

Now, if $f=\mathcal{U}_N$ we get that

$$G_N(\mathcal{U}_N,\mathcal{U}_N) = \frac{1}{|B(\sqrt{N})\cap\mathbb{Z}^2|}\sum_{i=1}^N \sum_{x,y\in B(\sqrt{N})\cap Z^2} q_{2i}(x-y) \leq C\,,$$

where we first used that $\sum_{x\in\mathbb{Z}^2} q_{2i}(x-y)=1$ and then the fact that the volume of $B(\sqrt{N})\cap\mathbb{Z}^2$ is of order N.

All together, we only need to get an upper bound on

$$\sigma^2(\beta)\mathbb{E}[Z_N^{\beta,\omega}(0)^2] = \sigma^2(\beta)\sum_{k=0}^{\infty}\sigma^2(\beta)^k \sum_{I\subseteq[\![1,N]\!]\,,\,|I|=k} u(I) = \sigma^2(\beta)\sum_{k=0}^{\infty}\big(\sigma^2(\beta)R_N\big)^k \mathrm{P}\big(\tau_k^{(N)}\le N\big)\,. \tag{4.10}$$

We now bound the probability appearing in the sum thanks to Chernoff's bound (note that it is equal to 0 if $k>N$): we have, for any $\hat\lambda>0$,

$$\mathrm{P}\big(\tau_k^{(N)}\le N\big) \le \mathrm{e}^{\hat\lambda}\mathrm{E}\big[\exp(-\tfrac{\hat\lambda}{N}T^{(N)})\big]^k\,. \tag{4.11}$$

To anticipate a bit, let us mention that we will choose $\hat\lambda = \frac{k}{\pi R_N-\vartheta} \le \frac{N}{\pi R_N-\vartheta}$, with $\pi R_N-\vartheta \sim \log N$ (recall $\vartheta \ll \log N$). To estimate the Laplace transform of $T^{(N)}$, we use the following Tauberian theorem, from [BGT89, Thm. 3.9.1].

Lemma 4.3. *For a sequence $(u(n))_{n\in\mathbb{N}}$ of positive numbers, define the quantities*

$$R(m) := \sum_{n=1}^{m} u(n) \qquad \textit{and} \qquad \hat R(\lambda) := \sum_{n=1}^{+\infty}\mathrm{e}^{-\lambda n}u(n)\,.$$

If there exist constants $a,b>0$ such that $aR(m)=\log m + b + o(1)$ as $m\to\infty$, then

$$a\hat R(\lambda) = \log\Big(\frac{1}{\lambda}\Big) + b - \gamma + \tilde o(1)\,,$$

as $\lambda\to 0$, where γ is the Euler–Mascheroni constant.

As a corollary, using (4.5),

$$\mathrm{E}\big[\exp(-\tfrac{\hat\lambda}{N}T^{(N)})\big] \le \frac{1}{\pi R_N}\sum_{n=1}^{\infty}\mathrm{e}^{-\frac{\hat\lambda}{N}n}u(n) = \frac{\log(\frac{N}{\hat\lambda}) - \alpha - \gamma + \tilde o_{\hat\lambda/N}(1)}{\pi R_N}\,,$$

where the last identity follow from Lemma 4.3, provided that $\hat\lambda/N\to 0$. Using again (4.5), this gives that

$$\mathrm{E}\big[\exp(-\tfrac{\hat\lambda}{N}T^{(N)})\big] \le \frac{\pi R_N + \log\hat\lambda - \gamma + o(1)}{\pi R_N}\,, \tag{4.12}$$

where $o(1)$ is a quantity that goes to 0 as soon as $N\to\infty$ and $\hat\lambda/N\to 0$. Going back to (4.11) and choosing $\hat\lambda=\hat\lambda_k = \frac{k}{\pi R_N-\vartheta}$, we get that for any $k\le N$:

$$\mathrm{P}\big(\tau_k^{(N)}\le N\big) \le \mathrm{e}^{\hat\lambda_k}\Big(\frac{\pi R_N - \log\hat\lambda_k - \gamma + o(1)}{\pi R_N}\Big)^k\,,$$

where the $o(1)$ in fact does not depend on k (since $\hat\lambda_k/N \le \frac{c}{\log N}$ uniformly for $k\le N$). Plugging this in (4.10) and using that $\sigma^2(\beta)R_N = \frac{\pi R_N}{\pi R_N-\vartheta}$, we get that

$$\begin{aligned}\sigma^2(\beta)\mathbb{E}[Z_N^{\beta,\omega}(0)^2] &\le \frac{\pi}{\pi R_N-\vartheta}\sum_{k=0}^{N}\mathrm{e}^{\hat\lambda_k}\Big(\frac{\pi R_N - \log\hat\lambda_k - \gamma + o(1)}{\pi R_N-\vartheta}\Big)^k \\ &\le \frac{\pi}{\pi R_N-\vartheta}\sum_{k=0}^{\infty}\mathrm{e}^{\hat\lambda_k}\mathrm{e}^{-\frac{k}{\pi R_N-\vartheta}(\log\hat\lambda_k - \vartheta + \gamma + o(1))}\,,\end{aligned}$$

where we have used the inequality $1-x\le \mathrm{e}^{-x}$.

Now, plugging $\hat{\lambda} = \frac{k}{\pi R_N - \vartheta}$, we get that

$$\sigma^2(\beta)\mathbb{E}[Z_N^{\beta,\omega}(0)^2] \le \frac{\pi}{\pi R_N - \vartheta} \sum_{k=1}^{\infty} \mathrm{e}^{(1+\vartheta-\gamma+o(1))\frac{k}{\pi R_N - \vartheta}} \cdot \Big(\frac{k}{\pi R_N - \vartheta}\Big)^{-\frac{k}{\pi R_N - \vartheta}}$$

$$= \frac{\pi}{r_N} \sum_{k=1}^{\infty} f\Big(\frac{k}{r_N}\Big) \qquad \text{where we set} \qquad \begin{cases} f(x) := \mathrm{e}^{(1+\vartheta-\gamma+o(1))x}\, x^{-x}\,, \\ r_N := \pi R_N - \vartheta\,. \end{cases}$$

Note that f is a unimodal function whose maximum is attained at some $x^* \in (0,\infty)$. If we define $i^* := \lfloor x^*\, r_N \rfloor$, and we partition the sum over k into intervals of the form $(ir_N, (i+1)r_N]$ for $i \in \mathbb{N}$, then for $i < i^*$ we have $\frac{k}{r_N} \le \frac{i+1}{r_N} \le \frac{i^*}{r_N} \le x^*$, while for $i > i^*$ we have $\frac{k}{r_N} \ge \frac{i}{r_N} \ge \frac{i^*+1}{r_N} \ge x^*$. Hence, we can estimate the sum as follows:

$$\text{for } i < i^*: \qquad \sum_{k \in (ir_N, (i+1)r_N]} f\Big(\frac{k}{r_N}\Big) \le \sum_{k \in (ir_N, (i+1)r_N]} f(i+1) \le \lceil r_N \rceil f(i+1)\,,$$

$$\text{for } i > i^*: \qquad \sum_{k \in (ir_N, (i+1)r_N]} f\Big(\frac{k}{r_N}\Big) \le \sum_{k \in (ir_N, (i+1)r_N]} f(i) \le \lceil r_N \rceil f(i)\,.$$

For $i = i^*$, since $\max_{x \in \mathbb{R}} f(x) = f(x^*)$, we bound the sum instead by

$$\sum_{k \in (i^* r_N, (i^*+1)r_N]} f\Big(\frac{k}{r_N}\Big) \le \lceil r_N \rceil f(x^*).$$

We end up with the following bound

$$\frac{1}{r_N} \sum_{k=1}^{\infty} f\Big(\frac{k}{r_N}\Big) \le \frac{\lceil r_N \rceil}{r_N} \Big(\sum_{i=1}^{\infty} f(i) + f(x^*) \Big) \le \Big(1 + \frac{1}{r_N}\Big)\Big(\sum_{i=1}^{\infty} f(i) + f(x^*) \Big).$$

Now we can use the fact that for all $i \ge 1$, we have $\frac{\mathrm{e}^i}{i^i} \le \frac{\mathrm{e}}{(i-1)!}$, so that

$$\sum_{i=1}^{\infty} f(i) = \sum_{i=1}^{\infty} \mathrm{e}^{(\vartheta-\gamma+o(1))i} \cdot \frac{\mathrm{e}^i}{i^i} \le \sum_{i=1}^{\infty} \mathrm{e}^{(\vartheta-\gamma+o(1))i} \cdot \frac{\mathrm{e}}{(i-1)!}$$

$$= \mathrm{e}^{\vartheta-\gamma+1+o(1)} \sum_{j=0}^{\infty} \frac{\mathrm{e}^{(\vartheta-\gamma+o(1))j}}{j!} = \mathrm{e}^{\vartheta-\gamma+1+o(1)} \mathrm{e}^{\mathrm{e}^{\vartheta-\gamma+o(1)}}.$$

Note that the last expression is equal to $\mathrm{e}^{\mathrm{e}^{\vartheta-\gamma+o(1)}}$ since $\vartheta \gg 1$. By direct computation, the maximum of f is attained at $x^* = \mathrm{e}^{1+\vartheta-\gamma+o(1)}$ and $f(x^*) = \mathrm{e}^{\mathrm{e}^{\vartheta-\gamma+o(1)}}$. This concludes the proof of Proposition 1.13. ☐

5. Control of the tilted variance

In this section, we control the tilted variance, *i.e.* we prove Proposition 3.9. First of all, we write $\widetilde{\mathbb{V}}\mathrm{ar}_x(X)$ as the sum of tilted covariances, that we split into two parts, namely the *diagonal* and the *off-diagonal* term:

$$\widetilde{\mathbb{V}}\mathrm{ar}_x(X) = \sum_{\ell=1}^{M} \widetilde{\mathbb{V}}\mathrm{ar}_x(X_{2\ell}) + 2 \sum_{\ell_1 < \ell_2} \widetilde{\mathbb{C}}\mathrm{ov}_x[X_{2\ell_1}, X_{2\ell_2}]\,.$$

The remaining of the proof consists in proving the following estimates.

Lemma 5.1 (Diagonal terms)**.** *There is a constant $C > 0$ such that, for any $2 \le j \le M = \mathrm{e}^{\eta}$, we have for N sufficiently large*

$$\sup_{|x| \le \sqrt{\tilde{N}}} \widetilde{\mathbb{V}}\mathrm{ar}_x(X_j) \le \frac{C}{j^2} \Big(\frac{1}{\vartheta - \eta} \mathrm{e}^{\vartheta-\eta} \Big)^2.$$

Lemma 5.2 (Off-diagonal terms)**.** *There is a constant $C>0$ such that, for any $j_2 \geq j_1 \geq 2$ with $j_2 - j_1 \geq 2$, we have*

$$\sup_{|x|\leq\sqrt{\tilde N}} \widetilde{\mathbb{C}\mathrm{ov}}_x[X_{j_1}, X_{j_2}] \leq \frac{C}{(j_2)^2}\Big(\frac{1}{\vartheta-\eta}\mathrm{e}^{\vartheta-\eta}\Big)^2 .$$

With these two claims, we finally have that

$$\widetilde{\mathbb{V}\mathrm{ar}}_x(X) \leq C\Big(\sum_{\ell=1}^M \frac{1}{\ell^2} + \sum_{1\leq\ell_1<\ell_2\leq M}\frac{1}{(\ell_2)^2}\Big)\Big(\frac{1}{\vartheta-\eta}\mathrm{e}^{\vartheta-\eta}\Big)^2 \leq C'\,(1+\log M)\Big(\frac{1}{\vartheta-\eta}\mathrm{e}^{\vartheta-\eta}\Big)^2 ,$$

where for the last inequality we have used that $\sum_{\ell=1}^M \frac{1}{\ell^2}$ is bounded by a constant and also that the sum over $\ell_1 \in [\![1, \ell_2[\![$ is bounded by $\sum_{\ell_1=1}^M \frac{1}{\ell_1} \leq \log M$. This concludes the proof of Proposition 3.9. □

It therefore remains to prove Lemma 5.1 and Lemma 5.2. We first deal with the off-diagonal term, *i.e.* Lemma 5.2, since it is a bit less technical than the diagonal term.

5.1. **Off-diagonal terms: proof of Lemma 5.2.** Let $j_2 > j_1 \geq 2$ with $j_2 - j_1 \geq 2$. Then, recalling the definition (3.17) of X_j and expanding the covariance, we can write explicitly:

$$\widetilde{\mathbb{C}\mathrm{ov}}_x[X_{j_1}, X_{j_2}] = \sum_{k_1,k_2=1}^{\log N} \sum_{\substack{A_1\subset I_{j_1}\times\mathbb{Z}^2, |A_1|=k_1 \\ A_2\subset I_{j_2}\times\mathbb{Z}^2, |A_2|=k_2}} q(A_1)q(A_2)\,\widetilde{\mathbb{C}\mathrm{ov}}_x[\xi(A_1),\xi(A_2)] .$$

Now, since the sets A_1, A_2 are disjoint, we get that

$$\begin{aligned}\widetilde{\mathbb{C}\mathrm{ov}}_x[\xi(A_1),\xi(A_2)] &= \widetilde{\mathbb{E}}_x[\xi(A_1\cup A_2)] - \widetilde{\mathbb{E}}_x[\xi(A_1)]\widetilde{\mathbb{E}}_x[\xi(A_2)] \\ &= \sigma^2(\beta)^{k_1+k_2}\big(q^{(x)}(A_1\cup A_2) - q^{(x)}(A_1)q^{(x)}(A_2)\big) ,\end{aligned}$$

using also that $\widetilde{\mathbb{E}}_x[\xi(A)] = \mathbb{E}[Z_N^{\beta,\omega}(x)\xi(A)] = \sigma^2(\beta)^{|A|}q^{(x)}(A)$, recall (3.11). All together, we have that $\widetilde{\mathbb{C}\mathrm{ov}}_x[X_{j_1}, X_{j_2}]$ is equal to

$$\sum_{k_1,k_2=1}^{\log N} \sigma^2(\beta)^{k_1+k_2} \sum_{\substack{A_1\subset I_{j_1}\times\mathbb{Z}^2, |A_1|=k_1 \\ A_2\subset I_{j_2}\times\mathbb{Z}^2, |A_2|=k_2}} q(A_1)q(A_2)\big(q^{(x)}(A_1\cup A_2) - q^{(x)}(A_1)q^{(x)}(A_2)\big) .$$

Here, denoting (s,u) the last point of A_1 and (t,v) the first point of A_2, we can write

$$\begin{gathered} q^{(x)}(A_1\cup A_2) = q^{(x)}(A_1)q_{t-s}(v-u)q(A_2') , \\ q^{(x)}(A_2) = q_t(v-x)q(A_2') , \qquad q(A_2) = q_t(v)q(A_2') ,\end{gathered}$$

with $A_2' = A_2 - (t,v)$ the set A_2 translated by its first point (with this point being removed). Hence, we have

$$q(A_1)q(A_2)\big(q^{(x)}(A_1\cup A_2) - q^{(x)}(A_1)q^{(x)}(A_2)\big) = q(A_1)q^{(x)}(A_1)\big(q_{t-s}(v-u) - q_t(v-x)\big)q_t(v)q(A_2')^2 .$$

Thus, summing over A_2' and k_2 and recalling the definition (4.3) of $\mathcal{V}_n$, we obtain that $\widetilde{\mathbb{C}\mathrm{ov}}_x[X_{j_1}, X_{j_2}]$ is equal to

$$\sum_{k_1=1}^{\log N} \sigma^2(\beta)^{k_1+1} \sum_{\substack{A_1\subset I_{j_1}\times\mathbb{Z}^2 \\ |A_1|=k_1}} \sum_{(t,v)\in I_{j_2}\times\mathbb{Z}^2} q(A_1)q^{(x)}(A_1)\big(q_{t-s}(v-u) - q_t(v-x)\big)q_t(v)\mathcal{V}_{j_2\tilde N - t} , \tag{5.1}$$

where we recall that (s,u) is the last point of $A_1 \subset I_{j_1}\times\mathbb{Z}^2$. Since $t \in I_{j_2} = [\![(j_2-1)\tilde N + 1, j_2\tilde N]\!]$, we can bound $\mathcal{V}_{j_2\tilde N - t} \leq \mathcal{V}_{\tilde N}$. Now, let us show that uniformly for $(s,u) \in I_{j_1}\times\mathbb{Z}^2$

$$\sum_{(t,v)\in I_{j_2}\times\mathbb{Z}^2} \big(q_{t-s}(v-u) - q_t(v-x)\big)q_t(v) \leq C\,\frac{j_1}{(j_2)^2} . \tag{5.2}$$

First, summing over $v \in \mathbb{Z}^2$, we get by Chapman–Kolmogorov that

$$\sum_{(t,v)\in I_{j_2}\times\mathbb{Z}^2} (q_{t-s}(v-u) - q_t(v-x))q_t(v) = \sum_{t\in I_{j_2}} q_{2t-s}(u) - q_{2t}(x)\,.$$

We can now bound $q_{2t-s}(u) - q_{2t}(x) \le q_{2t-s}(0) - q_{2t}(0) + q_{2t}(0) - q_{2t}(x)$, and control both terms separately.

For the first term, for any $t \in I_{j_2}$, $s \in I_{j_1}$, we have by the mean value theorem

$$q_{2t-s}(0) - q_{2t}(0) = u(t - s/2) - u(t) \le c\frac{s}{(t-s/2)^2} \le C\,\frac{j_1\tilde N}{(j_2\tilde N)^2}\,,$$

where the last inequality holds uniformly for $s \in I_{j_1}, t \in I_{j_2}$ since $2t-s \ge 2(j_2-1)\tilde N - j_1\tilde N \ge j_2\tilde N$, recalling that $j_2 - j_1 \ge 2$. For the other term, we use the local CLT (see e.g. [LL10, Thm. 2.1.1], in particular (2.5)), which gives that

$$\big|q_{2t}(0) - q_{2t}(x)\big| \le \frac{1}{2\pi t}\big|1 - \mathrm{e}^{-|x|^2/t}\big| + \frac{C}{t^2} \le C'\frac{|x|^2}{t^2} \le \frac{C''}{(j_2)^2\tilde N}\,,$$

uniformly for $|x| \le \sqrt{\tilde N}$ and $t \in I_{j_2}$. All together, we have shown that $q_{2t-s}(u) - q_{2t}(x) \le C\frac{j_1}{(j_2)^2}\tilde N^{-1}$ uniformly for $(s,u) \in I_{j_1}\times\mathbb{Z}^2$ and $t \in I_{j_2}$, so that summing over $t \in I_{j_2}$ we obtain (5.2).

Then, plugging (5.2) back into (5.1) (notice that all the terms are non-negative), we get that

$$\widetilde{\mathbb{C}}\mathrm{ov}_x\big[X_{j_1}, X_{j_2}\big] \le C\,\frac{j_1}{(j_2)^2}\,\sigma^2(\beta)\mathcal{V}_{\tilde N}\sum_{k_1=1}^{\log N}\sigma^2(\beta)^{k_1}\sum_{A_1\subset I_{j_1}\times\mathbb{Z}^2, |A_1|=k_1} q(A_1)q^{(x)}(A_1).$$

Now, as above, letting (n_1, x_1) be the first point in A_1 and $A_1' = A_1 - (n_1, x_1)$ the translated set (with (n_1, x_1) removed), we can write $q(A_1) = q_{n_1}(x_1)q(A_1')$ and $q^{(x)}(A_1) = q_{n_1}(x_1 - x)q(A_1')$. Therefore, summing over A_1' and k_1 gives, recalling the definition (4.3) of $\mathcal{V}_n$,

$$\sum_{k_1=1}^{\log N}\sigma^2(\beta)^{k_1}\sum_{A_1\subset I_{j_1}\times\mathbb{Z}^2, |A_1|=k_1} q(A_1)q^{(x)}(A_1) = \sigma^2(\beta)\sum_{(n_1,x_1)\in I_{j_1}\times\mathbb{Z}^2} q_{n_1}(x_1)q_{n_1}(x_1-x)\mathcal{V}_{j_1\tilde N - n_1}\,.$$

Bounding $\mathcal{V}_{j_1\tilde N - n_1} \le \mathcal{V}_{\tilde N}$ and using Chapman–Kolmogorov, this is bounded by

$$\sigma^2(\beta)\mathcal{V}_{\tilde N}\sum_{n_1\in I_{j_1}} q_{2n_1}(x) \le \frac{C}{j_1}\,\sigma^2(\beta)\mathcal{V}_{\tilde N}\,.$$

All together, we get that

$$\widetilde{\mathbb{C}}\mathrm{ov}_x\big[X_{j_1}, X_{j_2}\big] \le \frac{C}{(j_2)^2}\,\big(\sigma^2(\beta)\mathcal{V}_{\tilde N}\big)^2\,,$$

which gives the conclusion of Lemma 5.2 thanks to Lemma 4.2. □

5.2. **Diagonal term: proof of Lemma 5.1.** For the diagonal term, we will use the analogy with the computation of the third moment of $Z_N^{\beta,\omega}(f)$; some of our estimates are adaptations of [CSZ20]. Let us start by writing $\widetilde{\mathbb{E}}_x[X_j^2] = \mathbb{E}[X_j^2 Z_N^{\beta,\omega}(x)]$, so that using the definitions (3.17) of X_j and the decomposition (3.11) of $Z_N^{\beta,\omega}(x)$, we get that

$$\widetilde{\mathbb{E}}_x\big[X_j^2\big] = \sum_{\substack{A,A'\subseteq I_j\times\mathbb{Z}^2\\ 1\le|A|,|A'|\le\log N}}\ \sum_{B\subseteq[\![1,N]\!]\times\mathbb{Z}^2}\sigma(\beta)^{|A|+|A'|+|B|}q(A)q(A')q^{(x)}(B)\mathbb{E}\big[\xi(A)\xi(A')\xi(B)\big]\,.$$

First of all, we remark that *triple intersection give negligible contributions to the aforementioned sum*, see [CSZ20, Proposition 4.3].

Lemma 5.3. *We have that for any fixed* $1 < \vartheta < +\infty$, *if* $\sigma^2(\beta) = (R_N - \frac{\vartheta}{\pi})^{-1}$

$$\limsup_{N\to\infty} \sum_{\substack{A,A',B\subseteq[\![1,N]\!]\times\mathbb{Z}^2\\ B\neq A\triangle A'}} \sigma(\beta)^{|A|+|A'|+|B|} q(A)q(A')q^{(x)}(B)\mathbb{E}[\xi(A)\xi(A')\xi(B)] = 0\,.$$

In particular, we may focus on the above sum when restricted to $B = A\triangle A'$, which means only pairwise intersections. Hence, we set $C_1 = A\setminus A'$, $C_2 = A'\setminus A$ and $C_3 = A\cap A'$. Equivalently, $A = C_1\sqcup C_3$ and $A' = C_2\sqcup C_3$, $B = C_1\sqcup C_2$; note that $|A|+|A'|+|B| = 2(|C_1|+|C_2|+|C_3|)$. We can then write that $\widetilde{\mathbb{E}}_x[X_j^2]$ is $o(1)$ plus

$$\sum_{k,k'=1}^{\log N} \sum_{\substack{C_1,C_2,C_3\subseteq I_j\times\mathbb{Z}^2 \text{ disjoint}\\ |C_1|+|C_3|=k,|C_2|+|C_3|=k'}} \sigma^2(\beta)^{|C_1|+|C_2|+|C_3|} q(C_1\sqcup C_3)q(C_2\sqcup C_3)q^{(x)}(C_1\sqcup C_2)$$
$$\le \sum_{k,k'=1}^{\log N} \big(\sigma^2(\beta)R_{\tilde N}\big)^{k+k'} \mathcal{M}^{(x)}_{I_j,\tilde N}(k,k')\,,$$

where we have used that $|C_1|+|C_2|+|C_3| \le k+k'$ and $\sigma^2(\beta)R_{\tilde N}\ge 1$ (see e.g. (4.7)), and defined

$$\mathcal{M}^{(x)}_{I_j,\tilde N}(k,k') := \sum_{\substack{C_1,C_2,C_3\subseteq I_j\times\mathbb{Z}^2 \text{ disjoint}\\ |C_1|+|C_3|=k,|C_2|+|C_3|=k'}} \frac{1}{(R_{\tilde N})^{|C_1|+|C_2|+|C_3|}} q(C_1\sqcup C_3)q(C_2\sqcup C_3)q^{(x)}(C_1\sqcup C_2)\,.$$

We now use the following claim, that we prove below.

Claim 5.4. *There is a constant* $C>0$ *such that, for any* $j\ge 2$, *and any* $\tilde N$ *large enough,*

$$\sup_{k,k'\ge 1}\ \sup_{|x|\le\sqrt{\tilde N}} \mathcal{M}^{(x)}_{I_j,\tilde N}(k,k') \le \frac{C}{j^2 R_{\tilde N}^2}\,.$$

With Claim 5.4 at hand and using also that $\sigma^2(\beta)R_{\tilde N} \le e^{\frac{\vartheta-\eta+o(1)}{\pi R_N}}$ (see (4.8)), we can then bound

$$\widetilde{\mathbb{E}}_x[X_j^2] \le o(1) + \frac{C}{j^2R_{\tilde N}^2}\Big(\sum_{k=1}^{\log N} e^{k\frac{\vartheta-\eta}{\pi R_N}+o(1)}\Big)^2 \le \frac{C'}{j^2}\Big(\frac{e^{\vartheta-\eta}}{\vartheta-\eta}\Big)^2,$$

using also that $\pi R_N = \log N + \alpha + o(1)$ and $R_{\tilde N}\sim R_N$ as well. This concludes the proof of Lemma 5.1. □

5.2.1. *Proof of Claim 5.4.* To prove the claim we rewrite the sum by regrouping elements of C_1, C_2, C_3 into stretches of "same-type interaction". Let us now describe how we regroup elements of C_1, C_2, C_3 into stretches with labels, in order to define "interaction diagrams". We refer to Figure 1 for an illustration of such a diagram. In the following, we order elements by lexicographical order, *i.e.* by time, and we let $(m_i, z_i)_{1\le i\le |C_1\cup C_2\cup C_3|}$ be the ordered elements of $C_1\cup C_2\cup C_3$.

For the first stretch, we let $(a_1,x_1) = (m_1,z_1) = \inf\{C_1\cup C_2\cup C_3\}$ be the first element. We label this first stretch by $d_1 = C_r$ where C_r is the set such that $(a_1,x_1)\in C_r$, and we then add elements to the first stretch until these elements are no longer in d_1. More precisely, we let $k_1 = \sup\{k, (m_i,z_i)\in d_1\ \forall i\le k\}$ be the length of the first stretch, and $(b_1,y_1) = (m_{k_1}, z_{k_2})$ be its last point. We write $\mathcal{S}_1 := \{(m_i,z_i), 1\le i\le k_1\}$ which contains all the elements of the first stretch.

We then proceed iteratively to define the different stretches. If the stretches $\mathcal{S}_1,\dots,\mathcal{S}_{p-1}$ with respective lengths $k_1,\dots,k_{p-1}$ have been defined, the first element of the p-th stretch (if it exists), is then $(a_p,x_p) := \inf\{(C_1\cup C_2\cup C_3)\setminus(\mathcal{S}_1\cup\dots\cup\mathcal{S}_{p-1})\}$, which is in fact the element $(m_{k_1+\dots+k_{p-1}+1}, z_{k_1+\dots+k_{p-1}+1})$. We label the stretch by $d_p = C_r$ where C_r is the set to which

(a_p, x_p) belongs, and we define k_p the length of the stretch and (b_p, y_p) its last element exactly as above. The p-the stretch is then $\mathcal{S}_p := \{(m_i, z_i), k_1 + \cdots + k_{p-1} + 1 \leq i \leq k_1 + \cdots + k_p\}$.

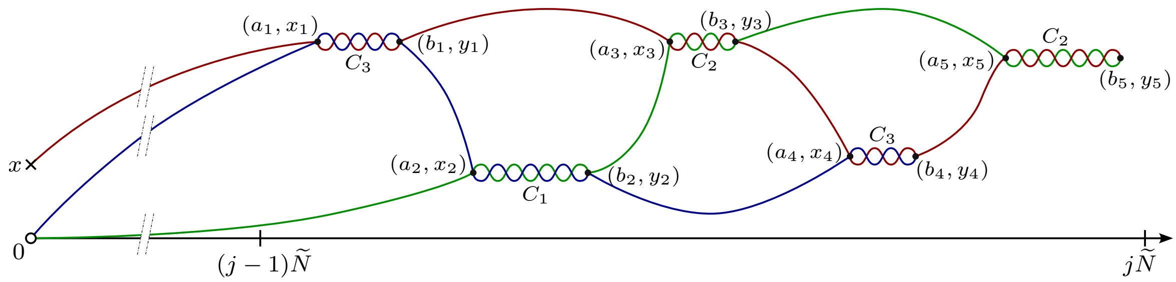

FIGURE 1. Illustration of an "interaction diagram". Pairwise interactions are regrouped in stretches of "same-type" interaction labeled by C_r, because of the term $q(C_r \sqcup C_{\hat{r}})q((C_r \sqcup C_{\check{r}}))$. A labeled diagram then corresponds to a collections of stretches with a label d_p, a length (cardinality) k_p and ordered starting and ending points (a_p, x_p), b_p, y_p. In the above diagram, there are $\ell = 5$ stretches.

We can then interpret the p-th stretch $\mathcal{S}_p$, of label $d_p = C_r$, as the contribution to the sum of the intersections $(C_r \sqcup C_{\hat{r}}) \cap (C_r \sqcup C_{\check{r}})$. More precisely, if we denote by $\ell \geq 2$ the total number of stretches, we notice that we can then write

$$
\begin{aligned}
& q(C_1 \sqcup C_3)q(C_2 \sqcup C_3)q^{(x)}(C_1 \sqcup C_2) \\
&\quad = q(a_1, x_1 - x_0^1)q(a_1, x_1 - x_0^2) \prod_{i=2}^{k_1} q(m_i - m_{i-1}, z_i - z_{i-1})^2 \\
&\qquad \cdot q(a_2 - b_1, x_2 - y_1)q(a_1, x_1 - x_0^2) \prod_{i=k_1+1}^{k_2} q(m_i - m_{i-1}, z_i - z_{i-1})^2 \\
&\qquad \cdot \prod_{p=3}^{\ell} q(a_p - b_{p-1}, x_p - y_{p-1})q(a_p - b_{p-2}, x_p - y_{p-2}) \prod_{i=k_1+\cdots+k_{p-1}+1}^{k_1+\cdots+k_p} q(m_i - m_{i-1}, z_i - z_{i-1})^2 \,,
\end{aligned}
$$

where $x_0^1, x_0^2, x_0^3 \in \{0, x\}$ and depend on whether the label of the first stretch is $d_1 = C_1$ or C_2 (in which case $x_0^1 = x_0^2 = 0$ and $x_0^3 = x$) or $d_1 = C_3$ (in which case $x_0^1 = x$, and $x_0^2 = x_0^3 = 0$); we refer to Figure 1 for an illustration.

In the big sum defining $\mathcal{M}^{(x)}_{I_j,\tilde{N}}$, we can in fact perform a partial sum inside each stretch: the (internal) contribution of stretch only depend on its length and starting and ending point. In particular, the contribution of the p-th stretch is then $Q^{*k_p}(b_p - a_p, y_p - x_p)$, where we defined

$$
Q(m, z) = Q^{(\tilde{N})}(m, z) := \frac{q(m, z)^2}{R_{\tilde{N}}} \mathbf{1}_{\{1 \leq m \leq \tilde{N}\}} \,.
$$

All together, we can rewrite the formula for $\mathcal{M}^{(x)}_{I_j,\tilde{N}}$ by summing over diagrams of labeled stretches the contribution of each diagram. Denoting $\ell \geq 2$ the number of *stretches* in the diagram, and decomposing over the label, length, starting and ending point of each stretch, we can therefore write

$$\mathcal{M}^{(x)}_{I_j,\tilde N} \le \sum_{\ell=2}^{\infty} \sum_{d_1,\dots,d_\ell \text{ labeling}} \sum_{\substack{k_1,\dots,k_\ell\ge 0\\ \mathcal{C}_{k,k'}(k_1,\dots,k_\ell)}} \sum_{\substack{a_1\le b_1<\dots<a_\ell\le b_\ell\in I_j\\ x_1,y_1,\cdots,x_\ell,y_\ell\in\mathbb{Z}^2}}$$

$$\sup_{|x_0^1|,|x_0^2|,|x_0^3|\le\sqrt{\tilde N}} \frac{q(a_1,x_1-x_0^1)q(a_1,x_1-x_0^2)}{R_{\tilde N}}\, Q^{*k_1}(b_1-a_1,y_1-x_1)$$

$$\cdot \frac{q(a_2,x_2-x_0^3)q(a_2-b_1,x_2-y_1)}{R_{\tilde N}}\, Q^{*k_2}(b_2-a_2,y_2-x_2)$$

$$\cdot \prod_{p=3}^{\ell} \frac{q(a_p-b_{p-1},x_p-y_{p-1})q(a_p-b_{p-2},x_p-y_{p-2})}{R_{\tilde N}}\, Q^{*k_p}(b_p-a_p,y_p-x_p)\,,$$

where $\mathcal{C}_{k,k'}(k_1,\dots,k_\ell)$ stands for the constraint that $|C_1|+|C_3|=k$, $|C_2|+|C_3|=k'$ (which can be read thanks to the labeling of the stretches). Note that this is only an upper bound because we have taken the supremum over x_0^1,x_0^2,x_0^3 in order to forget about the initial point.

We now use the following basic estimate: there exits a constant $\hat c>1$ such that

$$q(m,z)\le \sup_{y\in\mathbb{Z}^2} q(m,y)\le \hat c\, u(m)\,.$$

We then apply this estimate to the kernels $q(a_p-b_{p-2},x_p-y_{p-2})$, as well as $q(a_2,x_2-x_0^3)$ and $q(a_1,x_1-x_0^1)$. We can then sum over the space variable iteratively, starting from $y_\ell,x_\ell,y_{\ell-1},x_{\ell-1}$, etc. Defining

$$U_k(m)=\sum_{z\in\mathbb{Z}^2} Q^{*k}(m,z)=K^{*k}(m) \quad \text{ with } \quad K(m)=\sum_{z\in\mathbb{Z}^2} Q(m,z):=\frac{u(m)}{R_{\tilde N}}\mathbf{1}_{\{1\le m\le\tilde N\}}\,,$$

and using that $\sum_{x_p\in\mathbb{Z}^2} q(a_p-b_{p-1},x_p-y_{p-1})=1$, we then obtain

$$\mathcal{M}^{(x)}_{I_j,\tilde N} \le \sum_{\ell=2}^{\infty}(\hat c)^\ell \sum_{d_1,\dots,d_\ell \text{ labeling}} \sum_{\substack{k_1,\dots,k_\ell\ge 0\\ \mathcal{C}_{k,k'}(k_1,\dots,k_\ell)}} \sum_{a_1\le b_1<\dots<a_\ell\le b_\ell\in I_j}$$

$$\frac{u(a_1)u(a_2)}{R_{\tilde N}^2} U_{k_1}(b_1-a_1)U_{k_2}(b_2-a_2)\cdot\prod_{p=3}^{\ell} K(a_p-b_{p-2})U_{k_p}(b_i-a_i)\,.$$

We now sum over $b_\ell,b_{\ell-1}$ (these are free ends of the diagram, see Figure 1), and we use $\sum_{m=0}^{\infty} U_k(m)=1$. This is in fact a crucial step since now we can sum over $k_\ell,k_{\ell-1}$ and forget about the constraint $\mathcal{C}(k,k')$. We now sum over the other k_p's and introduce the notation $U(m):=\sum_{k\ge 0} U_k(m)$. Using also that $u(a_1)u(a_j)\le \frac{c}{j\tilde N}$ uniformly for $a_1,a_2\in I_j$, we get

$$\mathcal{M}^{(x)}_{I_j,\tilde N}(k,k')\le \frac{c^2}{j^2R_{\tilde N}^2}\sum_{\ell=2}^{\infty}(\hat c)^\ell \sum_{d_1,\cdots,d_\ell \text{ labeling}} J_\ell\,,$$

where we have set

$$J_\ell := \sum_{\substack{a_1\le b_1<\dots<a_{\ell-2}\\ \le b_{\ell-2}<a_{\ell-1}<a_\ell\in I_j}} \frac{1}{\tilde N}U(b_1-a_1)\frac{1}{\tilde N}U(b_2-a_2)\prod_{i=3}^{\ell-2}K(a_i-b_{i-2})U(b_i-a_i)\prod_{i=\ell-1}^{\ell}K(a_i-b_{i-2})\,. \tag{5.3}$$

Note that there are $3\cdot 2^{\ell-1}$ possible labelings. Note also that we can interpret J_ℓ as a probability, as follows; let us also stress that J_ℓ does not depend on j so we can take $j=1$. Indeed, let τ,τ' be independent renewals with step probability mass function $K(m)=\frac{u(m)}{R_{\tilde N}}\mathbf{1}_{\{1\le m\le\tilde N\}}$ and starting points uniform in $[\![1,\tilde N]\!]$. If we denote by $\mathcal{L}_{\tilde N}(\tau,\tau')$ the number of alternating stretches of τ,τ', then we can write $J_\ell=\mathrm{P}(\tau\cap\tau'=\emptyset,\mathcal{L}_{\tilde N}(\tau,\tau')\ge\ell)$. We refer to Figure 2 for an illustration.

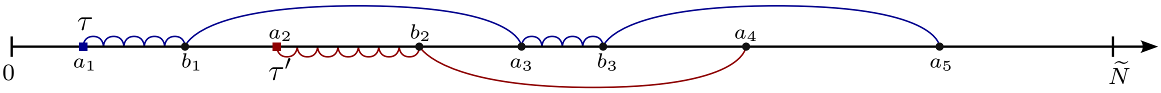

FIGURE 2. Illustration of the interpretation of the formula (5.3) for J_ℓ in terms of alternating stretches of two independent (and identically distributed) renewals τ, τ'. The first point is chosen uniformly in $[\![1, \tilde N]\!]$ and they then have inter-arrival distribution $K(m)$ defined above. The stretches alternate between τ and τ' and have starting and ending point denoted by a_p, b_p, in reference to the interaction diagrams (see Figure 1), and we denote by $\mathcal{L}_{\tilde N}(\tau,\tau')$ the number of different stretches: in the above picture we have $\ell = 5$. Note that the last stretches only have a starting point since we have already summed over their ending points (this allowed us to forget about the constraint $\mathcal{C}(k,k')$), see the formula (5.3) for J_ℓ.

In conclusion, we have, uniformly in $k, k' \geq 1$,

$$\sup_{|x|\le\sqrt{\tilde N}} \mathcal{M}^{(x)}_{I_j,\tilde N}(k,k') \le \frac{3c^2}{2j^2 R^2_{\tilde N}} \sum_{\ell=2}^{\infty} (2\hat c)^\ell J_\ell\,, \qquad \text{with} \qquad J_\ell := \mathrm{P}\big(\tau\cap\tau' = \emptyset, \mathcal{L}_{\tilde N}(\tau,\tau') \ge \ell\big)\,.$$

We can now conclude using the following claim, which controls the probability J_ℓ.

Claim 5.5. *For every $\varepsilon > 0$, there exists a constant $C_\varepsilon < +\infty$ such that*

$$J_\ell := \mathrm{P}\big(\tau\cap\tau' = \emptyset, \mathcal{L}_{\tilde N}(\tau,\tau') \ge \ell\big) \le C_\varepsilon\, \varepsilon^\ell \qquad \forall \ell \ge 2\,.$$

In particular, this gives that $\sum_{\ell=2}^\infty C^\ell J_\ell < +\infty$, or equivalently $\mathrm{E}\big[C^{\mathcal{L}_{\tilde N}(\tau,\tau')} \mathbf{1}_{\{\tau\cap\tau'=\emptyset\}}\big] < +\infty$, for any $C = 2\hat c$. This readily concludes the proof of Claim 5.4. ☐

5.2.2. *Proof of Claim 5.5.* The proof is similar to what is done in [CSZ20, Section 5.3], but we give the details for completeness. We actually directly show that $J_\ell := \mathrm{P}\big(\tau\cap\tau' = \emptyset, \mathcal{L}_{\tilde N}(\tau,\tau') \ge \ell\big)$ decays faster than exponentially in ℓ. So let us fix in the following $\ell \geq 2$.

We start by the definition (5.3), that we re-write as

$$J_\ell = \sum_{\substack{0<a_1\le b_1<\cdots<a_{\ell-2}\\ \le b_{\ell-2}<a_{\ell-1}<a_\ell\le\tilde N}} \frac{1}{\tilde N^2} \prod_{i=1}^{\ell-2} U(b_i - a_i) \prod_{i=3}^{\ell} K(a_i - b_{i-2})\,.$$

Now, by [CSZ19a, Theorem 1.4] (and recalling (4.4)-(4.5) for controlling $u(m)$ and $R_{\tilde N}$), we have that there is a constant $C > 1$ such that

$$U(m) \le C\,\frac{\log \tilde N}{\tilde N} \cdot G_0\Big(\frac{m}{\tilde N}\Big)\,, \qquad K(m) \le \frac{C}{\log \tilde N}\,\frac{1}{m}\,,$$

where $G_0(t) := \int_0^\infty \frac{1}{\Gamma(s+1)}\, s\, t^{s-1}\, \mathrm{e}^{-\gamma s}\,\mathrm{d}s$ for $t \in (0,1]$ is the so-called Dickman subordinator. (Note that by taking $\tilde N$ large we could make the constant C arbitrarily close to 1.)

Plugging this in the above formula (and noticing that all the $\log \tilde N$ cancel out), this gives

$$\begin{aligned} J_\ell &\le C^{\ell-2} \sum_{\substack{0<a_1\le b_1<\cdots<a_{\ell-2}\\ \le b_{\ell-2}<a_{\ell-1}<a_\ell\le\tilde N}} \frac{1}{\tilde N^2} \prod_{i=1}^{\ell-2} \frac{1}{\tilde N} G_0\Big(\frac{b_i-a_i}{\tilde N}\Big) \prod_{i=3}^{\ell} \frac{1}{\tilde N}\,\frac{\tilde N}{a_i - b_{i-2}} \\ &\le (C')^\ell \int\cdots\int_{\substack{0<s_1<t_1<\cdots<s_{\ell-2}\\ \le t_{\ell-2}<s_{\ell-1}<s_\ell<1}} \prod_{i=1}^{\ell-2} G_0(t_i - s_i) \prod_{i=3}^{\ell} \frac{1}{s_i - t_{i-2}}\,\mathrm{d}\mathbf{s}\,\mathrm{d}\mathbf{t}\,. \end{aligned}$$

With a change of variable $u_i = t_i - s_i$ for $1 \le i \le \ell - 2$, $u_\ell = s_\ell - s_{\ell-1}$, and $v_i = s_i - t_{i-1}$ for $i \le \ell - 1$, we get

$$J_\ell \le (C')^\ell \underset{\substack{u_i \in (0,1), v_i \in (0,1) \\ u_1 + \cdots + u_\ell + v_1 + \cdots + v_\ell < 1}}{\int \cdots \int} \prod_{i=1}^{\ell-2} G_0(u_i) \prod_{i=3}^{\ell} \frac{1}{v_i + u_{i-1} + v_{i-1}} \, \mathrm{d}\mathbf{u} \, \mathrm{d}\mathbf{v} \,.$$

Then, bounding $v_i + u_{i-1} + v_{i-1} \ge v_i + v_{i-1}$ and introducing a multiplier $\lambda > 0$ and using that $\prod_{i=1}^{\ell-2} \mathrm{e}^{\lambda u_i} \le \mathrm{e}^\lambda$, we get

$$J_\ell \le (C')^\ell \mathrm{e}^\lambda \Big(\int_0^1 G_0(u) \mathrm{e}^{-\lambda u} \, \mathrm{d}u \Big)^{\ell-2} \int_{(0,1)^{\ell-2}} \prod_{i=3}^{\ell} \frac{1}{v_i + v_{i-1}} \, \mathrm{d}\mathbf{v} \,.$$

On the one hand, by [CSZ20, Lemma 5.2], there exists $c < \infty$ such that for all $\lambda \ge 1$

$$\int_0^1 G_0(u) \mathrm{e}^{-\lambda u} \, \mathrm{d}u \le \frac{c}{2 + \log \lambda} \,,$$

so we end up with

$$J_\ell \le (C')^2 \mathrm{e}^\lambda \Big(\frac{cC'}{2 + \log \lambda} \Big)^{\ell-2} \int_{(0,1)^{\ell-2}} \prod_{i=3}^{\ell} \frac{1}{v_i + v_{i-1}} \, \mathrm{d}\mathbf{v} \,,$$

and it remains to control the last integral.

For this, define $\varphi^{(0)}(v) \equiv 1$ and by iteration $\varphi^{(k)}(v) = \int_{(0,1)} \frac{1}{v+u} \varphi^{(k-1)}(u) \, \mathrm{d}u$, so that the integral is equal to $\int_0^1 \varphi^{(\ell-2)}(v) \, \mathrm{d}v$. We now show by iteration that for any $k \ge 1$

$$\forall v \in (0,1) \qquad \varphi^{(k)}(v) \le \frac{\pi^k}{\sqrt{v}} \,.$$

The base case $k = 0$ is trivial, since $\varphi^{(0)}(v) = 1 \le \frac{1}{\sqrt{v}}$ for $v \in (0,1)$. For the inductive step, we assume that $\varphi^{(k-1)}(v) \le \frac{\pi^{k-1}}{\sqrt{v}}$ for all $v \in (0,1)$. Then, we have

$$\begin{aligned} \varphi^{(k)}(v) = \int_0^1 \frac{1}{v+u} \varphi^{(k-1)}(u) \, \mathrm{d}u &\le \pi^{k-1} \int_0^1 \frac{1}{\sqrt{u}(v+u)} \, \mathrm{d}u \,. \\ &\overset{t = u^2}{=} 2\pi^{k-1} \int_0^1 \frac{1}{v + t^2} \, \mathrm{d}t = \frac{2\pi^{k-1}}{\sqrt{v}} \arctan\Big(\frac{1}{\sqrt{v}} \Big) \le \frac{\pi^k}{\sqrt{v}}. \end{aligned}$$

Therefore, we obtain that

$$\int_{(0,1)^{\ell-2}} \prod_{i=3}^{\ell} \frac{1}{v_i + v_{i-1}} \, \mathrm{d}\mathbf{v} = \int_{(0,1)} \varphi^{(\ell-2)}(v) \, \mathrm{d}v \le 2\pi^{\ell-2} \,.$$

All together, we conclude that

$$J_\ell \le 2(C')^2 \mathrm{e}^\lambda \Big(\frac{cC'\pi}{2 + \log \lambda} \Big)^{\ell-2}$$

Taking $\lambda = \lambda_\varepsilon$ large enough concludes the proof of Claim 5.5. ☐

Appendix A. The coarse-graining procedure

In this section, we prove Proposition 2.4. First of all, let us present some "finite-volume criterion" result which explains how having a small fractional moment at some given time scale then triggers an exponential decay of the partition function at a larger time scale (but keeping the same starting distribution).

Lemma A.1 (Finite-volume criterion)**.** *If there is some scale N_0 such that*

$$\sup_{\varphi \in \mathcal{M}_1(B(\sqrt{N_0}))} \mathbb{E}\big[Z_{N_0}^{\beta,\omega}(\varphi)^{1/2}\big] \le \frac{1}{300}\,, \tag{A.1}$$

then for any $N \ge N_0$, we have that

$$\sup_{\varphi \in \mathcal{M}_1(B(\sqrt{N_0}))} \mathbb{E}\big[Z_{N}^{\beta,\omega}(\varphi)^{1/2}\big] \le \mathrm{e}^{-\lfloor N/N_0 \rfloor}\,.$$

Proof of Lemma A.1. We will prove the result only when N is an integer multiple of N_0, *i.e.* $N = mN_0$ for some $m \in \mathbb{N}$. For any integers $s < t$, for any probability measure μ on $\mathbb{Z}^2$ and any $B \subset \mathbb{Z}^2$, let us introduce the notation

$$Z_{s,t}^{\beta,\omega}(\mu; B) := \mathbf{E}_\mu\Big[\exp\Big(\sum_{n=s+1}^{t} (\beta\omega(n,S_n) - \lambda(\beta))\Big)\mathbf{1}_{\{S_t \in B\}}\Big]\,,$$

which is the partition function of a polymer with initial distribution μ at time s and constrained to end in B at time t. Let us also denote $Z_{s,t}^{\beta,\omega}(x,y)$ when μ is a Dirac mass at x and B is reduced to the set $\{y\}$.

Then, for some "squeleton" $\mathcal{Y} = (y_i)_{i\ge 1} \in (\mathbb{Z}^2)^{\mathbb{N}}$, let define a $\sqrt{N_0}$-scale coarse-grained partition function starting from $\varphi \in \mathcal{M}_1(B(\sqrt{N_0}))$ and with squeleton $\mathcal{Y}$ by setting

$$Z_{mN_0}^{\beta,\omega}(\varphi;\mathcal{Y}) = \sum_{x_0 \in B(0)} \mu(x_0) \sum_{x_1 \in B(y_1)} \cdots \sum_{x_m \in B(y_m)} \prod_{j=1}^{m} Z_{(j-1)N_0, jN_0}(x_{j-1}, x_j)\,,$$

where for simplicity we denoted $B(y) := B(y\sqrt{N_0}, \frac12\sqrt{N_0})$ the ball centered at $y\sqrt{N_0}$ of radius $\frac12\sqrt{N_0}$; note also that we have used the Markov property to write the product of point to point partition functions. Using the standard inequality $(\sum_i z_i)^{1/2} \le \sum_i z_i^{1/2}$ for non-negative $(z_i)_i$, we then get that for any $m \in \mathbb{N}$,

$$Z_{mN_0}^{\beta,w}(\varphi)^{1/2} = \Big(\sum_{(y_1,\dots,y_m)\in(\mathbb{Z}^2)^m} Z_{mN_0,\beta}^{\omega}(\varphi;\mathcal{Y})\Big)^{1/2} \le \sum_{(y_1,\dots,y_m)\in(\mathbb{Z}^2)^m} Z_{mN_0,\beta}^{\omega}(\varphi;\mathcal{Y})^{1/2}\,, \tag{A.2}$$

so that we are reduced to estimating a fractional moment along a squeleton $\mathcal{Y}$.

Notice now that we have some coarse-grained product structure for $Z_{kN_0}^{\beta,\omega}(\mathcal{Y})$: indeed, we can write

$$Z_{(k+1)N_0}^{\beta,\omega}(\varphi;\mathcal{Y}) = Z_{kN_0}^{\beta,\omega}(\varphi;\mathcal{Y})\; Z_{kN_0,(k+1)N_0}^{\beta,\omega}(\mu_{k,\varphi,\mathcal{Y}}^{\beta,\omega}; B(y_{k+1}))\,,$$

where $\mu_{k,\varphi,\mathcal{Y}}^{\beta,\omega}$ is the $\mathcal{Y}$-squeleton polymer probability distribution, supported on on $B(y_k)$, given by

$$\mu_{k,\varphi,\mathcal{Y}}^{\beta,\omega}(x) := \frac{1}{Z_{kN_0}^{\beta,\omega}(\varphi;\mathcal{Y})} \sum_{x_0 \in B(0)} \mu(x_0) \sum_{x_1 \in B(y_1)} \cdots \sum_{x_{k-1} \in B(y_{k-1})} \prod_{j=1}^{k} Z_{(j-1)N_0,jN_0}(x_{j-1},x_j)\mathbf{1}_{\{x_k = x\}}\,.$$

Therefore, taking the conditional expectation with respect to $\mathcal{F}_{kN_0}$ and using that $\mu_{k,\varphi,\mathcal{Y}}^{\beta,\omega}$ is $\mathcal{F}_{kN_0}$-measurable, we get that

$$\mathbb{E}\Big[Z_{(k+1)N_0}^{\beta,\omega}(\varphi;\mathcal{Y})^{1/2}\;\Big|\;\mathcal{F}_{kN_0}\Big] \le Z_{kN_0}^{\beta,\omega}(\varphi;\mathcal{Y})^{1/2} \sup_{\mu\in\mathcal{P}_1(B(y_k))} \mathbb{E}\Big[Z_{kN_0,(k+1)N_0}^{\beta,\omega}\big(\mu; B_{\tilde N}(y_{k+1})\big)^{1/2}\Big]\,.$$

Therefore, if we define

$$\mathcal{Q}(y) := \sup_{\mu \in \mathcal{P}_1(B(\sqrt{N_0}))} \mathbb{E}\Big[Z_{0,N_0}^{\beta,\omega}\big(\mu; B(y)\big)^{1/2}\Big]\,, \tag{A.3}$$

then by translation invariance we get by iteration that

$$\sup_{\varphi\in\mathcal{M}_1(B(\sqrt{N_0}))} \mathbb{E}\Big[Z_{mN_0}^{\beta,w}(\varphi;\mathcal{Y})^{1/2}\Big] \le \prod_{i=1}^{m} \mathcal{Q}(y_i - y_{i-1})\,,$$

so that plugged into (A.2) we get that

$$\mathbb{E}[(Z^{\beta,w}_{mN_0})^{1/2}] \le \sum_{(y_1,\dots,y_m)\in(\mathbb{Z}^2)^m} \prod_{i=1}^m \mathcal{Q}(y_i - y_{i-1}) = \Big(\sum_{y\in\mathbb{Z}^2} \mathcal{Q}(y)\Big)^m .$$

It therefore only remains to show that under (A.1) we have that $\sum_{y\in\mathbb{Z}^2} \mathcal{Q}(y) \le \mathrm{e}^{-1}$.

First of all, we always have that $Q(y) \le \frac{1}{300}$, thanks to Equation (A.1). On the other hand, simply applying Jensen's inequality, we have that

$$\mathbb{E}[Z^{\beta,\omega}_{0,N_0}(\mu; B(y))^{1/2}] \le \sqrt{\sum_{|x|\le\frac12\sqrt{N_0}} \mu(x)\mathbf{P}_x(S_{N_0}\in B_{\tilde N}(y))} \le \sqrt{\mathbf{P}(S_{N_0}\in B(y\sqrt{N_0},\sqrt{N_0}))},$$

where we have widened the ball around $y\sqrt{N_0}$ by $\frac12\sqrt{N_0}$ to account for the worst case scenario for the starting point $|x| \le \frac12\sqrt{N_0}$. Now, notice that $(\pm S^{(1)}_n \pm S^{(2)}_n)_{n\ge0}$ are standard simple random walks in dimension 1, so that

$$\mathbf{P}\Big(S_{N_0}\in B(y\sqrt{N_0},\sqrt{N_0})\Big) \le \mathbf{P}\Big(\mathrm{SRW}_{N_0} \ge (|y|_1-2)\sqrt{N_0}\Big) \le \mathrm{e}^{-(|y|_1-2)^2/2},$$

where the last inequality is standard.

Therefore, for any integer threshold $K\ge1$, we obtain that

$$\sum_{y\in\mathbb{Z}^2}\mathcal{Q}(y) \le \sum_{|y|_1\le K}\frac{1}{300} + \sum_{|y|_1>K}\mathrm{e}^{-(|y|_1-2)^2/4} = (2K^2+2K+1)\cdot\frac{1}{300} + \sum_{r>K} 4r\mathrm{e}^{-(r-2)^2/4}.$$

Now, it turns out that for $K=6$ the first term is bounded by $\frac{29}{100}$ and the second by $\frac{6}{100}$, with $\frac{35}{100} < \mathrm{e}^{-1}$. This concludes the proof. $\square$

Appendix B. Lower bound on the free energy

Let us prove the lower bound in (1.19) from Theorem 1.11, using the same strategy as in [BL17]. The idea is to start from the super-additivity of $\mathbb{E}[\log Z^\beta_N]$, which gives that

$$\mathtt{F}(\beta) = \sup_{N\ge1}\frac{1}{N}\mathbb{E}[\log Z^\beta_N],$$

see e.g. [Com17, Theorem 2.1].

We will apply this inequality for some specific $N_c = N_c(\beta)$ such that $\sigma^2(\beta)R_{N_c} = 1$ (in other words such that $\vartheta(\beta,N) := \pi R_N - \frac{\pi}{\sigma^2(\beta)} = 0$ in (1.15)), which therefore gives that

$$\mathtt{F}(\beta) \ge \frac{1}{N_c(\beta)}\mathbb{E}[\log Z^\beta_{N_c}] \ge c\,\mathrm{e}^{-\frac{\pi}{\sigma^2(\beta)}}\mathbb{E}[\log Z^\beta_{N_c}],$$

using that $\pi R_{N_c} = \log N_c + \alpha + o(1)$ as $N_c\to\infty$ (or $\beta\downarrow0$), see (1.10). What remains to prove is therefore the following lemma, which estimates $\mathbb{E}[\log Z^{\beta,\omega}_{N_c}]$.

Lemma B.1. *Let $N\ge1$ and let $\beta=\beta_c(N)$ be such that $\sigma^2(\beta_c)R_N = 1$. Then, there is some constant $C>0$ such that, for all $N\ge2$, we have*

$$\mathbb{E}[\log Z^{\beta_c(N)}_N] \ge -C(\log N)^4.$$

With this lemma at hand, and since $\beta_c(N_c(\beta)) = \beta$, this readily gives that $\mathbb{E}[\log Z^{\beta_c(N)}_N] \ge -C'\sigma^2(\beta)^{-4}$, using also that $\log N_c(\beta) = \frac{\pi}{\sigma^2(\beta)} + \alpha + o(1)$. This concludes the proof of the lower bound in Theorem 1.11.

Remark B.2. The bound in Lemma B.1 is of course not optimal, namely we expect that $\mathbb{E}[\log Z^{\beta_c(N),\omega}_N] \sim -c\log\log N$. In fact, one conjectures that $\lambda_N^{-1}(\log Z^{\beta_c(N),\omega}_N - \lambda_N^2)$ converges in distribution if $\lambda_N = \log\log N$ (we expect asymptotic normality in the lower quasi-critical regime, with $\lambda_N = \log(\frac{\log N}{|\vartheta_N|})$). Combined with super-additivity, this would give the lower bound

$-(cst.)\log(\frac{1}{\sigma^2(\beta)})\mathrm{e}^{-\pi/\sigma^2(\beta)}$ for the free energy. In fact, we believe that our upper bound in Theorem 1.11 is not completely sharp since we obtain it from an averaged starting point. We would therefore expect the following behavior for the free energy:

$$\mathtt{F}(\beta) \sim -c\log\Big(\frac{1}{\sigma^2(\beta)}\Big)\mathrm{e}^{-\pi/\sigma^2(\beta)} \qquad \text{as } \beta \downarrow 0\,.$$

Proof of Lemma B.1. The proof relies on concentration inequalities to estimate the left tail of $\log Z_{N^\beta}$. We use the following concentration inequality from [CTT17, Prop. 3.4].

Proposition B.3. *Assume that the environment is bounded, i.e. $|\omega| \le K$, and let f be a convex function. Then, there exists some constant $c > 0$ such that for any a, M and $t > 0$, we have*

$$\mathbb{P}(f(\omega) \ge a; |\nabla f| \le M)\mathbb{P}(f(\omega) \le a - t) \le 2\mathrm{e}^{-c\frac{t^2}{K^2M^2}}\,.$$

We will apply this result to $\log Z_N^\beta$, which is a convex function in ω, whose norm of the gradient is given by

$$|\nabla \log Z_N^\beta|^2 = \sum_{n=1}^N \sum_{|x|\le n} \Big(\frac{\partial}{\partial \omega_{n,x}} \log Z_N^\beta\Big)^2\,.$$

Our first lemma controls the first factor in Proposition B.3.

Lemma B.4. *Assume that $\sigma^2(\beta)R_N = 1$. Then, there is a constant $C > 0$ such that*

$$\mathbb{P}\Big(\log Z_N^\beta \ge -1; |\nabla \log Z_N^\beta|^2 \le C(\log N)^3\Big) \ge \frac{1}{C\log N}\,.$$

Then, applying Proposition B.3 with $a = -1$ and $M = \sqrt{C}(\log N)^{3/2}$, we get that for a bounded environment $|\omega| \le K$,

$$\mathbb{P}(\log Z_N^\beta \le -1 - t) \le 2C\log N\mathrm{e}^{-\frac{c}{C}\frac{t^2}{K^2(\log N)^3}}\,.$$

We can in fact reduce to a bounded environment with a large constant $K = (\log N)^{3/2}$: define $\tilde\omega_{n,x} = \omega_{n,x}\mathbf{1}_{\{|\omega_{n,x}|\le(\log N)^{3/2}\}}$, and note that

$$\mathbb{P}(\tilde\omega_{n,x} = \omega_{n,x}\ \forall n \in [\![1,N]\!], \forall |x| \le n) \le N^3\mathbb{P}(|\omega| \ge (\log N)^{3/2}) \le N^3\mathrm{e}^{-c_0(\log N)^{3/2}}\,.$$

Therefore,

$$\mathbb{P}(\log Z_N^{\beta,\omega} \le -1-t) \le \mathbb{P}(\log Z_N^{\beta,\tilde\omega} \le -1-t) + \mathbb{P}(\tilde\omega \ne \omega) \le 2C\log N\mathrm{e}^{-\frac{c}{C}\frac{t^2}{(\log N)^6}} + N^3\mathrm{e}^{-c_0(\log N)^{3/2}}\,.$$

where we have applied Proposition B.3 with $K = (\log N)^{3/2}$, $a = -1$, M (well, we need to check that $\tilde\sigma^2(\beta)R_N = 1 + O(\frac{1}{\log N})$, but that should be ok).

Then, using that $-\mathbb{E}[\log Z_N^\beta] \le 1 + \int_1^\infty \mathbb{P}(-\log Z_N^\beta \ge u)\,\mathrm{d}u$, we can split the integral into two parts. The first part is

$$\int_1^{N^2} \mathbb{P}(\log Z_N^\beta \le -1 - u)\,\mathrm{d}u \le C'(\log N)^4 + N^5\mathrm{e}^{-c_0(\log N)^{3/2}}$$

where we have used the upper bound found above. For the remaining part, we use the very rough bound: for $u \ge 2\lambda(\beta)N$

$$\begin{aligned}\mathbb{P}(\log Z_N^\beta \le -u) &\le \mathbb{P}(\beta N \min\{\omega_{n,x}, n \in [\![1,N]\!], |x| \le n\} - \lambda(\beta)N \le -u)\\ &\le \mathbb{P}\Big(\min\{\omega_{n,x}, n \in [\![1,N]\!], |x| \le n\} \le -\frac{1}{2}\frac{u}{\beta N}\Big) \le N^3\mathrm{e}^{-c_0u/2\beta N}\end{aligned}$$

Thus, the second part of the integral $\int_{N^2}^\infty \mathbb{P}(\log Z_N^\beta \le -1 - u)\,\mathrm{d}u$ is bounded by $c\beta N^4\mathrm{e}^{-c_0N/2\beta}$, which is negligible compared to the first term. This concludes the proof. □

Proof of Lemma B.4. First of all, let us write

$$\begin{aligned}\mathbb{P}\big(\log Z_N^\beta \geq -1; |\nabla \log Z_N^\beta|^2 \leq C(\log N)^3\big) \\ = \mathbb{P}\big(Z_N^\beta \geq \mathrm{e}^{-1}\big) - \mathbb{P}\big(Z_N^\beta \geq \mathrm{e}^{-1}; |\log Z_N^\beta|^2 > C(\log N)^3\big).\end{aligned}$$

For the first term, we use Paley–Zygmund inequality to get that

$$\mathbb{P}(Z_N^\beta \geq \mathrm{e}^{-1}) \geq (1-\mathrm{e}^{-1})^2 \frac{1}{\mathbb{E}[(Z_N^\beta)^2]} \geq \frac{c}{\log N},$$

where we have used that, at criticality, $\mathbb{E}[(Z_N^\beta)^2] \leq c \log N$. For the second term, a straightforward calculation gives that

$$|\nabla \log Z_N^\beta|^2 = \frac{\beta^2}{(Z_N^\beta)^2} \mathbf{E}^{\otimes 2}\Big[\sum_{n=1}^N \mathbf{1}_{\{S_n=\tilde S_n\}} \mathrm{e}^{\sum_{n=1}^N \beta(\omega_{n,S_N}+\omega_{n,\tilde S_N})-2\lambda(\beta)}\Big].$$

Bounding $\frac{1}{(Z_N^\beta)^2} \leq \mathrm{e}^2$ on the event $Z_N^\beta \geq \mathrm{e}^{-1}$, we get that, applying also Markov's inequality

$$\mathbb{P}\Big(Z_N^\beta \geq \mathrm{e}^{-1}; |\nabla \log Z_N^\beta|^2 > C(\log N)^3\Big) \leq \frac{e^2}{C(\log N)^3} \mathbf{E}^{\otimes 2}\Big[\beta^2 \sum_{n=1}^N \mathbf{1}_{\{S_n=\tilde S_n\}} \mathrm{e}^{\lambda_2(\beta)\sum_{n=1}^N \mathbf{1}_{\{S_n=\tilde S_n\}}}\Big],$$

with $\lambda_2(\beta) = \lambda(2\beta) - \lambda(\beta)$. Then, we can use that, at criticality, we have the following bound, that we prove below

Claim B.5. *Assume that $\sigma^2(\beta) R_N \leq \mathrm{e}^{\frac{\vartheta}{\log N}}$ for some $\vartheta \in \mathbb{R}_+$. Then there is a constant $C = C(\vartheta)$ such that*

$$\mathbf{E}^{\otimes 2}\Big[\beta^2 \sum_{n=1}^N \mathbf{1}_{\{S_n=\tilde S_n\}} \mathrm{e}^{\lambda_2(\beta)\sum_{n=1}^N \mathbf{1}_{\{S_n=\tilde S_n\}}}\Big] \leq C\,(\log N)^2.$$

All together, this gives that

$$\mathbb{P}\Big(\log Z_N^\beta \geq -1; |\nabla \log Z_N^\beta|^2 \geq C(\log N)^3\Big) \geq \frac{c}{\log N} - \frac{\mathrm{e}^2 C'}{C\log N} \geq \frac{c}{2\log N},$$

provided that we had fixed C large enough. ☐

Proof of Claim B.5. Recalling that $\sigma^2(\beta) = \mathrm{e}^{\lambda_2(\beta)} - 1$, we can perform the following chaos expansion:

$$\begin{aligned}&\mathbf{E}^{\otimes 2}\Big[\sum_{n=1}^N \mathbf{1}_{\{S_n=\tilde S_n\}} (1+\sigma^2(\beta))^{\sum_{n=1}^N \mathbf{1}_{\{S_n=\tilde S_n\}}}\Big] \\ &\quad = \sum_{k=0}^\infty \sigma^2(\beta)^k \sum_{1\leq n_1<\cdots<n_k\leq N} \sum_{n=1}^N \mathbf{P}^{\otimes 2}\big(S_{n_i} = S_{n_i}\ \forall i \in \{1,\dots,k\}, S_n = \tilde S_n\big).\end{aligned} \tag{B.1}$$

Now, we consider two contributions. First, if $n \in \{n_1,\dots,n_k\}$, this gives a term

$$\sum_{k=0}^\infty k\sigma^2(\beta)^k \sum_{1\leq n_1<\cdots<n_k\leq N} \prod_{i=1}^k u(n_i - n_{i-1}).$$

where k is simply a combinatorial factor due to the choice of index $i \in \{1,\dots,k\}$ such that $n = n_i$. Second, if $n \notin \{n_1,\dots,n_k\}$, this gives a term

$$\sum_{k=0}^\infty (k+1)\sigma^2(\beta)^k \sum_{1\leq n_1<\cdots<n_{k+1}\leq N} \prod_{i=1}^k u(n_i - n_{i-1}),$$

where the combinatorial factor is due to the choice of interval (n_{i-1}, n_i) in which n falls. All together, after a change of index for the second term, the left-hand side in (B.1) is equal to

$$(1+\sigma^2(\beta)^{-1}) \sum_{k=0}^{\infty} k\sigma^2(\beta)^k \sum_{1\le n_1<\cdots<n_k\le N} \prod_{i=1}^{k} u(n_i-n_{i-1}) .$$

Noting that $\beta^2(1+\sigma^2(\beta)^{-1})$ is bounded by a constant, we focus on sum. We use the following upper bound, see [CSZ19a, Lemma 5.4]: there is a constant $c>0$ such that, for every $k\ge 1$

$$\frac{1}{(R_N)^k} \sum_{1\le n_1<\cdots<n_k\le N} \prod_{i=1}^{k} u(n_i-n_{i-1}) \le \mathrm{e}^{-c\frac{k}{\log N}\log^+(\frac{k}{\log N})} .$$

With this bound at hand, we get that

$$\begin{aligned} \sum_{k=0}^{\infty} k\sigma^2(\beta)^k \sum_{1\le n_1<\cdots<n_k\le N} \prod_{i=1}^{k} u(n_i-n_{i-1}) &\le \sum_{k=0}^{\infty} k\big(\sigma^2(\beta)R_N\big)^k \mathrm{e}^{-c\frac{k}{\log N}\log^+(\frac{k}{\log N})} \\ &\le (\log N)^2 \times \frac{1}{\log N}\sum_{k=0}^{\infty} \frac{k}{\log N}\, \mathrm{e}^{\vartheta\frac{k}{\log N}-c\frac{k}{\log N}\log^+(\frac{k}{\log N})} , \end{aligned}$$

where we have also used that $\sigma^2(\beta)R_N \le \mathrm{e}^{\vartheta/\log N}$. The last term converges to $\int_0^\infty t\,\mathrm{e}^{\vartheta t - ct\log_+(t)}\,\mathrm{d}t$ by a Riemann approximation, so it is in particular bounded. This concludes the proof. □

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