THE CRITICAL 2D KPZ PROCESS

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Abstract. This work constructs a non-trivial solution to the ill-defined two-dimensional Kardar-Parisi-Zhang (KPZ) equation at criticality: a distribution-valued stochastic process on the plane, which we call the *Critical 2D KPZ Process*. It arises as the limit in law of centred log partition functions of two-dimensional Gaussian directed polymers at criticality, viewed as discretised KPZ solutions. It is also a measurable function of the Critical 2D Stochastic Heat Flow (SHF) via a renormalised Cole–Hopf transformation, obtained as the limit of the centred logarithm of the spatially mollified SHF as the smoothing is removed. The Critical 2D KPZ process is *non-Gaussian* and its covariance displays a log-log blow-up near the diagonal. Gaussian fluctuations emerge when averaging on shrinking spatial scales.

AI statement. The results presented here were obtained through extensive interactions with generative AI (ChatGPT 5.5, 5.6 and 6); see Appendix C for more details. The proof builds on key recent and older advances in our field, combined in novel ways with substantial AI input. We view this manuscript as a "community project" that represents more than a decade's collective effort. We have worked on making the presentation as accessible as possible, highlighting the main ideas and their relations to the literature.

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1. Introduction and main results

The Kardar–Parisi–Zhang (KPZ) equation [KPZ86] describes the evolution of a randomly growing interface. Formally, it can be written as

$$\partial_t h(t,x) = \frac{1}{2}\Delta h(t,x) + \frac{1}{2}|\nabla h(t,x)|^2 + \beta\xi(t,x), \qquad t>0, \quad x\in\mathbb{R}^d. \tag{1.1}$$

Here h is the interface height, ξ is a space-time white noise, and $\beta>0$ is the noise strength. The equation is a central model in the KPZ universality class, which includes random growth models, interacting particle systems and directed polymers. Its one-dimensional theory has been studied extensively, with precise results on fluctuation laws and scaling limits; see the surveys [Cor12, QS15].

The Cole–Hopf transformation $h=\log u$ connects (1.1) to the multiplicative stochastic heat equation (SHE)

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \beta u(t,x)\xi(t,x). \tag{1.2}$$

The roughness of white noise makes these equations singular: even in dimension one, the height is not differentiable and $|\nabla h|^2$ has no classical meaning, although the multiplicative SHE admits an Itô solution. Major advances in singular stochastic partial differential equations (SPDEs) include Hairer's construction of the one-dimensional KPZ equation [Hai13] and theory of regularity structures [Hai14], paracontrolled distributions [GIP15], and the renormalisation-group approaches of Kupiainen [Kup16] and Duch [Duc25]. Energy solutions provide another approach for one-dimensional stochastic Burgers/KPZ equations in stationary settings [GJ14, GP18]. Dimension two is critical for both KPZ and the multiplicative SHE, and lies outside the scope of these general subcritical constructions.

In recent years, there has been significant progress in making sense of the solution of the SHE in the critical dimension $d=2$. Bertini and Cancrini [BC98] first computed the limiting covariance of the solution of mollified 2D SHE in what would later be called the critical window. Motivated by the study of the directed polymer model and its phase transition, Caravenna, Sun, and Zygouras [CSZ17b] showed that the regularised 2D SHE exhibits a phase transition on an *intermediate-disorder scale*. Below the critical point, the one-point distribution of the solution converges to a log-normal limit, while at and above the critical point, it converges to 0. Below the same critical point, the solutions of the regularised SHE and the KPZ equation, regarded as random fields, were shown to have Gaussian fluctuations that belong to the Edwards-Wilkinson universality class [CSZ17b, CD20, CSZ20, Gu20]. In a finer critical window around the critical point, [CSZ19b] showed that third moments of the regularised SHE converge to finite limits, and Gu, Quastel, and Tsai [GQT21] subsequently extended it to all finite moments. However, these moments diverge too fast to show that the solution of the regularised SHE in the critical window converges to a unique limit.

The convergence to a unique limit, named the *Critical 2D Stochastic Heat Flow* (SHF), was finally achieved in [CSZ23] for the field of directed polymer partition functions, which is the solution of the 2D SHE regularised through space-time discretisation. Thus the SHF gives a meaning to the critical 2D SHE. Subsequently, Tsai [Tsa24] gave an axiomatic characterisation of the SHF and showed that it is also the limit of the noise-mollified 2D SHE. Recently, there have also been interesting universality results for the SHF [DHP26].

There has also been recent progress on other critical singular SPDEs, including Gaussian scaling limits for the two-dimensional anisotropic KPZ and stochastic Burgers equations [CET23, CGT24, CMT25, CKM26]. Gaussian fluctuation results have also been obtained for the weakly coupled four-dimensional Anderson model [GR26, DS26].

Making sense of the solution of 2D KPZ at criticality, however, remains a challenge. One cannot simply take the logarithm of the SHF because it is a singular random measure [CSZ25]. A recent result of Gu and Tsai [GT26] created a significant opening towards defining the Critical 2D KPZ Process. They showed that the logarithm of the spatial average of the SHF on scale ε, after suitable deterministic centring, has asymptotically Gaussian fluctuations of order $\sqrt{\log\log\varepsilon^{-1}}$.

We focus here on the directed polymer partition functions with Gaussian disorder, which solve a space-time discretised stochastic heat equation, while their logarithms are corresponding discrete KPZ solutions. Our main result is to construct the *Critical 2D KPZ Process* as the unique limit of the logarithms of these partition functions, after centring and without any further scaling. We also show that the centred logarithm of the mollified SHF converges to the same field. These convergences hold jointly at finitely many fixed positive times, hence they define a *limit process*, which is a measurable function of the SHF.

Two inputs play a central role in the construction. Liu and Xie [LX26] recently made a crucial observation that a Gaussian convexity property [AC15, Che26] can be applied to the directed polymer with Gaussian disorder and hence to the SHF. This yields sharp control on negative moments of the polymer partition functions as well as of the mass of the SHF on any ball. Another fundamental ingredient is Gu and Tsai's approximation and expansion of the SHF as a product of independent contributions over a geometric sequence of disjoint time intervals [GT26], which can be transferred to polymer partition functions.

A key novel ingredient in the proof is to introduce a factorisation at two times of the partition functions. This allows us to control sharply the effect of erasing the disorder in a small time interval, first at a local level (by covariance bounds derived from Gaussian convexity coupled to sharp moment estimates), then at a global level (by using the approximation from [GT26] to control how discrepancy in the initial condition is propagated in time through the SHF). A sketch of the proof strategy is given in Section 2.

We now define the relevant objects and the critical window, and we state the main results. An overview of earlier work and connections with the present paper is given in Section 1.4.

1.1. Directed polymers and the Critical 2D Stochastic Heat Flow. Let us introduce the directed polymer model, focusing on its partition functions: we refer to Comets [Com17] and the recent survey [Zyg24] for general background.

Let $\omega = (\omega(j,z))_{j\in\mathbb{N},z\in\mathbb{Z}^2}$ be independent standard Gaussian random variables, with law denoted $\mathbb{P}$ and expectation $\mathbb{E}$. These variables play the role of a random environment, or *disorder*. Let also S be the simple symmetric random walk on $\mathbb{Z}^2$. We write $\mathrm{P}_{a,z}$ and $\mathrm{E}_{a,z}$ for its law and expectation from $S_a = z$, and we abbreviate $\mathrm{P}_z := \mathrm{P}_{0,z}$, $\mathrm{E}_z := \mathrm{E}_{0,z}$ and $\mathrm{P} = \mathrm{P}_{0,0}$, $\mathrm{E} := \mathrm{E}_{0,0}$.

For integers $0 \leqslant a \leqslant b$ and $w, z \in \mathbb{Z}^2$, let us define the directed polymer *point-to-point partition functions* by

$$Z^{\beta}_{a,b}(w,z) := \mathrm{E}_{a,w}\Big[\exp\Big\{ \sum_{j=a+1}^{b} \big(\beta\omega(j,S_j) - \tfrac{1}{2}\beta^2\big)\Big\} \mathbf{1}_{\{S_b=z\}}\Big], \tag{1.3}$$

with $Z^{\beta}_{a,a}(w,z) = \mathbf{1}_{\{w=z\}}$. *Point-to-plane partition functions* are obtained by removing the final constraint $\mathbf{1}_{\{S_b=z\}}$, that is

$$Z^{\beta}_{a,b}(w) := \mathrm{E}_{a,w}\Big[\exp\Big\{ \sum_{j=a+1}^{b} \big(\beta\omega(j,S_j) - \tfrac{1}{2}\beta^2\big)\Big\}\Big] = \sum_{z\in\mathbb{Z}^2} Z^{\beta}_{a,b}(w,z)\,. \tag{1.4}$$

Note that $Z^{\beta}_{a,b}(\cdot)$ are *random* functions, depending on the realisation ω of the disorder. These partition functions can be seen as a discretised version of the stochastic heat equation (1.2), see (1.11) below; we refer to [CSZ20, Section 1.2] and [CSZ26, Sections 1.1–1.3].

The expected replica overlap between two independent random walks S, S' is

$$R_N := \sum_{n=1}^{N} \mathrm{P}(S_n = S'_n) = \sum_{n=1}^{N} \mathrm{P}(S_{2n} = 0) = \frac{1}{\pi} \log N + O(1)\,. \tag{1.5}$$

We set the disorder strength $\beta = \beta_N$ in the critical window (and omit it from notation):

$$\sigma_N^2 := \mathrm{e}^{\beta_N^2} - 1 := \frac{1}{R_N}\Big(1 + \frac{\vartheta}{\log N}\Big), \tag{1.6}$$

where $\vartheta \in \mathbb{R}$ is fixed (or can be replaced by a deterministic $\vartheta_N \to \vartheta$). Under diffusive rescaling of space-time, partition functions in the critical window have a universal scaling limit: a process $\mathscr{Z}^{\vartheta} = (\mathscr{Z}^{\vartheta}_{s,t}(\mathrm{d}x, \mathrm{d}y))_{0 \leqslant s < t}$ of random measures on $\mathbb{R}^2 \times \mathbb{R}^2$ called the *Critical 2D Stochastic Heat Flow (SHF)*, see [CSZ23]. Let us recall this convergence. We normalise the SHF mean by the standard heat kernel:

$$\mathbb{E}[\mathscr{Z}^{\vartheta}_{s,t}(\mathrm{d}x, \mathrm{d}y)] = g_{t-s}(y - x)\,\mathrm{d}x\,\mathrm{d}y\,, \qquad \text{with} \qquad g_u(z) := \frac{1}{2\pi u} \exp\Big(-\frac{|z|^2}{2u}\Big)\,.$$

For $t \geqslant 0$, denote by $[t] := 2\lfloor t/2 \rfloor$ an even integer approximation, so that $t \leqslant [t] < t + 2$. To account for the periodicity of the simple random walk, define the even lattice

$$\mathbb{Z}^2_{\mathrm{even}} := \{z = (z_1, z_2) \in \mathbb{Z}^2 : z_1 + z_2 \in 2\mathbb{Z}\}$$

and, for $x \in \mathbb{R}^2$, let $[x]$ be a nearest point of $\mathbb{Z}^2_{\mathrm{even}}$ (with a fixed choice to break ties). Given $N \in \mathbb{N}$, we partition $\mathbb{R}^2$ into cells of diameter $O(N^{-1/2})$ defined by

$$\mathtt{C}_N(z) := \{x \in \mathbb{R}^2 : [\sqrt{N/2}\, x] = z\}, \qquad z \in \mathbb{Z}^2_{\mathrm{even}}\,. \tag{1.7}$$

Note that each cell $\mathtt{C}_N(z)$ is a twisted square with area $2 \cdot (\sqrt{2/N})^2 = 4/N$.

For continuous space-time parameters $0 \leqslant s < t$ and $x, y \in \mathbb{R}^2$, we define the *diffusively rescaled point-to-point partition functions* by

$$\mathcal{Z}^{\vartheta,(N)}_{s,t}(x, y) := \frac{N}{4} Z^{\beta_N}_{[Ns],[Nt]}\big([\sqrt{N/2}\, x], [\sqrt{N/2}\, y]\big)\,, \tag{1.8}$$

where the scale $\sqrt{N/2}$ accounts for the coordinate variance $1/2$ of the simple random walk. Similarly, recalling (1.4), the *diffusively rescaled point-to-plane partition functions* are

$$\mathcal{Z}^{\vartheta,(N)}_{s,t}(x) := Z^{\beta_N}_{[Ns],[Nt]}([\sqrt{N/2}\, x]) = \int_{\mathbb{R}^2} \mathcal{Z}^{\vartheta,(N)}_{s,t}(x, y)\,\mathrm{d}y\,. \tag{1.9}$$

Note that $\mathcal{Z}^{\vartheta,(N)}_{s,t}(x, y)$ and $\mathcal{Z}^{\vartheta,(N)}_{s,t}(x)$ are random functions on $\mathbb{R}^2 \times \mathbb{R}^2$ and $\mathbb{R}^2$, piecewise constant on cells $\mathtt{C}_N(z)$. As $N \to \infty$ they become very *intermittent* and one cannot expect to obtain a non-trivial limit in a space of functions. However, they do have a limit as *generalised functions* or *distributions*, after integration against suitable test functions. Since partition functions are nonnegative, one can actually work in the simpler space of locally finite *measures*. This is the main result of [CSZ23], which we recall as follows[1].

[1]Note that in [CSZ23] the convention for the partition functions is slightly different, since it does not include the disorder variable at the terminal time. This however does not affect the result.

Theorem A ([CSZ23], Theorems 1.1 and 9.1)**.** *In the critical window* (1.6)*, jointly for any finite collection of pairs $0 \leqslant s < t$, we have the convergence in distribution*

$$\mathcal{Z}_{s,t}^{\vartheta,(N)}(x,y)\,\mathrm{d}x\,\mathrm{d}y \xrightarrow[N\to\infty]{d} \mathscr{Z}_{s,t}^{\vartheta}(\mathrm{d}x,\mathrm{d}y)\,, \qquad \mathcal{Z}_{s,t}^{\vartheta,(N)}(x)\,\mathrm{d}x \xrightarrow[N\to\infty]{d} \mathscr{Z}_{s,t}^{\vartheta}(\mathrm{d}x)\,, \quad (1.10)$$

in the vague topology of measures on $\mathbb{R}^2\times\mathbb{R}^2$ and $\mathbb{R}^2$ (i.e., integrating against compactly supported test functions).

The diffusively rescaled partition functions $\mathcal{Z}_t^{\vartheta,(N)}$ from (1.8) provide a space-time discretisation of the stochastic heat equation (1.2) on a lattice of mesh $O(N^{-1})$ in time and $O(N^{-1/2})$ in space. Indeed, a discretised solution of (1.2) with initial condition $u_0:\mathbb{R}^2\to\mathbb{R}^+$ is given by the "averaged-to-point" partition function

$$\mathcal{Z}_t^{\vartheta,(N)}(u_0,x) := \int_{\mathbb{R}^2} \mathrm{d}y\, u_0(y)\, \mathcal{Z}_{0,t}^{\vartheta,(N)}(y,x)\,. \quad (1.11)$$

For this reason, its limit in law as $N\to\infty$,

$$\mathscr{Z}_t^{\vartheta}(u_0,\mathrm{d}x) := \int_{\mathbb{R}^2} u_0(y)\, \mathscr{Z}_{0,t}^{\vartheta}(\mathrm{d}y,\mathrm{d}x), \quad (1.12)$$

provides a non-trivial "candidate solution" to the 2D stochastic heat equation (1.2).

Remark 1.1 (Noise-mollified SHE)**.** *Instead of discretising the stochastic heat equation by polymer partition functions, one can also regularise* (1.2) *by mollifying the noise in space, i.e. replacing ξ by $\xi_\varepsilon=\rho_\varepsilon*\xi$ with $\rho_\varepsilon(x)=\varepsilon^{-2}\rho(x/\varepsilon)$, where ρ is a smooth compactly supported probability density. This equation is well-posed in the Itô sense and, for $\vartheta\in\mathbb{R}$, we can consider its fundamental solutions $u_{s,t}(x,y)=u_{s,t}^{\vartheta,(\varepsilon)}(x,y)$ with noise strength in the critical window from* [BC98]*, more precisely $\beta^2=\beta_\varepsilon^2(\vartheta):=\frac{2\pi}{|\log\varepsilon|}(1+\frac{\theta+c_\rho}{2|\log\varepsilon|})$ for an explicit constant c_ρ* [Tsa24, Equation (1.6)] *and $\theta=\vartheta-\gamma$ with γ the Euler–Mascheroni constant:*

$$\partial_t u_{s,t}(x,y) = \tfrac{1}{2}\Delta_y u_{s,t}(x,y) + \beta_\varepsilon(\vartheta)\,\xi_\varepsilon(t,y)\,u_{s,t}(x,y), \qquad u_{s,s}(x,\cdot)=\delta_x(\cdot)\,. \quad (1.13)$$

It was proved by Tsai in [Tsa24, Proposition 1.5] *that the Critical 2D SHF $(\mathscr{Z}_{s,t}^{\vartheta}(\mathrm{d}x,\mathrm{d}y))_{0\leqslant s\leqslant t}$ is also the limit of the noise-mollified SHE solution $(u_{s,t}^{\vartheta,(\varepsilon)}(x,y)\,\mathrm{d}x\,\mathrm{d}y)_{0\leqslant s\leqslant t}$ as $\varepsilon\downarrow 0$.*

1.2. Main result: the Critical 2D KPZ Process. Recall that the Cole–Hopf transformation $h=\log u$ formally maps the KPZ equation (1.1) to the stochastic heat equation (1.2). Then, in view of (1.11) and (1.12) above, there are at least two possible ways to construct a "candidate solution" of the 2D KPZ equation (1.1).

A first approximation of the sought-after KPZ solution is the logarithm of partition functions (1.11), viewed as discretised solutions of (1.2), *with suitable centring*: for an initial condition $h_0:\mathbb{R}^2\to\mathbb{R}$, we define

$$\begin{aligned}\mathcal{H}_t^{\vartheta,(N)}(h_0,x) &:= \log\mathcal{Z}_t^{\vartheta,(N)}(\mathrm{e}^{h_0},x) \;-\; c_t^{\vartheta,(N)}\\ &\qquad\text{with}\quad c_t^{\vartheta,(N)} := \mathbb{E}\log\mathcal{Z}_t^{\vartheta,(N)}(\mathbf{1},x)\,,\end{aligned} \quad (1.14)$$

where the centring $c_t^{\vartheta,(N)}$ does not depend on x, by translation invariance. Subtracting $c_t^{\vartheta,(N)}$ is a form of *renormalisation* and amounts to centring $\log\mathcal{Z}_t^{\vartheta,(N)}(\mathrm{e}^{h_0},x)$ in the special case $h_0=\mathbf{0}$. We stress that this is uniformly close to centring for general $h_0\in L^\infty$: one can easily obtain that

$$\big|c_t^{\vartheta,(N)} - \mathbb{E}\big[\log\mathcal{Z}_t^{\vartheta,(N)}(\mathrm{e}^{h_0},x)\big]\big| \leqslant \|h_0\|_\infty\,, \quad (1.15)$$

see Section 4.4 for details. Let us also stress that we show in Lemma 4.3 (and the following discussion) that

$$c_t^{\vartheta,(N)} = \mathbb{E}\log \mathscr{Z}_t^{\vartheta,(N)}(\mathrm{e}^{h_0}, x) + O(1) = -\tfrac{1}{2}\log\log N + O(1) \qquad \text{as } N \to \infty\,. \tag{1.16}$$

A second natural way to make sense of the KPZ solution is to apply the Cole–Hopf transformation $h = \log u$ to the continuum "SHF solution" $\mathscr{Z}_t^{\vartheta}(u_0, \mathrm{d}x)$ in (1.12). This is a measure, but we can regularise it by removing the noise in a small time interval of size $\delta > 0$ (we call this a temporal mollification), or equivalently convolving with the heat kernel

$$g_v^{(x)}(y) := g_v(y-x) := \frac{1}{2\pi v}\,\mathrm{e}^{-\frac{|y-x|^2}{2v}}\,. \tag{1.17}$$

We thus define an approximate renormalised logarithm of the SHF by[2]

$$\begin{aligned} \mathscr{H}_t^{\vartheta,(\delta)}(h_0, x) \;&:=\; \log \mathscr{Z}_{t-\delta}^{\vartheta}\big(\mathrm{e}^{h_0}, g_\delta^{(x)}\big) \;-\; c_t^{\vartheta,(\delta)} \\ &\qquad \text{with} \quad c_t^{\vartheta,(\delta)} := \mathbb{E}\big[\log \mathscr{Z}_{t-\delta}^{\vartheta}\big(\mathbf{1}, g_\delta^{(x)}\big)\big]\,. \end{aligned} \tag{1.18}$$

We will justify in Section 4.2 that this is a locally integrable function of x. As in the discrete case, $c_t^{\vartheta,(\delta)}$ provides an approximate centring for $h_0 \in L^\infty$: replacing $\mathbf{1}$ with e^{h_0} changes the expected logarithm by at most $\|h_0\|_\infty$, uniformly in δ and x, exactly as in (1.15).

Our main result proves that both approximations (1.14) and (1.18) converge to the same limiting stochastic process, which gives a non-trivial notion of solution to the ill-defined two-dimensional KPZ equation (1.1) at criticality. Convergence does not take place in a space of functions but of genuine distributions. These can no longer be identified with measures, so we use local versions of negative Sobolev spaces $H^{-a}_{\mathrm{loc}}(\mathbb{R}^2)$, defined in (6.1).

Theorem 1.2 (Critical 2D KPZ Process)**.** *Fix $\vartheta \in \mathbb{R}$. There exists a distribution-valued process on $\mathbb{R}^2$, labeled by initial condition $h_0 \in L^\infty(\mathbb{R}^2) \cap C(\mathbb{R}^2)$, which we denote by*

$$\mathscr{H}^{\vartheta} = (\mathscr{H}_t^{\vartheta}(h_0, \cdot))_{t \geqslant 0}$$

and call the Critical 2D KPZ Process*. It is a measurable function of the SHF, arising as the limit in probability of regularised log-SHF, see* (1.18)*: jointly at finitely many $t > 0$ and h_0,*

$$\forall\, a > 0\,, \qquad \mathscr{H}_t^{\vartheta,(\delta)}(h_0, \cdot\,) \xrightarrow[\delta \downarrow 0]{\mathbb{P}} \mathscr{H}_t^{\vartheta}(h_0, \cdot\,) \qquad \text{in } H^{-a}_{\mathrm{loc}}(\mathbb{R}^2)\,. \tag{1.19}$$

Convergence holds in $L^p(\mathbb{P})$, $p \in [1, \infty)$, when integrating both sides against a test function $\varphi \in C_c^\infty(\mathbb{R}^2)$.

The same process $\mathscr{H}^{\vartheta}$ arises as the limit in law of renormalised log-partition functions, see (1.14)*, with Gaussian disorder and critical tuning* (1.6)*: jointly at finitely many $t > 0$ and h_0,*

$$\forall\, a > 0\,, \qquad \mathcal{H}_t^{\vartheta,(N)}(h_0, \cdot\,) \xrightarrow[N \to \infty]{d} \mathscr{H}_t^{\vartheta}(h_0, \cdot\,) \qquad \text{in } H^{-a}_{\mathrm{loc}}(\mathbb{R}^2)\,. \tag{1.20}$$

Let us stress that Theorem 1.2 builds *jointly* the Critical 2D KPZ Process with different (bounded) initial conditions h_0 from the *same* mollified Critical 2D SHF, with a common centring $c_t^{\vartheta,(\delta)}$ that does not depend on the initial condition, see (1.18). We also show below in Remark 9.1 that if h_0, h_0' are bounded Lipschitz initial conditions, then *deterministically*, the approximating fields $\mathscr{H}_t^{\vartheta,(\delta)}$ verify, for any $\delta > 0$ and any $\psi \in C_c^\infty$,

$$\Big|\mathscr{H}_t^{\vartheta,(\delta)}(h_0, \psi) - \mathscr{H}_t^{\vartheta,(\delta)}(h_0', \psi)\Big| \leqslant \|\psi\|_1 \,\big\|h_0' - h_0\big\|_\infty\,.$$

[2]One could replace the time $t-\delta$ by t in (1.18), which amounts to (spatially) regularising the SHF solution $\mathscr{Z}_t^{\vartheta}(u_0, \cdot)$ by convolving with the heat kernel g_δ. This does not affect our main result, see Corollary 6.2.

This can be thought of as a reflection of the comparison principle for Hamilton–Jacobi equations in this singular setting. This property passes to the limit thanks to Theorem 1.2, and in particular we have for any test function $\psi \in C_c^\infty$ that

$$|\mathscr{H}_t^\vartheta(h_0,\psi) - \mathscr{H}_t^\vartheta(h_0',\psi)| \leqslant \|\psi\|_1 \, \|h_0' - h_0\|_\infty \, .$$

We thus deduce that $\mathscr{H}_t^\vartheta(h_0,\cdot)$ is continuous w.r.t. the (bounded Lipschitz) initial condition, and additionally that all the 2D KPZ processes $\mathscr{H}_t^\vartheta(h_0,\cdot)$ with different bounded initial conditions h_0 remain *deterministically* within $O(1)$ distance.

Remark 1.3 (Critical 2D KPZ Process from noise-mollified SHE)**.** *Another approximation of the KPZ solution can be obtained from the fundamental solutions $u_{s,t}^{\vartheta,(\varepsilon)}(x,y)$ of the space-mollified stochastic heat equation* (1.13) *in the critical window, which converge to the SHF, see Remark 1.1. Writing*

$$u_{s,t}^{\vartheta,(\varepsilon)}(u_0,x) := \int_{\mathbb{R}^2} u_0(y)\, u_{s,t}^{\vartheta,(\varepsilon)}(y,x)\,\mathrm{d}y \, ,$$

taking logarithm and a suitable centring, we define, for an initial condition $h_0 \in L^\infty(\mathbb{R}^2)$,

$$\begin{aligned} h_t^{\vartheta,(\varepsilon)}(h_0,x) &:= \log u_t^{\vartheta,(\varepsilon)}(\mathrm{e}^{h_0},x) - c_t^{\vartheta,(\varepsilon)} \\ &\text{with} \quad c_t^{\vartheta,(\varepsilon)} := \mathbb{E}\log u_t^{\vartheta,(\varepsilon)}(\mathbf{1},x) \, . \end{aligned} \tag{1.21}$$

By Itô's formula, $h(t,x) = h_t^{\vartheta,(\varepsilon)}(h_0,x)$ solves the following renormalised 2D KPZ equation*:*

$$\begin{aligned} \partial_t h(t,x) \; &= \frac{1}{2}\Delta h(t,x) + \frac{1}{2}|\nabla h(t,x)|^2 + \beta_\varepsilon(\vartheta)\,\xi_\varepsilon(t,x) \; - \; C_t^{\vartheta,(\varepsilon)} \\ &\text{with} \qquad C_t^{\vartheta,(\varepsilon)} := \frac{\beta_\varepsilon(\vartheta)^2}{2}\varepsilon^{-2}\,\|\rho\|_{L^2}^2 \; + \; \frac{\mathrm{d}}{\mathrm{d}t}c_t^{\vartheta,(\varepsilon)} \, . \end{aligned} \tag{1.22}$$

With minor technical adaptations to the proof, we expect that the Critical 2D KPZ Process $\mathscr{H}^\vartheta = (\mathscr{H}_t^\vartheta(h_0,\cdot))_{t\geqslant 0}$ is also the limit in distribution of $(h_t^{\vartheta,(\varepsilon)}(h_0,\cdot))_{t\geqslant 0}$ as $\varepsilon \downarrow 0$, jointly at finitely many $t > 0$. We omit these adaptations for brevity.

1.3. First properties: local structure and non-Gaussianity. In this section, we focus for simplicity on the flat initial condition $h_0 = \mathbf{0}$ and we denote $\mathscr{H}_t^\vartheta(\cdot) := \mathscr{H}_t^\vartheta(\mathbf{0},\cdot)$. We next examine the law of the limit and the spatial singularities revealed by averaging it on small scales. For a nonnegative $\varphi \in C_c^\infty(\mathbb{R}^2)$ let us put

$$\varphi_\varepsilon^{(x)}(y) := \varepsilon^{-2}\varphi((y-x)/\varepsilon).$$

Theorem 1.4 (Local structure)**.** *Fix $t > 0$ and $\vartheta \in \mathbb{R}$. The random distribution $\mathscr{H}_t^\vartheta$ of Theorem 1.2, with flat initial condition $h_0 = \mathbf{0}$, has the following properties.*

(1) *For every fixed $x \in \mathbb{R}^2$ and nonnegative $\varphi \in C_c^\infty(\mathbb{R}^2)$, we have*

$$\frac{\mathscr{H}_t^\vartheta(\varphi_\varepsilon^{(x)})}{\sqrt{\log\log(1/\varepsilon)}} \implies \mathcal{N}\big(0,(\textstyle\int\varphi)^2\big) \qquad \text{as } \varepsilon \downarrow 0 \, . \tag{1.23}$$

(2) *There is a nonnegative radial covariance kernel $\mathsf{C}_t^\vartheta \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$, continuous off zero, such that for real $f, g \in C_c^\infty(\mathbb{R}^2)$,*

$$\mathrm{Cov}\big(\mathscr{H}_t^\vartheta(f), \mathscr{H}_t^\vartheta(g)\big) = \int_{\mathbb{R}^2\times\mathbb{R}^2} f(x)g(y)\mathsf{C}_t^\vartheta(x-y)\,\mathrm{d}x\,\mathrm{d}y \, . \tag{1.24}$$

Moreover, the covariance kernel has a $\log\log$ *blow-up near the diagonal:*

$$\lim_{|z|\downarrow 0} \frac{\mathsf{C}_t^{\vartheta}(z)}{\log\log(1/|z|)} = 1. \tag{1.25}$$

As a corollary of Theorem 1.4-(1), we show that the Critical 2D KPZ Process cannot locally be a signed measure.

Proposition 1.5. *Almost surely,* $\mathscr{H}_t^{\vartheta}$ *cannot be represented as a locally finite signed Radon-measure. In particular, it cannot be represented by a locally integrable function and it cannot be spatially constant.*

More importantly, the next result shows that the Critical 2D KPZ Process cannot be Gaussian, in the stronger sense that no one-dimensional marginal obtained by the action of $\mathscr{H}_t^{\vartheta}$ on a nonnegative C_c^{∞} test function can be Gaussian.

Theorem 1.6 (Non-Gaussianity)**.** *For every* $t > 0$, $\vartheta \in \mathbb{R}$ *and nonzero nonnegative* $\psi \in C_c^{\infty}(\mathbb{R}^2)$, *the random variable* $\mathscr{H}_t^{\vartheta}(\psi)$ *is non-Gaussian, including exclusion of degenerate Gaussian laws.*

Remark 1.7 (Comparison with renormalisation group predictions)**.** *Theorems 1.2 and 1.6 yield a non-Gaussian limit in the critical window* (1.6) *with fixed* ϑ. *This may seem in contrast with the renormalisation group calculation of Frey and Täuber* [FT94]*: the cubic term in their function* β_g *in (3.48) would predict that a non-trivial scaling limit requires a negative shift of* ϑ *of order* $\log\log N$. *However, that formula contains a misprint: in dimension* $d = 2$ *the cubic terms cancel out*[3]*, leading to* $\beta_g = -g^2 + O(g^4)$. *This formula, already stated in* [FT94, Eq. (3.64)] *and recovered in* [Wie97, Eq. (44)]*, shows that the two-loop renormalisation group calculation is consistent with the emergence of a non-Gaussian limit in the critical window, with no additional* $\log\log$ *tuning, as in our main results.*

Let us end this section by observing that the Critical 2D KPZ Process inherits the following rotation-translation invariance and scaling covariance properties from the SHF.

Proposition 1.8. *Let us fix any finite collection* $(t_i, \varphi_i)_{1\leqslant i\leqslant k}$ *of times and test functions. The Critical 2D KPZ Process* $\mathscr{H}^{\vartheta}$, *with flat initial condition* $h_0 = \mathbf{0}$, *has the following properties:*

(1) *It is rotation-translation invariant in law: for any* $z \in \mathbb{R}^2$, $\alpha \in \mathbb{R}$,

$$\left(\mathscr{H}_{t_i}^{\vartheta}(T_z^{\alpha}\varphi_i)\right)_{i=1}^{k} \overset{\text{dist}}{=} \left(\mathscr{H}_{t_i}^{\vartheta}(\varphi_i)\right)_{i=1}^{k},$$

where $T_z^{\alpha}\varphi(\cdot) = \varphi(z + \mathrm{e}^{i\alpha}\cdot)$ *denotes the rotated-translated version of* φ.

(2) *It satisfies the following covariance relation: for all* $\mathtt{a} > 0$

$$\left(\mathscr{H}_{\mathtt{a}t_i}^{\vartheta}(S_{\mathtt{a}}\varphi_i)\right)_{i=1}^{k} \overset{\text{dist}}{=} \left(\mathscr{H}_{t_i}^{\vartheta+\log\mathtt{a}}(\varphi_i)\right)_{i=1}^{k},$$

where $S_{\mathtt{a}}\varphi(\cdot) = \frac{1}{\mathtt{a}}\varphi(\frac{\cdot}{\sqrt{\mathtt{a}}})$ *denotes the scaled version of* φ.

Proof. The distribution of the SHF is translation invariant, see [CSZ23, Theorem 1.2], and it is also rotation invariant, as follows from the rotational invariance of the mollified SHE (with a radial mollifier ρ) which is preserved by taking the limit in distribution [Tsa24, Proposition 1.5]. Then, the Critical 2D KPZ Process $\mathscr{H}^{\vartheta}$ inherits these invariance properties, thanks to the (joint) convergence in distribution from Theorem 1.2.

[3]The coefficient of $g^2 F_\nu(d)$ inside the parentheses of (3.48) should rather be $+3(d-1)/(8d)$, cf. equations (3.33) and (3.46)–(3.47) therein.

As far as the second covariance property is concerned, one can use the scaling covariance relation of the SHF from [CSZ23, Theorem 1.2]. Recalling $\mathscr{H}_t^{\vartheta,(\delta)}(\varphi)$ from (1.18) (with flat initial condition $h_0 = \mathbf{0}$), and observing that $g_{\mathtt{a}\delta}^{(x)}(\cdot) = S_{\mathtt{a}} g_{\delta}^{(x/\sqrt{\mathtt{a}})}(\cdot)$, this yields that for any $\delta > 0$ and $\mathtt{a} > 0$,

$$\left(\mathscr{H}_{\mathtt{a}t_i}^{\vartheta,(\mathtt{a}\delta)}(S_{\mathtt{a}}\varphi_i)\right)_{i=1}^{k} \overset{\text{dist}}{=} \left(\mathscr{H}_{t_i}^{\vartheta+\log \mathtt{a},(\delta)}(\varphi_i)\right)_{i=1}^{k}.$$

The desired covariance property then follows, thanks to the (joint) convergence in distribution from Theorem 1.2. □

1.4. Relation to earlier work. In this section, we review contributions from the literature that played an important role towards the main results in this paper.

Our convergence proof in Theorem 1.2 first applies a localisation argument that was used in [CSZ20, Section 2.1] to prove Edwards-Wilkinson limit for the 2D KPZ equation in the subcritical regime (see also [Gu20]). The main idea is to approximate the field by a local field, whose covariance has fast spatial decay. In the critical window we consider here, such an approximation by a local field is no longer accurate. Nevertheless, we use the decomposition (2.24) to separate a local term, which is analogous to the field considered in the sub-critical regime, and a global term that is estimated separately by adapting techniques from [GT26] and applying a Gaussian convexity property [Che26, LX26].

A key ingredient in our analysis is the geometric block factorisation that had been used by Gu and Tsai [GT26] to prove a central limit theorem for the average of the SHF on shrinking spatial scales. Such decomposition on exponential time scales first appeared in the study of 2D SHE and directed polymer in the subcritical regime [CSZ17b], and the idea of approximating by products of independent block partition functions on different time scales was developed by Cosco and Donadini [CD25b] (see also [CSZ25] for a different decomposition in the so-called quasi-critical window). Gu and Tsai [GT26] extended the approximation idea from [CD25b] to the critical window and developed an expansion to control the error of approximation, which was applied to the SHF. We will adapt this expansion to lattice polymers and use it to track how the partition function depends on the polymer endpoint distribution at an intermediate time. In particular, we compare two such distributions evolved through the same later disorder, obtaining Proposition 4.4. Together with negative-moment and logarithmic-moment bounds, this comparison controls the global term in (2.24).

Passing from partition functions to their logarithms requires control of small values, which we obtain via negative moments. Liu and Xie [LX26] obtained sharp bounds, up to multiplicative constants, for positive and negative moments of spatial averages of the SHF over shrinking balls, in terms of explicit powers of their corresponding second moments. In our proof we need sharper moment estimates of this sort: Proposition 4.1 shows that, for a suitably averaged partition function over a short time interval, the ratio of the second moment to the negative-first moment is close to one. We also use bounds for the centred fourth moments of partition functions from [CSZ25], based on the operator estimates of Caravenna, Cottini and Rossi [CCR25], to control the errors in the factorisation and in the comparison of positive and negative moments.

As in [LX26], Gaussian convexity also plays an important role here, controlling lower tails at criticality, as well as obtaining spatial decorrelations. In addition, Gaussian convexity is also used to prove non-Gaussianity of the Critical 2D KPZ Process. This convexity structure is preserved, in a distributional sense, under weak limits [LX26]. Together with the representation of the resulting laws as convex functions of a standard Gaussian [FMP25],

this yields a rigidity principle: Gaussianity would force certain limiting polymer endpoint measures to be deterministic. Using the flow property established by Clark and Mian [CM26], we show that this is incompatible with the singular spatial structure of the SHF [CSZ25].

Notational conventions. We distinguish the random-walk law and expectation P, E from the disorder law and expectation $\mathbb{P}, \mathbb{E}$. For real-valued random variables, the L^p norm is denoted by $\|X\|_{L^p} = (\mathbb{E}|X|^p)^{1/p}$. For functions on $\mathbb{R}^2$ or $\mathbb{Z}^2$, we denote by $\|f\|_p$ the L^p norm with respect to the Lebesgue measure or counting measure, respectively.

In the rest of the article, the precise values of the constants c, C may change from line to line; we may write c_ϑ, C_ϑ if we want to highlight that the constants may depend on some parameter ϑ. We also write $f = O(g)$ if there is a constant $C < +\infty$ such that $f(x) \leqslant Cg(x)$ and we write $f \asymp g$ if $f = O(g)$ and $g = O(f)$; we write $f = O_\vartheta(g)$ or $f \asymp_\vartheta g$ if we want to highlight that the constants may depend on some parameter ϑ.

2. Strategy of the proof and organisation of the paper

In this section, we outline the strategy of the proof of our main result Theorem 1.2 and we describe the organisation of the paper. We first define some notation.

2.1. Notation. The transition probabilities of the simple random walk S on $\mathbb{Z}^2$ are

$$q_n(z) := \mathrm{E}_0[\mathbf{1}_{\{S_n = z\}}], \qquad q_n^{(w)}(z) := q_n(w, z) := q_n(z - w) = \mathrm{E}_w[\mathbf{1}_{\{S_n = z\}}]\,. \tag{2.1}$$

Let $\mathcal{F}_{m,n}$ be the sigma field generated by the environment $\omega = (\omega(j, z))$ on time $(m, n]$:

$$\mathcal{F}_{m,n} := \sigma\{\omega(j, z) : m < j \leqslant n,\ z \in \mathbb{Z}^2\}\,. \tag{2.2}$$

Recall the point-to-point partition functions $Z^\beta_{a,b}(w, z)$ from (1.3), for points $w, z \in \mathbb{Z}^2$. For an initial measure ν and a terminal function f on $\mathbb{Z}^2$, set

$$\begin{aligned} Z^\beta_{a,b}(\nu, f) &:= \sum_{w,z} \nu(w)\, Z^\beta_{a,b}(w, z)\, f(z)\,, \\ Z^\beta_{a,b}(\nu, z) := \sum_w \nu(w) Z^\beta_{a,b}(w, z)\,, &\qquad Z^\beta_{a,b}(w, f) := \sum_z Z^\beta_{a,b}(w, z) f(z)\,. \end{aligned} \tag{2.3}$$

Point-to-plane partition functions are obtained for $f = \mathbf{1}$ (often omitted from notation):

$$Z^\beta_{a,b}(w) := Z^\beta_{a,b}(w, \mathbf{1}) = \sum_z Z^\beta_{a,b}(w, z)\,, \qquad Z^\beta_{a,b}(\nu) := Z^\beta_{a,b}(\nu, \mathbf{1}) = \sum_{w,z} \nu(w)\, Z^\beta_{a,b}(w, z)\,. \tag{2.4}$$

We henceforth set $\beta = \beta_N$ in the critical window (1.6) *and omit it from notation.*

Combining the notation introduced so far, we define *a smoothed version $\widehat{Z}_{a,b}(x)$ of the point-to-plane partition function $Z_{a,b}(x)$*, where we average out disorder until time a:

$$\widehat{Z}_{a,b}(x) := \mathbb{E}[Z_{0,b}(x) \mid \mathcal{F}_{a,b}] = Z_{a,b}(q_a^{(x)}, \mathbf{1}) = \sum_y q_a(y - x) Z_{a,b}(y, \mathbf{1}). \tag{2.5}$$

Thus the walk evolves without disorder up to time a, while later disorder remains.

The polymer endpoint distributions, also called endpoint profiles, starting from a point $w \in \mathbb{Z}^2$ or a probability ν on $\mathbb{Z}^2$ are

$$\mu^{(w)}_{a,b}(z) := \frac{Z^\beta_{a,b}(w, z)}{Z^\beta_{a,b}(w)}\,, \qquad \mu^{(\nu)}_{a,b}(z) := \frac{Z^\beta_{a,b}(\nu, z)}{Z^\beta_{a,b}(\nu)}\,. \tag{2.6}$$

Starting from a single point corresponds to $\nu = \delta_w = q_0^{(w)}$.

We will use functional notation for the SHF, analogous to (2.3) and (2.4):

$$\begin{aligned} \mathscr{Z}^{\vartheta}_{s,t}(\varphi,\psi) &:= \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\, \mathscr{Z}^{\vartheta}_{s,t}(\mathrm{d}x,\mathrm{d}y)\, \psi(y)\,, \\ \mathscr{Z}^{\vartheta}_{s,t}(\varphi) &:= \mathscr{Z}^{\vartheta}_{s,t}(\varphi,\mathbf{1}) = \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\, \mathscr{Z}^{\vartheta}_{s,t}(\mathrm{d}x,\mathrm{d}y)\,. \end{aligned} \tag{2.7}$$

When the starting time is 0, we often omit it from notation and simply write

$$\mathcal{F}_n := \mathcal{F}_{0,n}\,, \qquad Z^{\beta}_n := Z^{\beta}_{0,n}\,, \qquad \mu^{(w)}_n := \mu^{(w)}_{0,n}\,, \qquad \mathscr{Z}^{\vartheta}_t := \mathscr{Z}^{\vartheta}_{0,t}\,.$$

2.2. Strategy of the proof of Theorem 1.2. We describe the main ideas in the proof of the convergence results (1.19) and (1.20), with both sides evaluated on a test function $\varphi \in C_c(\mathbb{R}^2)$. We focus for simplicity on time $t = 1$ and initial condition $h_0 = \mathbf{0}$; the extension to finitely many times and test functions and to an initial condition $h_0 \in L^\infty(\mathbb{R}^2) \cap C(\mathbb{R}^2)$ follows the same strategy and is detailed in Section 5.

Recall $\mathscr{H}^{\vartheta,(\delta)}_t$ from (1.18) and $\mathcal{H}^{\vartheta,(N)}_t$ from (1.14), and let us omit the dependence on ϑ in the notation from now on. We actually prefer to work with *time-reversed versions* (the distribution is unchanged[4]): let us call $\mathscr{H}^{(\delta)}(\varphi)$ the δ-regularised log-SHF averaged over the starting point, with flat terminal condition:

$$\mathscr{H}^{(\delta)}(\varphi) \;:=\; \int_{\mathbb{R}^2} \varphi(x) \Big\{ \log \mathscr{Z}_{\delta,1}\big(g^{(x)}_\delta\big) \;-\; \mathbb{E}\big[\log \mathscr{Z}_{\delta,1}\big(g^{(0)}_\delta\big)\big]\Big\}\,\mathrm{d}x\,. \tag{2.8}$$

We similarly let $\mathcal{H}^{(N)}(\varphi)$ be the renormalised point-to-plane log-partition function averaged over the starting point: restricting for simplicity to even values of N, we define

$$\begin{aligned} \mathcal{H}^{(N)}(\varphi) \;&:=\; \int_{\mathbb{R}^2} \varphi(x) \Big\{ \log \mathcal{Z}^{(N)}_1(x) \;-\; \mathbb{E}\big[\log \mathcal{Z}^{(N)}_1(0)\big]\Big\}\,\mathrm{d}x \\ &=\; \sum_{z\in\mathbb{Z}^2_{\mathrm{even}}} \varphi^{(N)}(z) \Big\{ \log Z_N(z) \;-\; \mathbb{E}\big[\log Z_N(0)\big]\Big\}\,, \end{aligned} \tag{2.9}$$

where in the last equality, recalling the cells $\mathtt{C}_N(z)$ from (1.7), we identify $\mathcal{Z}^{(N)}_1(x) = Z_N(z)$ for $x \in \mathtt{C}_N(z)$ (see (1.9) and (1.4)) and we define

$$\varphi^{(N)}(z) \;:=\; \int_{\mathtt{C}_N(z)} \varphi(x)\,\mathrm{d}x \;\approx\; \tfrac{4}{N}\varphi\Big(\tfrac{z}{\sqrt{N/2}}\Big)\,. \tag{2.10}$$

We now state a preliminary version of Theorem 1.2 which retains the essential challenges.

Theorem 2.1 (Critical 2D KPZ Process, preliminary version)**.** *For any test function $\varphi \in C_c(\mathbb{R}^2)$ there exists a random variable $\mathscr{H}(\varphi) \in L^2$ such that, in the critical scaling* (1.6)

$$\mathscr{H}^{(\delta)}(\varphi) \;\xrightarrow[\delta\downarrow 0]{L^2}\; \mathscr{H}(\varphi)\,, \qquad \mathcal{H}^{(N)}(\varphi) \;\xrightarrow[N\to\infty]{d}\; \mathscr{H}(\varphi)\,. \tag{2.11}$$

We will prove the two convergences in (2.11) *simultaneously.* The idea is to interpolate between the δ-regularised log-SHF $\mathscr{H}^{(\delta)}(\varphi)$ and the log-partition function $\mathcal{H}^{(N)}(\varphi)$ by

[4]Strictly speaking, this is not true in the discrete setting *i.e.* for the polymer partition functions defined in (1.3): time-reversal changes the convention of the absence or presence of a disorder variable at the starting or end point. This however does not affect the results.

means of a δ-regularised log-partition function: we thus define, recalling $\widehat{Z}$ from (2.5),

$$\begin{aligned}\mathcal{H}^{(N;\delta)}(\varphi) &:= \sum_{z\in\mathbb{Z}^2_{\mathrm{even}}} \varphi^{(N)}(z)\Big\{\log \widehat{Z}_{m,N}(z) - \mathbb{E}\big[\log \widehat{Z}_{m,N}(0)\big]\Big\} \\ &= \sum_{z\in\mathbb{Z}^2_{\mathrm{even}}} \varphi^{(N)}(z)\Big\{\log Z_{m,N}(q_m^{(z)}) - \mathbb{E}\big[\log Z_{m,N}(q_m^{(0)})\big]\Big\} \quad \text{with } m := [\delta N]\,.\end{aligned} \tag{2.12}$$

Here disorder is removed until time $m = [\delta N]$, while the distribution $q_m^{(z)}$ converges to $g_\delta^{(x)}$ after spatial rescaling. Intuitively, we will show that

$$\mathscr{H}^{(\delta)}(\varphi) \underset{\substack{\text{as } N\to\infty\\ \text{for } \textit{fixed } \delta}}{\overset{d}{\approx}} \mathcal{H}^{(N;\delta)}(\varphi) \underset{\substack{\text{as } \delta\downarrow 0\\ \textit{uniformly} \text{ for large } N}}{\overset{L^2}{\approx}} \mathcal{H}^{(N)}(\varphi)\,, \tag{2.13}$$

as formulated precisely in the next two results.

Proposition 2.2 (Log-polymer to log-SHF)**.** *Jointly for finitely many $\delta \in (0,1)$, we have*

$$\mathcal{H}^{(N;\delta)}(\varphi) \xrightarrow[N\to\infty]{d} \mathscr{H}^{(\delta)}(\varphi)\,, \tag{2.14}$$

and the limiting variable $\mathscr{H}^{(\delta)}(\varphi)$ belongs to L^2.

Proposition 2.3 (Log-polymer smoothing)**.** *We have*

$$\varepsilon_\delta := \limsup_{N\to\infty} \big\|\mathcal{H}^{(N)}(\varphi) - \mathcal{H}^{(N;\delta)}(\varphi)\big\|_{L^2} \xrightarrow[\delta\downarrow 0]{} 0\,. \tag{2.15}$$

More precisely there is $C_\varphi < \infty$ such that, for all sufficiently small $\delta > 0$,

$$\varepsilon_\delta \leqslant \frac{C_\varphi}{|\log\delta|^{1/40}}\,. \tag{2.16}$$

Proposition 2.2 is not surprising: the convergence of polymer partition functions to the SHF, see Theorem A, suggests that for fixed $\delta > 0$ we should have convergence in distribution of $\mathcal{H}^{(N;\delta)}(\varphi)$ to $\mathscr{H}^{(\delta)}(\varphi)$ as $N \to \infty$. Turning this intuition into a proof will require us to control the *lower tail* of partition functions, in order to deal with the singularity of the logarithm at zero. The details are provided in Subsection 4.2.

Proposition 2.3 is much more challenging: the strategy for its proof, which is the technical core of this paper, is described in the next Subsection 2.3.

Combining both propositions, we can now easily prove Theorem 2.1.

Proof of Theorem 2.1. For fixed $\delta, \delta' > 0$, combining (2.14) and Portmanteau we get

$$\big\|\mathscr{H}^{(\delta)}(\varphi) - \mathscr{H}^{(\delta')}(\varphi)\big\|_{L^2} \leqslant \limsup_{N\to\infty} \big\|\mathcal{H}^{(N;\delta)}(\varphi) - \mathcal{H}^{(N;\delta')}(\varphi)\big\|_{L^2} \leqslant \varepsilon_\delta + \varepsilon_{\delta'}\,, \tag{2.17}$$

where we used the triangle inequality and the definition of ε_δ, see (2.15). Since $\varepsilon_\delta \to 0$, this shows that the random variables $(\mathscr{H}^{(\delta)}(\varphi))_{\delta\in(0,1)}$ are Cauchy in L^2 as $\delta \downarrow 0$, hence they converge in L^2 to a limiting random variable $\mathscr{H}(\varphi)$, which must be a measurable function of the SHF. This proves the first convergence in (2.11).

We next define $d(X,Y) := \sup|\mathbb{E}[\psi(X)] - \mathbb{E}[\psi(Y)]|$ where the supremum runs over 1-bounded and 1-Lipschitz functions $\psi : \mathbb{R} \to \mathbb{R}$, namely $|\psi(x) - \psi(y)| \leqslant |x - y|$ and $|\psi(x)| \leqslant 1$ for all $x, y \in \mathbb{R}$. Note that $d(X,Y)$ is a distance between the *laws* of random variables X, Y which metrises convergence in distribution and we can write

$$d\big(\mathcal{H}^{(N)}(\varphi), \mathscr{H}(\varphi)\big) \leqslant d\big(\mathcal{H}^{(N)}(\varphi), \mathcal{H}^{(N;\delta)}(\varphi)\big) + d\big(\mathcal{H}^{(N;\delta)}(\varphi), \mathscr{H}^{(\delta)}(\varphi)\big) + d\big(\mathscr{H}^{(\delta)}(\varphi), \mathscr{H}(\varphi)\big)\,.$$

We have $d\big(\mathcal{H}^{(N;\delta)}(\varphi), \mathscr{H}^{(\delta)}(\varphi)\big) \to 0$ as $N \to \infty$ for any fixed $\delta > 0$, by Proposition 2.2. Since $d(X,Y) \leqslant \mathbb{E}[|X-Y|] \leqslant \|X-Y\|_{L^2}$ for $X = \mathcal{H}^{(N)}(\varphi)$ and $Y = \mathcal{H}^{(N;\delta)}(\varphi)$, it follows by the definition (2.15) of ε_δ that

$$\limsup_{N\to\infty} d\big(\mathcal{H}^{(N)}(\varphi), \mathscr{H}(\varphi)\big) \;\leqslant\; \varepsilon_\delta \;+\; d\big(\mathscr{H}^{(\delta)}(\varphi), \mathscr{H}(\varphi)\big)\,.$$

Both terms in the right hand side vanish as $\delta \downarrow 0$, by Proposition 2.3 and the first part of the proof. This proves the second convergence in (2.11). □

2.3. Strategy of the proof of Proposition 2.3. We need to show that $\mathcal{H}^{(N)}(\varphi)$ and $\mathcal{H}^{(N;\delta)}(\varphi)$ are close in L^2 for small $\delta > 0$, uniformly over large $N \in \mathbb{N}$.

We first give some intuition. By (2.9) and (2.12), we should compare the partition function $Z_N(z)$ and its smoothed version $\widehat{Z}_{m,N}(z)$ for $m = [\delta N]$. By definition

$$\widehat{Z}_{m,N}(z) = Z_{m,N}(q_m^{(z)})\,, \tag{2.18}$$

and we can express $Z_N(x)$ similarly: by (2.6), a Chapman-Kolmogorov decomposition gives

$$Z_N(z) = \sum_w Z_m(z,w)\, Z_{m,N}(w) = Z_m(z) \sum_w \mu_m^{(z)}(w)\, Z_{m,N}(w) = Z_m(z)\, Z_{m,N}(\mu_m^{(z)})\,. \tag{2.19}$$

The problem is then to compare two partition functions $Z_{m,N}(\mu_m^{(z)})$ and $Z_{m,N}(q_m^{(z)})$, which are over the same time interval $(m,N]$ but started with different initial conditions at time m, namely, the polymer endpoint distribution $\mu_m^{(z)}$ and the random walk endpoint distribution $q_m^{(z)}$.

The key idea is to introduce an intermediate time n with $m \ll n \ll N$. We will choose

$$m = [\delta N]\,, \qquad n = [\sqrt{\delta}\, N]\,, \tag{2.20}$$

so that $1 \ll m \ll n \ll N$ as $N \to \infty$ and then $\delta \downarrow 0$. Let us decompose the two partition functions $Z_N(z)$ and $\widehat{Z}_{m,N}(z)$ at the same time n, as in (2.19):

$$Z_N(z) \;=\; Z_n(z)\, Z_{n,N}(\mu_n^{(z)})\,, \tag{2.21}$$

$$\widehat{Z}_{m,N}(z) \;=\; \widehat{Z}_{m,n}(z)\, Z_{n,N}(\widehat{\mu}_{m,n}^{(z)})\,, \tag{2.22}$$

where $\mu_n^{(z)}$ and $\widehat{\mu}_{m,n}^{(z)}$ are the polymer endpoint distributions associated with the partition functions $Z_n(z)$ and $\widehat{Z}_{m,n}(z) = Z_{m,n}(q_m^{(z)})$:

$$\mu_n^{(z)}(w) = \frac{Z_{0,n}(z,w)}{Z_n(z)} \qquad \text{and} \qquad \widehat{\mu}_{m,n}^{(z)}(w) = \frac{Z_{m,n}(q_m^{(z)},w)}{\widehat{Z}_{m,n}(z)} \tag{2.23}$$

To prove Proposition 2.3 (recalling $\mathcal{H}^{(N)}(\varphi)$ and $\mathcal{H}^{(N;\delta)}(\varphi)$ from (2.9) and (2.12)), we need to control the fluctuations of the field of the difference of the logarithms

$$\log Z_N(z) - \log \widehat{Z}_{m,N}(z) \;=\; L_{m,n}(z) + G_{m,n,N}(z), \tag{2.24}$$

where we used the decompositions (2.21) and (2.22) to separate the difference of the local terms

$$L_{m,n}(z) \;:=\; \log Z_n(z) - \log \widehat{Z}_{m,n}(z)\,, \tag{2.25}$$

and the difference of the global terms

$$G_{m,n,N}(z) \;:=\; \log \frac{Z_{n,N}(\mu_n^{(z)})}{Z_{n,N}(\widehat{\mu}_{m,n}^{(z)})}\,. \tag{2.26}$$

Here are the key heuristics that explain why both the local difference and global difference are small after spatial averaging, and what role our choice of n plays.

- *Both* $\log Z_n(z)$ *and* $\log \widehat{Z}_{m,n}(z)$ *are small in* L^2 *after centring and spatial averaging, without the need for cancellation between them.* The reason is that they depend on disorder up to time $n \ll N$, so their spatial correlations decay on the spatial scale $\sqrt{n}$, much smaller than the spatial averaging scale $\sqrt{N}$. Such a localisation argument has also been used in the study of the subcritical 2D KPZ [CSZ20, Proposition 2.2]. The key new ingredient here is (3.12), which bounds the covariance of log-partition functions and is the consequence of a powerful Gaussian convexity theorem [Che26, Theorem 1].
- *The distributions* $\mu_n^{(z)}$ *and* $\widehat{\mu}_{m,n}^{(z)}$ *at time* n *are close.* The reason is that, while they are far apart at time m, cf. $\mu_m^{(z)}$ and $\widehat{\mu}_{m,m}^{(z)} = q_m^{(z)}$, they evolve through the same disorder on the time interval $(m, n]$, and since n/m is large, this common evolution largely washes out their initial discrepancy. This is proved through an adaptation of the Gu–Tsai factorisation [GT26, Proposition 2.1].
- *The partition functions* $Z_{n,N}(\mu_n^{(z)})$ *and* $Z_{n,N}(\widehat{\mu}_{m,n}^{(z)})$ *are close in* L^2. The reason is that partition functions $Z_{n,N}(\mu)$ depend linearly on their initial profile μ, so the part common to the two profiles gives the same contribution, and their difference only sees the small residual discrepancy left at time n, which we know to be small.

We now formulate the above heuristics into precise technical lemmas that we will verify in order to conclude the proof of Proposition 2.3.

Centring the field in (2.24) and averaging in space gives

$$\mathcal{H}^{(N)}(\varphi) \;=\; \mathcal{H}^{(N;\delta)}(\varphi) \;+\; \mathcal{L}^{(N;\delta)}(\varphi) \;+\; \mathcal{G}^{(N;\delta)}(\varphi), \tag{2.27}$$

where we set, recalling also the definition (2.10) of $\varphi^{(N)}$,

$$\mathcal{L}^{(N;\delta)}(\varphi) \;:=\; \sum_{z\in\mathbb{Z}^2_{\text{even}}} \varphi^{(N)}(z)\left\{L_{m,n}(z) - \mathbb{E}[L_{m,n}(0)]\right\}, \tag{2.28}$$

$$\mathcal{G}^{(N;\delta)}(\varphi) \;:=\; \sum_{z\in\mathbb{Z}^2_{\text{even}}} \varphi^{(N)}(z)\left\{G_{m,n,N}(z) - \mathbb{E}[G_{m,n,N}(0)]\right\}. \tag{2.29}$$

The proof of Proposition 2.3 follows directly from the next two results, which are the key technical ingredients of the paper. Their proofs are detailed in Subsections 4.3 and 4.5.

Proposition 2.4 (Local approximation)**.** *For all* $\varphi \in C_c(\mathbb{R}^2)$*, there is a constant* $C_\varphi > 0$ *such that, for all sufficiently small* δ*,*

$$\limsup_{N\to\infty} \mathbb{E}\Big[\mathcal{L}^{(N;\delta)}(\varphi)^2\Big] \leqslant \frac{C_\varphi}{|\log\delta|^{1/4}}. \tag{2.30}$$

Proposition 2.5 (Global approximation)**.** *For all* $\varphi \in C_c(\mathbb{R}^2)$*, there is a constant* $C_\varphi > 0$ *such that, for all sufficiently small* δ*,*

$$\limsup_{N\to\infty} \mathbb{E}\Big[\mathcal{G}^{(N;\delta)}(\varphi)^2\Big] \leqslant \frac{C_\varphi}{|\log\delta|^{1/20}}. \tag{2.31}$$

Taking square roots in (2.30) and (2.31), and using (2.27), we obtain (2.16).

Let us comment on the proof of Proposition 2.4, presented in Subsection 4.3. In the local term $L_{m,n}(z)$, see (2.25), the two logarithms are not small at a fixed starting point z, nor is their difference. However, after centring and spatial averaging as in (2.28), each logarithm separately becomes small in L^2. To implement this argument at criticality and

control covariances of log-partition functions, we exploit a novel covariance bound (3.12) that follows from the same Gaussian convexity theorem [Che26, Theorem 1] that was used in [LX26] to derive moment comparison bounds and Gaussian concentration of measure for the directed polymer model. To conclude the covariance bound, a key ingredient is Proposition 4.1, whose proof adapts a factorisation technique from [GT26].

We next comment on the global approximation, Proposition 2.5. Unlike the logarithms in (2.25), the global term $G_{m,n,N}(z)$ in (2.26) is small in L^2 for any fixed starting point z, for a different reason. The two endpoint distributions $\mu_n^{(z)}$ and $\widehat{\mu}_{m,n}^{(z)}$, which start from different profiles at time m, evolve through the same disorder on $(m, n]$. When n/m is large, this common evolution washes out their initial discrepancy. This is quantified in Proposition 4.4, whose proof adapts the same factorisation technique from [GT26, Proposition 2.1], which identifies the common part of the two endpoint distributions and isolates the residual dependence on their initial profiles. To transfer this control to the logarithms, we use negative-moment bounds for the partition functions, based again on Gaussian convexity [Che26, LX26], together with the logarithmic-moment bounds in Proposition 4.5. Together, these estimates allow us to pass from the comparison of partition functions to their logarithms and prove Proposition 2.5.

2.4. Organisation of the rest of the article. Let us now give a brief outline on how the rest of the paper is organised.

- Section 3 gives Gaussian-convexity inequalities and their polymer consequences, recalling some key results from [Che26] and [LX26]. Crucially, it derives a new covariance bound (3.12) for log-partition functions, which is controlled through the quotient of second and negative-first moments.

- Section 4 proves the convergence in Proposition 2.2 and also the local and global approximations of Propositions 2.4-2.5. These last results rely on quantitative inputs developed in Section 7, notably: (i) a sharp moment comparison (Proposition 4.1) to control the local approximation; (ii) a sharp endpoint estimate of the difference between $\mu_n^{(x)}$ and $\widehat{\mu}_{m,n}^{(x)}$ (Proposition 4.4) to control the global approximation.

- Section 5 extends Theorem 2.1 to joint convergence against test functions at finitely many times and to non-flat initial condition.

- Section 6 extends the joint convergence of Section 5 to convergence in local Sobolev spaces. It concludes the proof of Theorem 1.2 and constructs the process $\mathscr{H}^{\vartheta}$.

- Section 7 proves important estimates that are used Section 4. It introduces the geometric block factorisation of Gu–Tsai [GT26] and adapts some of their results on the block expansion to our discrete setting. This is the main ingredient (together with the moment inequalities of Section 3 and sharp second moment estimates) that allows us to prove Proposition 4.1 and 4.4, respectively in Subsections 7.4 and 7.5.

- Section 8 proves Theorem 1.4. We first prove the local normality of the Critical 2D KPZ Process, which essentially relies on Gu–Tsai's ("local") log-normality result for the SHF [GT26] together with some control when taking a spatial average. Proposition 1.5 can then be derived from this local normality with diverging variance. Finally, we conclude the proof of Theorem 1.4 by showing the $\log\log$ divergence of its covariances near the diagonal, through the study of the spectral measure associated with the covariance kernel.

- Finally, Section 9 shows the non-Gaussianity of the Critical 2D KPZ Process. It relies on the Ehrhard-type property of the process, together with Gaussian rigidity with respect

to the initial condition (see Remark 9.1, Lemmas 9.2 and 9.3). A consequence of this is that if $\mathscr{H}_t^\vartheta(\mathbf{0},\psi)$ were Gaussian, then any weak limit of polymer endpoint distribution would be deterministic and non-zero (see Lemmas 9.4 and 9.5). We show that this contradicts the singularity of the SHF with respect to the Lebesgue measure, see [CSZ25].

• Additionally, we collect in Appendix several technical results. Appendix A gives a proof of a key technical estimate for the block expansion from Section 7.2, namely Proposition 7.4 (it is analogous to [GT26, Corollary 3.5] but cannot be directly adapted from it). Appendix B provides several important moment estimates, in particular a sharp second moment estimate stated in Lemma 7.6.

3. Gaussian convexity estimates

The estimates in this section convert positive moments into bounds on negative moments and covariances of logarithms. We present general results for convex functions of a Gaussian vector, which we then specialise to our directed polymers framework.

3.1. Positive and negative moments. Given a random variable F such that $\mathbb{E}[\mathrm{e}^{\lambda F}] < \infty$ for all $\lambda > 0$, we define for all $q \in \mathbb{R}$

$$\Phi(q) = \Phi_F(q) := \begin{cases} q^{-1} \log \mathbb{E}[\mathrm{e}^{qF}] & q \neq 0\,, \\ \mathbb{E}[F] & q = 0\,. \end{cases} \tag{3.1}$$

When F is a convex function of a Gaussian vector, we are going to exploit the following convexity property proved by Chen [Che26, Theorem 1] (convexity for $q > 0$ was established earlier in [AC15, Proposition 1], and an alternative proof based on Ehrhard's inequality was recently given in [FMP25, Section 3]).

Theorem 3.1 (Gaussian convexity property)**.** *Let $F = F(G)$ be a convex function of a finite-dimensional Gaussian vector G such that $\mathbb{E}[\mathrm{e}^{\lambda F}] < \infty$ for all $\lambda > 0$. Then the function $\Phi(q)$ for $q \in \mathbb{R}$ is convex.*

Liu and Xie [LX26] applied Theorem 3.1 to the directed polymer model and the Critical 2D Stochastic Heat Flow (SHF) and derived many consequences, including bounds on variance, concentration of measure, moment comparisons, etc. Here we focus on the following negative-moment comparison result: a positive second moment controls all negative-moments. It is convenient to normalise F so that $\mathbb{E}[\mathrm{e}^F] = 1$ (just replace F by $F - \log \mathbb{E}[\mathrm{e}^F]$).

Lemma 3.2 (Approximate log-normal bounds)**.** *For F as in the assumptions of Theorem 3.1 with $\mathbb{E}[\mathrm{e}^F] = 1$, we have*

$$\mathrm{Var}[F] \leqslant -2\,\mathbb{E}[F]\,, \tag{3.2}$$

$$-\mathbb{E}[F] \leqslant \frac{1}{2} \log \mathbb{E}[\mathrm{e}^{2F}]\,, \tag{3.3}$$

and also

$$\log \mathbb{E}[\mathrm{e}^{-pF}] \leqslant -p\,\mathbb{E}[F] + \frac{p^2}{2}\,\mathrm{Var}[F]\,, \qquad p \geqslant 0\,, \tag{3.4}$$

$$\mathbb{E}[\mathrm{e}^{-pF}] \leqslant \mathbb{E}[\mathrm{e}^{2F}]^{p(p+1)/2}, \qquad p \geqslant 0. \tag{3.5}$$

If F is Gaussian, so that e^F is log-normal, then all inequalities above are equalities.

Proof. Since $\mathbb{E}[\mathrm{e}^{qF}] = 1 + q\,\mathbb{E}[F] + (1+o(1))\frac{q^2}{2}\,\mathbb{E}[F^2]$ as $q \downarrow 0$, we get that

$$\Phi'(0) = \frac{1}{2}\,\mathrm{Var}[F]\,.$$

Recall that $\Phi(0) = \mathbb{E}[F]$ and $\Phi(1) = 0$. Convexity gives $\Phi(1) \leqslant \frac{1}{2}(\Phi(0) + \Phi(2))$, that is (3.3).

The supporting line at 0 gives $\Phi(q) \geqslant \Phi(0) + \Phi'(0)\,q$. For $q = 1$ this yields (3.2), while for $q = -p$ this yields (3.4) (after multiplication by $-p$ and reversal of the inequality).

Finally, starting from (3.4) and applying (3.2) and (3.3), we obtain (3.5):

$$\log \mathbb{E}[\mathrm{e}^{-pF}] \leqslant (p + p^2)(-\mathbb{E}[F]) \leqslant \frac{p(p+1)}{2}\,\log \mathbb{E}[\mathrm{e}^{2F}]\,. \qquad \square$$

3.2. Covariance comparison. It is useful to define the following quantity:

$$\Delta(F) := \Phi(1) - \Phi(0) - \Phi'(0) = -\mathbb{E}F - \frac{1}{2}\,\mathrm{Var}[F]\,, \tag{3.6}$$

which can be thought of as measuring the convexity of the normalised log moment-generating function Φ at 0. Since $\Delta(F) = 0$ when F is Gaussian (recall that we use the normalisation $\mathbb{E}[\mathrm{e}^F] = 1$), $\Delta(F)$ illustrates the "Gaussianity defect" through the convex distortion of F, or the "log-Gaussianity defect" of e^F.

Lemma 3.3 (Gaussianity defect)**.** *For F as in the assumptions of Theorem 3.1 with $\mathbb{E}[\mathrm{e}^F] = 1$, we have*

$$0 \leqslant \Delta(F) \leqslant \log \frac{\mathbb{E}[\mathrm{e}^{2F}]}{\mathbb{E}[\mathrm{e}^{-F}]}\,. \tag{3.7}$$

Proof. The first inequality is simply (3.2). We next apply (3.4) for $p = 1$ and (3.3) to get

$$\log \mathbb{E}[\mathrm{e}^{-F}] \leqslant -\mathbb{E}[F] + \frac{1}{2}\,\mathrm{Var}[F] = -2\,\mathbb{E}[F] - \Delta(F) \leqslant \log \mathbb{E}[\mathrm{e}^{2F}] - \Delta(F)\,, \tag{3.8}$$

which is just a rewriting of (3.7). $\square$

We now present a simple consequence of the Gaussian convexity property in Theorem 3.1, which will play a key role in controlling the covariance of logarithms of our random fields.

Lemma 3.4 (Covariance bound)**.** *Let $F_x = F_x(G)$, $F_y = F_y(G)$ be convex functions of the same Gaussian vector, as in the assumptions of Theorem 3.1, with $\mathbb{E}[\mathrm{e}^{F_x}] = \mathbb{E}[\mathrm{e}^{F_y}] = 1$. Then*

$$\mathrm{Cov}[F_x, F_y] \;\leqslant\; \log \mathbb{E}[\mathrm{e}^{F_x + F_y}] + \Delta(F_x) + \Delta(F_y)\,. \tag{3.9}$$

If the Gaussian vector G has independent coordinates G_i, and F_x, F_y are coordinatewise nondecreasing functions, then $\mathrm{Cov}[F_x, F_y] \geqslant 0$.

Proof. Define

$$H \;:=\; F_x + F_y - \log \mathbb{E}[\mathrm{e}^{F_x + F_y}]\,,$$

so that H is a convex function of G with $\mathbb{E}[\mathrm{e}^H] = 1$. Then $\mathrm{Var}[H] \leqslant -2\,\mathbb{E}[H]$ by (3.2), i.e.

$$\mathrm{Var}[F_x] + \mathrm{Var}[F_y] + 2\;\mathrm{Cov}[F_x, F_y] \leqslant -2\,\mathbb{E}[F_x] - 2\,\mathbb{E}[F_y] + 2\,\log \mathbb{E}[\mathrm{e}^{F_x + F_y}]\,,$$

which gives (3.9) after rearranging the terms. If G has a product Gaussian law, the lower bound $\mathrm{Cov}[F_x, F_y] \geqslant 0$ follows by the FKG inequality for nondecreasing functions. $\square$

3.3. Ehrhard-type random variables. Since the SHF arises as a weak limit of lattice partition functions, we need a formulation of Gaussian convexity in terms of laws that are preserved under weak convergence. Following Liu and Xie [LX26, Definition 2.4], we introduce the Ehrhard-type class[5].

Definition 3.5 (Ehrhard-type random variables)**.** We call a real-valued random variable X and its distribution *Ehrhard type* if either X is constant, or if $\mathbb{P}(X < a) < 1$ for every $a \in \mathbb{R}$ and $a \mapsto \Phi^{-1}(\mathbb{P}(X < a))$ is concave, where Φ is the standard Gaussian distribution function.

We recall the following consequence of [FMP25, Theorem 4.3] and [LX26, Corollary 2.2 and Lemmas 2.5–2.6].

Lemma 3.6. *Every finite convex function of a finite-dimensional Gaussian vector is of Ehrhard type. A real-valued random variable X is of Ehrhard type if and only if it admits a representation*

$$X \overset{\mathrm{d}}{=} h(G), \qquad G \sim \mathcal{N}(0,1), \qquad h : \mathbb{R} \to \mathbb{R} \text{ nondecreasing and convex.} \tag{3.10}$$

The class of Ehrhard-type laws is closed under weak convergence.

Remark 3.7. *This representation implies that Lemmas 3.2, and 3.3 apply if F is replaced by any Ehrhard-type random variable with proper centring. Lemma 3.4 also holds if F_x and F_y therein are replaced by any Ehrhard-type random variables that are convex functions of the same Gaussian vector or obtained as joint weak limits (with convergence of moments) of convex functions of the same Gaussian vectors.*

3.4. Consequences for directed polymers and the SHF. The logarithms of the partition functions $\widehat{Z}_{s,n}(x)$ in (2.5) are convex and nondecreasing in the (independent) Gaussian disorder variables. Lemma 3.3 therefore gives

$$\begin{aligned} 0 &\leqslant -\mathbb{E}\log \widehat{Z}_{s,n}(x) - \tfrac{1}{2}\operatorname{Var}(\log \widehat{Z}_{s,n}(x)) \\ &= \Delta(\log \widehat{Z}_{s,n}(x)) \leqslant \log \frac{\mathbb{E}[\widehat{Z}_{s,n}(x)^2]}{\mathbb{E}[\widehat{Z}_{s,n}(x)^{-1}]}. \end{aligned} \tag{3.11}$$

On the other hand, Lemma 3.4 gives

$$\begin{aligned} 0 &\leqslant \operatorname{Cov}(\log \widehat{Z}_{s,n}(x), \log \widehat{Z}_{s,n}(y)) \\ &\leqslant \log \mathbb{E}[\widehat{Z}_{s,n}(x)\widehat{Z}_{s,n}(y)] + \Delta(\log \widehat{Z}_{s,n}(x)) + \Delta(\log \widehat{Z}_{s,n}(y)). \end{aligned} \tag{3.12}$$

The bounds (3.11)–(3.12), together with the moment comparison in Proposition 4.1 below, will be used to prove the local approximation in Proposition 2.4. The general covariance bound in Lemma 3.4 also gives the estimates needed in Lemma 6.3, which will yield the Sobolev tightness in Lemma 6.4.

By taking the weak limit of polymer partition functions, it was shown in the proof of Theorem 1.5 in [LX26, Section 4] that the logarithm of SHF $\log \mathscr{Z}^{\vartheta}_{s,t}(f,g)$ with $f, g \geqslant 0$ is also Ehrhard type. Nonnegative linear combinations of log partition functions are again convex functions of the same Gaussian disorder, and hence their weak limits, such as the fields in Theorem 1.2 integrated against nonnegative test functions, are also Ehrhard type. For example,

$$F_\delta(x) := \log \mathscr{Z}^{\vartheta}_{\delta,1}(g_\delta^{(x)}, \mathbf{1}), \qquad 0 < \delta < 1, \tag{3.13}$$

[5]Our definition of Ehrhard type differs from [LX26] by including constant random variables.

is Ehrhard type. Since $\mathbb{E}[\mathrm{e}^{F_\delta(x)}] = 1$, the extension of the Gaussian convexity property to Ehrhard-type laws in [LX26, Theorem 2.9] gives (3.2)–(3.3), and hence

$$0 \leqslant \mathrm{Var}(F_\delta(x)) \leqslant -2\mathbb{E}[F_\delta(x)] \leqslant \log \mathbb{E}[\mathrm{e}^{2F_\delta(x)}] \leqslant \log\log(1/\delta) + C_\vartheta. \tag{3.14}$$

Here ϑ is fixed, and the bound holds uniformly in x for sufficiently small δ. The last inequality follows from $\mathbb{E}[\mathrm{e}^{2F_\delta(x)}] \leqslant 1 + C_\vartheta \log(1/\delta)$, obtained by passing the lattice second-moment bound (4.5) to the SHF limit at fixed δ, using Fatou's lemma. This upper bound will be needed in Section 8.

Let us conclude this section by observing that, by Lemma 3.6 and Theorem 1.2, even though the Critical 2D KPZ Process is non-Gaussian, it is of Ehrhard type.

Proposition 3.8. *For any $t > 0$, $\vartheta \in \mathbb{R}$ and $\varphi \in C_c^\infty(\mathbb{R}^2)$ with $\varphi \geqslant 0$, the random variable $\mathscr{H}_t^\vartheta(\varphi)$ is of Ehrhard type.*

4. Proofs of the convergence estimates

In this section, we prove Propositions 2.2, 2.4 and 2.5 from Section 2, which complete the proof of Theorem 2.1. The proof of our main result Theorem 1.2 will be completed in the next Sections 5 and 6.

4.1. Preliminaries: some moment bounds. Let us start with a few useful second moment estimates. First, notice that by [CSZ19a, Proposition A.2], we have that for any fixed $t > 0$, $\mathbb{E}[Z_{[Nt]}(0)^2] \sim \nu(te^{\vartheta-\gamma}) \log N$ as $N \to \infty$, with ν the Volterra function. We also have the following uniform second moment bound on the point-to-plane partition function: for any $1 \leqslant k \leqslant N$

$$\sigma_N^2 \mathbb{E}[Z_k(0)^2] \leqslant \frac{C_\vartheta}{\log(1 + N/k)}\,, \tag{4.1}$$

see Claim B.1.

For deterministic probability measures ν, ν' on $\mathbb{Z}^2$, we also have the following *first-collision* decomposition of covariances using their chaos representation, cf. [CSZ19b, Equations (3.2)–(3.3)]:

$$\mathbb{E}[Z_{s,n}(\nu) Z_{s,n}(\nu')] - 1 = \sum_{r=s+1}^{n} \sum_{x,y \in \mathbb{Z}^2} \nu(x)\nu'(y) q_{2(r-s)}(y-x) \sigma_N^2 \mathbb{E}[Z_{n-r}(0)^2]\,.$$

In particular, recalling the definition (2.5) of $\widehat{Z}_{s,n}(x)$, we get that

$$\mathbb{E}[\widehat{Z}_{s,n}(x)\widehat{Z}_{s,n}(y)] - 1 = \sum_{r=s+1}^{n} q_{2r}(y-x)\, \sigma_N^2 \mathbb{E}[Z_{n-r}(0)^2]\,. \tag{4.2}$$

As a consequence, since $\mathbb{E}[Z_{n-r}(0)^2] \leqslant \mathbb{E}[Z_n(0)^2]$, we get from (4.1) that

$$\mathbb{E}[\widehat{Z}_{s,n}(x)\widehat{Z}_{s,n}(y)] - 1 \leqslant \frac{C}{\log(1+N/n)} \sum_{r=s+1}^{n} q_{2r}(y-x) \leqslant C \frac{\mathsf{G}_n(y-x)}{\log(1+N/n)}\,, \tag{4.3}$$

where $\mathsf{G}_n(z) = \sum_{r=1}^n q_{2r}(z)$ is the Green function of the random walk up to time n. Note that $\mathsf{G}_n(0) = R_n$ with R_N defined in (1.5). In particular, taking $x = y = 0$ and using $q_{2r}(0) \leqslant C/r$, we obtain, for $1 \leqslant s \leqslant n \leqslant N$,

$$\mathbb{E}[\widehat{Z}_{s,n}(0)^2] \leqslant 1 + C \frac{R_n - R_s}{\log(1+N/n)} \leqslant 1 + C \frac{\log(n/s)}{\log(1+N/n)}. \tag{4.4}$$

The first inequality also holds for $s = 0$, with $R_0 = 0$.

For $m = [\delta N]$, Lemma 3.2 and (4.1) also imply that

$$\mathbb{E}[\widehat{Z}_{m,N}(x)^{-1}] \leqslant \mathbb{E}[\widehat{Z}_{m,N}(x)^2] \leqslant 1 + C \sum_{r=m+1}^{N} q_{2r}(0) \leqslant 1 + C \log\Big(\frac{N}{m}\Big) \leqslant 1 + C \log(1/\delta)\,, \tag{4.5}$$

where we have used the local limit theorem (see e.g. [LL10, Theorem 2.3.11]) to get that $q_{2r}(0) \leqslant C/r$ for any $r \geqslant 1$.

4.2. Convergence with initial disorder removed: proof of Proposition 2.2. From now on, for $0 < \delta < 1$ fixed, we let

$$m = [\delta N]\,.$$

For $f \in L^1(\mathbb{R}^2)$, write $f^{(N)}(z) = \int_{\mathtt{C}_N(z)} f(x)\,\mathrm{d}x$, as in (2.10). As a consequence of Theorem A, jointly for finitely many pairs $0 \leqslant s < t$ and test functions $\varphi \in C_c(\mathbb{R}^2)$, we have

$$Z_{[Ns],[Nt]}(\varphi^{(N)}) = \int_{\mathbb{R}^2} \varphi(x) \mathscr{Z}^{\vartheta,(N)}_{s,t}(x)\,\mathrm{d}x \ \xrightarrow[N\to\infty]{d}\ \mathscr{Z}^{\vartheta}_{s,t}(\varphi)\,. \tag{4.6}$$

Step 1: convergence of partition functions. It follows by (4.6) that for any $x \in \mathbb{R}^2$, as $N \to \infty$

$$Z_{m,N}((g^{(x)}_\delta)^{(N)}) \ \xrightarrow[N\to\infty]{d}\ \mathscr{Z}^{\vartheta}_{\delta,1}(g^{(x)}_\delta)\,. \tag{4.7}$$

Strictly speaking, we cannot use (4.6) since $g^{(x)}_\delta$ does not have compact support, but we can apply the convergence (4.6) to $g^{(x)}_{\delta,A} \coloneqq \chi_A g^{(x)}_\delta$ with χ_A some smooth function with compact support equal to 1 on the ball of radius A. We thus get $Z_{m,N}((g^{(x)}_{\delta,A})^{(N)}) \xrightarrow{d} \mathscr{Z}^{\vartheta}_{\delta,1}(g^{(x)}_{\delta,A})$, and we may use the L^1 bounds

$$\mathbb{E}\big[|Z_{m,N}(f^{(N)}) - Z_{m,N}(h^{(N)})|\big] \leqslant \|f-h\|_1 \quad \text{and} \quad \mathbb{E}\big[|\mathscr{Z}^{\vartheta}_{\delta,1}(f) - \mathscr{Z}^{\vartheta}_{\delta,1}(h)|\big] \leqslant \|f-h\|_1 \tag{4.8}$$

applied to $f = g^{(x)}_\delta$ and $h = g^{(x)}_{\delta,A}$ to conclude the proof of (4.7), by letting $A \to \infty$.

Then, using again (4.8), letting $x_N \coloneqq [\sqrt{N/2}x]$ we also have that

$$\sup_x \mathbb{E}\big[|Z_{m,N}(q^{(x_N)}_m) - Z_{m,N}((g^{(x)}_\delta)^{(N)})|\big] \leqslant \sup_x \sum_{y\in\mathbb{Z}^2_{\mathrm{even}}} \Big|q^{(x_N)}_m(y) - \int_{\mathtt{C}_N(y)} g^{(x)}_\delta(u)\,\mathrm{d}u\Big|\,, \tag{4.9}$$

which goes to 0 as $N \to \infty$, by the local limit theorem (see [LL10, Theorem 2.3.11]). We can thus deduce from (4.7) that we have

$$\widehat{Z}_{m,N}(x_N) = Z_{m,N}(q^{(x_N)}_m) \ \xrightarrow[N\to\infty]{d}\ \mathscr{Z}^{\vartheta}_{\delta,1}(g^{(x)}_\delta)\,. \tag{4.10}$$

We also stress that the convergence holds jointly for finitely many pairs (δ, x).

Step 2: Convergence for the truncated logarithms. Define the logarithm truncated to $[-R, R]$, by setting

$$\log^{(R)}(u) = \max\{-R, \min\{\log u, R\}\} \quad (u > 0), \qquad \log^{(R)}(0) = -R. \tag{4.11}$$

We now show the convergence

$$\sum_{x\in\mathbb{Z}^2_{\mathrm{even}}} \varphi^{(N)}(x) \log^{(R)} \widehat{Z}_{m,N}(x) \ \xrightarrow[N\to\infty]{d}\ \int_{\mathbb{R}^2} \varphi(x) \log^{(R)} \mathscr{Z}^{\vartheta}_{\delta,1}(g^{(x)}_\delta)\mathrm{d}x\,. \tag{4.12}$$

Let $\varepsilon \in (0,1)$ be fixed, and partition the support of φ into a finite number $k_\varepsilon \leqslant C_\varphi \varepsilon^{-2}$ of cells $(\mathtt{C}_i)_{1 \leqslant i \leqslant k_\varepsilon}$, all cells being of diameter at most ε. For each cell $\mathtt{C}_i$, let $x_i \in \mathtt{C}_i$, and notice that a straightforward calculation shows that

$$\sup_{x \in \mathtt{C}_i} \| g_\delta^{(x)} - g_\delta^{(x_i)} \|_1 \leqslant C\varepsilon/\delta \,,$$

Hence, by (4.8) and (4.9) and since $\log^{(R)}$ is a Lipschitz function, we have (recalling that $x_N = [\sqrt{N/2}x]$)

$$\sup_{1 \leqslant i \leqslant k_\varepsilon} \sup_{x \in \mathtt{C}_i} \mathbb{E}\big[\big| \log^{(R)} \widehat{Z}_{m,N}(x_N) - \log^{(R)} Z_{m,N}((g_\delta^{(x_i)})^{(N)}) \big| \big] \leqslant C_{\delta,R}\, \varepsilon + o_N(1) \,.$$

This shows in particular that

$$\begin{aligned} \mathbb{E}\Big[\Big| \sum_{x \in \mathbb{Z}^2_{\mathrm{even}}} \varphi^{(N)}(x) \log^{(R)} \widehat{Z}_{m,N}(x) - \sum_{i=1}^{k_\varepsilon} \Big(\int_{\mathtt{C}_i} \varphi(u) \mathrm{d}u \Big) \log^{(R)} Z_{m,N}((g_\delta^{(x_i)})^{(N)}) \Big| \Big] \\ \leqslant C_{R,\varphi,\delta}\, \varepsilon + o_N(1) \,. \end{aligned} \tag{4.13}$$

Then, using the convergence (4.7) and since $\log^{(R)}$ is a bounded continuous function, we get that $\log^{(R)} Z_{m,N}((g_\delta^{(x_i)})^{(N)})$ converges in distribution to $\log^{(R)} \mathscr{Z}^\vartheta_{\delta,1}(g_\delta^{(x_i)})$ jointly for all $i \leqslant k_\varepsilon$. This thus gives that

$$\sum_{i=1}^{k_\varepsilon} \Big(\int_{\mathtt{C}_i} \varphi(u) \mathrm{d}u \Big) \log^{(R)} Z_{m,N}((g_\delta^{(x_i)})^{(N)}) \xrightarrow[N \to \infty]{d} \sum_{i=1}^{k_\varepsilon} \Big(\int_{\mathtt{C}_i} \varphi(u) \mathrm{d}u \Big) \log^{(R)} \mathscr{Z}^\vartheta_{\delta,1}(g_\delta^{(x_i)}) \,. \tag{4.14}$$

Similarly as above, we also have that

$$\sup_{1 \leqslant i \leqslant k_\varepsilon} \sup_{x \in \mathtt{C}_i} \mathbb{E}\big[\big| \log^{(R)} \mathscr{Z}^\vartheta_{\delta,1}(g_\delta^{(x)}) - \log^{(R)} \mathscr{Z}^\vartheta_{\delta,1}(g_\delta^{(x_i)}) \big| \big] \leqslant \mathtt{C}_{\delta,R}\, \varepsilon \,,$$

so we obtain

$$\mathbb{E}\Big[\Big| \int_{\mathbb{R}^2} \varphi(x) \log^{(R)} \mathscr{Z}^\vartheta_{\delta,1}(g_\delta^{(x)}) \mathrm{d}x - \sum_{i=1}^{k_\varepsilon} \Big(\int_{\mathtt{C}_i} \varphi(u) \mathrm{d}u \Big) \log^{(R)} \mathscr{Z}^\vartheta_{\delta,1}(g_\delta^{(x_i)}) \Big| \Big] \leqslant C_{R,\varphi,\delta}\, \varepsilon \,. \tag{4.15}$$

All together, combining the convergence (4.14) with (4.13)–(4.15), letting $N \to \infty$ then $\varepsilon \downarrow 0$, this concludes the proof of (4.12).

Since both terms in (4.12) are bounded (by $R\|\varphi\|_1$), this also gives convergence of expectations

$$\sum_{x \in \mathbb{Z}^2_{\mathrm{even}}} \varphi^{(N)}(x) \mathbb{E}\big[\log^{(R)} \widehat{Z}_{m,N}(x) \big] \xrightarrow[N \to \infty]{} \int_{\mathbb{R}^2} \varphi(x) \mathbb{E}\big[\log^{(R)} \mathscr{Z}^\vartheta_{\delta,1}(g_\delta^{(x)}) \big] \mathrm{d}x \,. \tag{4.16}$$

Once more, let us stress that the convergences (4.12)–(4.16) hold jointly for any finite collection of fixed values of δ.

Step 3: Removing the truncation. Since we have that

$$|\log u - \log^{(R)}(u)| \leqslant \mathrm{e}^{-R}(u + u^{-1}), \qquad u > 0,$$

we get thanks to (4.5) that

$$\mathbb{E}\big[|\log \widehat{Z}_{m,N}(x) - \log^{(R)} \widehat{Z}_{m,N}(x)| \big] \leqslant \mathrm{e}^{-R}(2 + C_\delta) \,.$$

This thus gives that, for any $R \geqslant 0$ and $N \geqslant 1$,

$$\mathbb{E}\Big[\Big| \mathcal{H}^{(N;\delta)}(\varphi) - \sum_{x \in \mathbb{Z}^2_{\mathrm{even}}} \varphi^{(N)}(x) \{ \log^{(R)} \widehat{Z}_{m,N}(x) - \mathbb{E} \log^{(R)} \widehat{Z}_{m,N}(x) \} \Big| \Big] \leqslant C_{\delta,\varphi} \mathrm{e}^{-R} \,.$$

For the continuous counterpart, using (4.10) and Portmanteau theorem, we also get from (4.5) that $\mathbb{E}[\mathscr{Z}^{\vartheta}_{\delta,1}(g^{(x)}_{\delta})^{-1}] \leqslant 1 + C_\delta$. In particular, we have that $\mathscr{Z}^{\vartheta}_{\delta,1}(g^{(x)}_{\delta})$ is a.s. positive and also

$$\mathbb{E}\big[|\log \mathscr{Z}^{\vartheta}_{\delta,1}(g^{(x)}_{\delta}) - \log^{(R)} \mathscr{Z}^{\vartheta}_{\delta,1}(g^{(x)}_{\delta})|\big] \leqslant \mathrm{e}^{-R}(2 + C_\delta)\,.$$

This thus gives that for any $R \geqslant 0$

$$\mathbb{E}\bigg[\Big|\mathscr{H}^{(\delta)}(\varphi) - \int_{\mathbb{R}^2} \varphi(x)\{\log^{(R)} \mathscr{Z}^{\vartheta}_{\delta,1}(g^{(x)}_{\delta}) - \mathbb{E}\log^{(R)} \mathscr{Z}^{\vartheta}_{\delta,1}(g^{(x)}_{\delta})\}\,\mathrm{d}x\Big|\bigg] \leqslant C_{\delta,\varphi}\mathrm{e}^{-R}\,.$$

Moreover, $|\log u|^2 \leqslant C(u + u^{-1})$, together with the first moment $\mathbb{E}[\mathscr{Z}^{\vartheta}_{\delta,1}(g^{(x)}_{\delta})] = 1$ and the negative-first-moment bound above, gives

$$\mathbb{E}\big[|\mathscr{H}^{(\delta)}(\varphi)|^2\big] \leqslant C_\delta \|\varphi\|_1^2.$$

Using (4.12)–(4.16) and then letting $R \to \infty$ this proves the convergence (2.14). Once again, the convergence applies jointly for any finite collection of fixed values of δ. □

4.3. Local approximation: proof of Proposition 2.4. In this section, we consider the local approximation error from (2.25), that we rewrite as follows, recalling (2.5):

$$L_{m,n}(x) = \log \widehat{Z}_{0,n}(x) - \log \widehat{Z}_{m,n}(x). \tag{4.17}$$

We will apply the covariance bound (3.12), for which the key technical input is the following one-point moment comparison, proved in Section 7.4 below. It is a consequence of a Gaussian convexity property [Che26, Theorem 1] and the resulting moment comparison bound in Lemma 3.3 on

$$\Delta(F) := -\mathbb{E}F - \tfrac{1}{2}\operatorname{Var}(F).$$

It also relies on the block factorisation of Gu and Tsai [GT26] that we detail in Section 7.

Proposition 4.1 (One-point moment comparison)**.** *Recall $\widehat{Z}_{s,n}(x) = Z_{s,n}(q^{(x)}_s, \mathbf{1})$ from (2.5), with $q^{(x)}_0 = \delta_x$. For every $0 < T_0 < T_1 < \infty$, there is a constant $C_{T_0,T_1} < \infty$ such that, for all sufficiently large N, uniformly in x and even integers $0 \leqslant s \leqslant n$ with $T_0 N \leqslant n \leqslant T_1 N$,*

$$0 \leqslant \Delta(\log \widehat{Z}_{s,n}(x)) \leqslant \log \frac{\mathbb{E}[\widehat{Z}_{s,n}(x)^2]}{\mathbb{E}[\widehat{Z}_{s,n}(x)^{-1}]} \leqslant C_{T_0,T_1}. \tag{4.18}$$

Moreover, there exist constants $\delta_0 \in (0,1)$ and $C < \infty$ such that, for every $0 < \delta \leqslant \delta_0$ and $n = [\sqrt{\delta}N]$, uniformly in x and even integers $0 \leqslant s \leqslant n$,

$$\log \frac{\mathbb{E}[\widehat{Z}_{s,n}(x)^2]}{\mathbb{E}[\widehat{Z}_{s,n}(x)^{-1}]} \leqslant \frac{C}{|\log \delta|^{1/4}} + o_N(1). \tag{4.19}$$

All constants may be chosen uniformly when ϑ ranges over a fixed compact interval, and $o_N(1)$ tends to zero as $N \to \infty$ at fixed δ, uniformly in s, x and in ϑ over that interval.

The small-time bound (4.19) quantifies how $\log \widehat{Z}_{s,n}(x)$ is close to a normal variable. In this sense it complements the asymptotic log-normality for locally averaged SHF proved in [GT26, Theorem 1.1]. This quantitative comparison, with $N \to \infty$ before $\delta \downarrow 0$, is what makes the two Δ terms in (3.12) vanish.

Proof of Proposition 2.4. Recalling (2.20), we take

$$m = [\delta N]\,, \qquad n = [\sqrt{\delta} N]\,.$$

For $s \in \{0, m\}$, using spatial stationarity and the covariance bound (3.12) together with the bound $\log u \leqslant u - 1$, we get

$$\begin{aligned} 0 &\leqslant \mathrm{Cov}\Big(\log \widehat{Z}_{s,n}(x), \log \widehat{Z}_{s,n}(y)\Big) \\ &\leqslant \mathbb{E}[\widehat{Z}_{s,n}(x)\widehat{Z}_{s,n}(y)] - 1 + C|\log\delta|^{-1/4} + o_N(1). \end{aligned} \tag{4.20}$$

where we have also used Proposition 4.1 to bound $\Delta(\log \widehat{Z}_{s,n}(x))$ and $\Delta(\log \widehat{Z}_{s,n}(y))$.

Now, recalling (4.3) and that $n = [\sqrt{\delta}N]$, we obtain that for $s \in \{0, m\}$

$$\mathrm{Var}\Big(\sum_x \varphi^{(N)}(x) \log \widehat{Z}_{s,n}(x)\Big) = \sum_{x,y} \varphi^{(N)}(x)\varphi^{(N)}(y)\,\mathrm{Cov}\Big(\log \widehat{Z}_{s,n}(x), \log \widehat{Z}_{s,n}(y)\Big) \tag{4.21}$$

$$\leqslant \frac{C}{|\log\delta|}\sum_{x,y} |\varphi^{(N)}(x)||\varphi^{(N)}(y)|\mathsf{G}_n(x-y) + \Big(\frac{C}{|\log\delta|^{1/4}} + o_N(1)\Big)\sum_{x,y}|\varphi^{(N)}(x)||\varphi^{(N)}(y)|\,.$$

All together, we get that

$$\limsup_{N\to\infty} \mathrm{Var}\Big(\sum_x \varphi^{(N)}(x)\log \widehat{Z}_{s,n}(x)\Big) \leqslant \frac{C\mathcal{E}_{\sqrt{\delta}}(\varphi)}{|\log\delta|} + \frac{C\|\varphi\|_1^2}{|\log\delta|^{1/4}}\,,$$

where

$$\mathcal{E}_{\sqrt{\delta}}(\varphi) = \int_{\mathbb{R}^2\times\mathbb{R}^2} |\varphi(x)||\varphi(y)| K_{\sqrt{\delta}}(x-y)\mathrm{d}x\mathrm{d}y\,, \tag{4.22}$$

with

$$K_\epsilon(z) = 4\int_0^\epsilon g_{2s}(z)\,\mathrm{d}s = \int_0^\epsilon \frac{1}{\pi s}\mathrm{e}^{-|z|^2/(4s)}\,\mathrm{d}s.$$

This is the limit of $\mathsf{G}_n([\sqrt{N/2}\,z])$ for $z \neq 0$ and $n/N \to \epsilon = \sqrt{\delta}$, with the spatial scaling from the definition (2.10) of $\varphi^{(N)}$, see (1.7). Since $\int_{\mathbb{R}^2} K_\epsilon(z)\,\mathrm{d}z = 4\epsilon$, we have $\mathcal{E}_{\sqrt{\delta}}(\varphi) \leqslant 4\sqrt{\delta}\|\varphi\|_\infty\|\varphi\|_1$, which gives a negligible contribution (we have kept this term in order to keep track of the dependence of the constants on the function φ, see Remark 4.2 below).

We now apply these estimates to $\mathcal{L}^{(N;\delta)}(\varphi)$, the object in Proposition 2.4: recalling (2.28) and (4.17), we have that

$$\begin{aligned} \mathcal{L}^{(N;\delta)}(\varphi) = &\sum_x \varphi^{(N)}(x)\big\{\log \widehat{Z}_{0,n}(x) - \mathbb{E}\log \widehat{Z}_{0,n}(0)\big\} \\ &- \sum_x \varphi^{(N)}(x)\big\{\log \widehat{Z}_{m,n}(x) - \mathbb{E}\log \widehat{Z}_{m,n}(0)\big\}\,. \end{aligned}$$

Since $\mathrm{Var}(X-Y) \leqslant 2\,\mathrm{Var}(X) + 2\,\mathrm{Var}(Y)$, applying the estimate (4.21) for $s = 0$ and $s = m$, and recalling that $n = [\sqrt{\delta}N]$, we obtain that

$$\limsup_{N\to\infty} \mathbb{E}\big[|\mathcal{L}^{(N;\delta)}(\varphi)|^2\big] \leqslant C_\varphi|\log\delta|^{-1/4}\,. \tag{4.23}$$

This proves Proposition 2.4. □

Remark 4.2. *In fact, we may keep track of the dependence of the bound on the function φ: we have indeed obtained the following*

$$\limsup_{N\to\infty} \mathbb{E}|\mathcal{L}^{(N;\delta)}(\varphi)|^2 \leqslant \frac{C\mathcal{E}_{\sqrt{\delta}}(\varphi)}{|\log\delta|} + \frac{C\|\varphi\|_1^2}{|\log\delta|^{1/4}}\,. \tag{4.24}$$

In particular, this inequality holds even if the test function φ depends on δ. For instance we can apply (4.24) *to* $\varphi_{\sqrt{\delta}}(\cdot) = \delta^{-1}\varphi(\cdot/\sqrt{\delta})$*, which verifies* $\|\varphi_{\sqrt{\delta}}\|_1 = \|\varphi\|_1$*. Scaling and splitting the time integral defining* $K_{1/\sqrt{\delta}}$ *at time* 1 *give*

$$\mathcal{E}_{\sqrt{\delta}}(\varphi_{\sqrt{\delta}}) = \mathcal{E}_{1/\sqrt{\delta}}(\varphi) \leqslant C\|\varphi\|_\infty\|\varphi\|_1 + C|\log\delta|\,\|\varphi\|_1^2.$$

(The bound $\mathcal{E}_{\sqrt{\delta}}(\varphi_{\sqrt{\delta}}) \leqslant 4\sqrt{\delta}\|\varphi_{\sqrt{\delta}}\|_\infty\|\varphi_{\sqrt{\delta}}\|_1$ *instead diverges as* $\delta^{-1/2}$*.)*

4.4. Intermezzo: about the deterministic centring. As a corollary of Proposition 4.1 (and of the Gaussian convexity property), we are able to identify the order of the deterministic centring constant in (1.14).

Lemma 4.3. *Let* $t > 0$ *be fixed. As* $N \to \infty$*, we have that*

$$\mathbb{E}\big[\log Z_{[Nt]}(0)\big] = -\tfrac{1}{2}\log\log N + O_{\vartheta,t}(1).$$

Proof. By the Gaussian convexity property, we have

$$\tfrac{1}{2}\log\mathbb{E}[Z_{[Nt]}(0)^{-1}] \leqslant -\mathbb{E}[\log Z_{[Nt]}(0)] \leqslant \tfrac{1}{2}\log\mathbb{E}[Z_{[Nt]}(0)^2]\,.$$

The first bound follows from the first inequality in (3.8) and the variance bound (3.2), and the second is (3.3). Since $\widehat{Z}_{0,[Nt]}(0) = Z_{[Nt]}(0)$, Proposition 4.1-(4.18) shows that

$$\Big|\mathbb{E}[\log Z_{[Nt]}(0)] + \tfrac{1}{2}\log\mathbb{E}[Z_{[Nt]}(0)^2]\Big| \leqslant C\,.$$

It thus only remains to estimate $\log\mathbb{E}[Z_{[Nt]}(0)^2]$, but this is standard. Indeed, from [CSZ19a, Proposition A.2], we have that $\mathbb{E}[Z_{[Nt]}(0)^2] \sim C_{\vartheta,t}\log N$ as $N \to \infty$. We thus have that $\log\mathbb{E}[Z_N(0)^2] = \log\log N + O_{\vartheta,t}(1)$, which proves the lemma. □

To recover the centring $c_t^{\vartheta,(N)}$ in (1.14), by time reversal and translation invariance, we have

$$c_t^{\vartheta,(N)} := \mathbb{E}\big[\log \mathcal{Z}_t^{\vartheta,(N)}(\mathbf{1},x)\big] = \mathbb{E}[\log Z^{\beta_N}_{[Nt]}(0)] = -\tfrac{1}{2}\log\log N + O_{\vartheta,t}(1)\,,$$

by Lemma 4.3. We can now compare the centring $c_t^{\vartheta,(N)}$ with that for a general initial condition $h_0 \in L^\infty$. By time reversal, we again have that

$$\mathbb{E}\big[\log \mathcal{Z}_t^{\vartheta,(N)}(\mathrm{e}^{h_0},x)\big] - c_t^{\vartheta,(N)} = \mathbb{E}\bigg[\log\frac{\mathcal{Z}_t^{\vartheta,(N)}(x,\mathrm{e}^{h_0})}{\mathcal{Z}_t^{\vartheta,(N)}(x,\mathbf{1})}\bigg] = \mathbb{E}\big[\log\mu^{(x)}_{[Nt]}(\mathrm{e}^{h_0})\big]$$

where $\mu^{(x)}_{[Nt]}$ is the polymer measure started from x. Since this is a probability measure, it readily yields $|c_t^{\vartheta,(N)} - \mathbb{E}[\log\mathcal{Z}_t^{\vartheta,(N)}(\mathrm{e}^{h_0},x)]| \leqslant \|h_0\|_\infty$, that is (1.15).

4.5. Global approximation: proof of Proposition 2.5. Recall the definition (2.26) of the global approximation error between times n and N, namely

$$G_{m,n,N}(x) = \log Z_{n,N}(\mu_n^{(x)}) - \log Z_{n,N}(\widehat{\mu}^{(x)}_{m,n}).$$

Note that both partition functions use the same disorder on $(n,N]$ and differ only in their starting distributions at time n. We rely on the following crucial Proposition 4.4, which controls the dependence on the starting distributions (see (7.44) for a more precise statement), by bounding the difference of the two partition functions and their second moments, on an event of high probability. Its proof again relies on the block decomposition of Gu and Tsai [GT26] and is postponed to Subsection 7.5. Recall that $m = [\delta N]$ and $n = [\sqrt{\delta}N]$.

Proposition 4.4 (Comparison of the partition functions)**.** *There exist constants $\delta_0 \in (0,1)$ and $C < \infty$ such that for every $0 < \delta \leqslant \delta_0$ and all sufficiently large N, there are events $\mathcal{G}_{N,\delta}(x) \in \mathcal{F}_n$ such that the following properties hold. First, uniformly in x, we have*

$$\mathbb{P}\big(\mathcal{G}_{N,\delta}(x)^c\big) \leqslant \frac{C}{|\log\delta|}\,. \tag{4.25}$$

Second, we have the crucial comparison bound

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{G}_{N,\delta}(x)}|Z_{n,N}(\mu_n^{(x)}) - Z_{n,N}(\widehat{\mu}_{m,n}^{(x)})|\big] \leqslant \frac{C}{|\log\delta|}\,, \tag{4.26}$$

Moreover, for $\nu = \mu_n^{(x)}$ or $\nu = \widehat{\mu}_{m,n}^{(x)}$,

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{G}_{N,\delta}(x)}Z_{n,N}(\nu)^2\big] \leqslant C|\log\delta|\,. \tag{4.27}$$

Finally, for every bounded deterministic terminal function f, we also have

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{G}_{N,\delta}(x)}|Z_{n,N}(\mu_n^{(x)},f) - Z_{n,N}(\widehat{\mu}_{m,n}^{(x)},f)|\big] \leqslant C\|f\|_\infty\frac{\sqrt{\log|\log\delta|}}{|\log\delta|}. \tag{4.28}$$

The constants may depend on ϑ, but are uniform over ϑ in a compact set.

We also stress that the event $\mathcal{G}_{N,\delta}(x)$ is in fact explicit (see (7.54)). It is on this good event that we are able to control the difference between $\mu_n^{(x)}$ and $\widehat{\mu}_{m,n}^{(x)}$. Aside from (4.25) which already appear in some form in [GT26, Corollary 4.2] (we include the proof here to keep the paper self-contained), the main novelty here is (4.26) (and (4.27)), which controls how the partition function on $(n,N]$ is affected by switching the initial condition from $\mu_n^{(x)}$ to $\widehat{\mu}_{m,n}^{(x)}$. It strongly relies on Lemma 7.7, which compares μ_n and $\widehat{\mu}_{m,n}$ with q_n (up to a leading correction) under some semi-norm that averages the disorder on $(n,N]$.

We also need the following proposition to control $G_{m,n,N}$ on the event $\mathcal{G}_{N,\delta}(x)^c$. It controls the moments of $\log Z_{n,N}(\nu)$. Its proof is postponed to Subsection 7.6.

Proposition 4.5 (Logarithmic moments)**.** *Fix $p \geqslant 1$. For every sufficiently small $\delta > 0$, the following bound holds for both $\nu = \mu_n^{(x)}$ and $\nu = \widehat{\mu}_{m,n}^{(x)}$:*

$$\limsup_{N\to\infty}\sup_x \|\log Z_{n,N}(\nu)\|_{L^p} \leqslant C_p\log|\log\delta|\,. \tag{4.29}$$

The constant C_p may depend on ϑ but not on δ. For a positive deterministic terminal function f with $\|\log f\|_\infty < \infty$, the same bound holds with $Z_{n,N}(\nu)$ replaced by $Z_{n,N}(\nu,f)$ and the right hand side replaced by $C_p\log|\log\delta| + \|\log f\|_\infty$.

An important technical novelty in the proof is good control of the left tail of $\log Z_{n,N}(\nu)$ (the right tail is in fact easily controlled), see Lemma 7.8 below. It greatly improves on the left-tail obtained in [GT26, Equation (5.2)]. The Gaussian convexity result from Section 3 plays a central role since it allows us to control negative moments of $Z_{n,N}(\nu)$ in terms of its second moment, thanks to Lemma 3.2-(3.5).

Remark 4.6. *The moment bounds* (4.5) *and the convergence* (4.10) *imply that, for every $p \geqslant 1$ and all sufficiently small $\delta > 0$,*

$$\|\log \mathscr{Z}_{\delta,1}^{\vartheta}(g_\delta^{(x)})\|_{L^p} \leqslant C_p\log|\log\delta|\,. \tag{4.30}$$

Indeed, for either $X = \widehat{Z}_{m,N}(x)$ or $X = \mathscr{Z}_{\delta,1}^{\vartheta}(g_\delta^{(x)})$, we have $\mathbb{E}[X] = 1$ and $\mathbb{E}[X^{-1}] \leqslant 1 + C\log(1/\delta)$, where the bound in the SHF case follows from weak convergence and Fatou.

Hence, for $u \geqslant 0$,

$$\mathbb{P}(|\log X| > u) \leqslant \min\{1, C|\log\delta|\mathrm{e}^{-u}\}.$$

Integrating this tail bound proves (4.30)*. By Minkowski's inequality and the centring in* (2.8)*, we also obtain*

$$\|\mathscr{H}^{(\delta)}(\varphi)\|_{L^p} \leqslant C_p\|\varphi\|_1 \log|\log\delta|\,.$$

We now have the tools to conclude the proof of Proposition 2.5.

Proof of Proposition 2.5. Fix $\delta > 0$ sufficiently small and x, and write $\mathcal{G} = \mathcal{G}_{N,\delta}(x)$. We estimate $G_{m,n,N}(x)$ separately on $\mathcal{G}$ and $\mathcal{G}^c$.

First of all, for $u, v > 0$ and $0 < \alpha < 1$, a straightforward calculation gives that

$$|\log u - \log v| \leqslant C_\alpha |u-v|^\alpha (u^{-\alpha} + v^{-\alpha}). \tag{4.31}$$

Let us fix α (we take $\alpha = 1/3$ below). We get that

$$\begin{aligned}
&\mathbb{E}[|G_{m,n,N}(x)|\mathbf{1}_{\mathcal{G}}] \\
&\quad\leqslant C_\alpha \mathbb{E}\big[\mathbf{1}_{\mathcal{G}}\big|Z_{n,N}(\mu_n^{(x)}) - Z_{n,N}(\widehat{\mu}_{m,n}^{(x)})\big|^\alpha \big(Z_{n,N}(\mu_n^{(x)})^{-\alpha} + Z_{n,N}(\widehat{\mu}_{m,n}^{(x)})^{-\alpha}\big)\big] \\
&\quad\leqslant 2C_\alpha \mathbb{E}\big[\mathbf{1}_{\mathcal{G}}\big|Z_{n,N}(\mu_n^{(x)}) - Z_{n,N}(\widehat{\mu}_{m,n}^{(x)})\big|\big]^\alpha \max_{\nu = \mu_n^{(x)}, \widehat{\mu}_{m,n}^{(x)}} \mathbb{E}\big[\mathbf{1}_{\mathcal{G}} Z_{n,N}(\nu)^{-\alpha/(1-\alpha)}\big]^{1-\alpha},
\end{aligned} \tag{4.32}$$

using Hölder's inequality for the last line.

The first term can be controlled by Proposition 4.4-(4.26). For the second term, we use that with either starting distribution $\nu = \mu_n^{(x)}$ or $\nu = \widehat{\mu}_{m,n}^{(x)}$ (which are $\mathcal{F}_n$-measurable), the partition function $Z_{n,N}(\nu)$ has conditional mean $\mathbb{E}[Z_{n,N}(\nu) \mid \mathcal{F}_n] = 1$, and that $\log Z_{n,N}(\nu)$ is convex in the Gaussian disorder on $(n, N]$. We may thus apply Lemma 3.2-(3.5) with $p = \alpha/(1-\alpha)$, to obtain that

$$\mathbb{E}\big[Z_{n,N}(\nu)^{-\alpha/(1-\alpha)} \mid \mathcal{F}_n\big] \leqslant \mathbb{E}\big[Z_{n,N}(\nu)^2 \mid \mathcal{F}_n\big]^{\alpha/2(1-\alpha)^2}.$$

Since $\mathcal{G} \in \mathcal{F}_n$, we thus get that

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{G}} Z_{n,N}(\nu)^{-\alpha/(1-\alpha)}\big] \leqslant \mathbb{E}\big[\mathbf{1}_{\mathcal{G}}\mathbb{E}[Z_{n,N}(\nu)^2 \mid \mathcal{F}_n]^{\alpha/2(1-\alpha)^2}\big] \leqslant \mathbb{E}\big[\mathbf{1}_{\mathcal{G}} Z_{n,N}(\nu)^2\big]^{\alpha/2(1-\alpha)^2},$$

having used Hölder's inequality for the last step and bounded $\mathbb{P}(\mathcal{G}) \leqslant 1$. Using Proposition 4.4-(4.27), this gives

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{G}} Z_{n,N}(\nu)^{-\alpha/(1-\alpha)}\big]^{1-\alpha} \leqslant C|\log\delta|^{\alpha/2(1-\alpha)}.$$

Going back to (4.32) and using also (4.26), we thus end up with

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{G}}|G_{m,n,N}(x)|\big] \leqslant C|\log\delta|^{-\alpha}|\log\delta|^{\alpha/2(1-\alpha)} \leqslant C|\log\delta|^{-1/12}, \tag{4.33}$$

where we have chosen $\alpha = 1/3$ for the last inequality (the optimal value would be attained at $\alpha = 1 - 2^{-1/2}$).

On $\mathcal{G}^c$, Cauchy–Schwarz inequality gives

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{G}^c}|G_{m,n,N}(x)|\big] \leqslant \mathbb{P}(\mathcal{G}^c)^{1/2}\big\{\|\log Z_{n,N}(\mu_n^{(x)})\|_{L^2} + \|\log Z_{n,N}(\widehat{\mu}_{m,n}^{(x)})\|_{L^2}\big\}.$$

Using Proposition 4.4-(4.25) and Proposition 4.5 with $p = 2$, we conclude that

$$\limsup_{N\to\infty} \sup_x \mathbb{E}\big[\mathbf{1}_{\mathcal{G}^c}|G_{m,n,N}(x)|\big] \leqslant \frac{C\log|\log\delta|}{|\log\delta|^{1/2}}. \tag{4.34}$$

This is bounded by $C|\log\delta|^{-1/12}$ for small δ. Adding (4.33) and (4.34) therefore yields

$$\limsup_{N\to\infty} \sup_x \mathbb{E}|G_{m,n,N}(x)| \leqslant \frac{C}{|\log\delta|^{1/12}}. \tag{4.35}$$

Now, applying again Proposition 4.5, we get that

$$\begin{aligned}\limsup_N \sup_x \|G_{m,n,N}(x)\|_{L^2} &\leqslant \limsup_N \sup_x \|G_{m,n,N}(x)\|_{L^1}^{1/3} \|G_{m,n,N}(x)\|_{L^4}^{2/3} \\ &\leqslant \frac{C}{|\log\delta|^{1/36}} (\log|\log\delta|)^{2/3} \leqslant \frac{C'}{|\log\delta|^{1/40}}\,.\end{aligned} \tag{4.36}$$

Finally, recalling the definition (2.29) of $\mathcal{G}^{(N;\delta)}(\varphi)$, by the triangle inequality we get

$$\|\mathcal{G}^{(N;\delta)}(\varphi)\|_{L^2} \leqslant 2 \sum_{x\in\mathbb{Z}^2_{\text{even}}} |\varphi^{(N)}(x)| \|G_{m,n,N}(x)\|_{L^2} \leqslant 2\|\varphi\|_{L^1} \sup_x \|G_{m,n,N}(x)\|_{L^2}\,. \tag{4.37}$$

Therefore, (4.36) proves Proposition 2.5. □

Remark 4.7. *We now extend the bound* (4.36) *to non-constant terminal weights. For a positive deterministic terminal function f with $\|\log f\|_\infty < \infty$, define*

$$G_{m,n,N}(x,f) := \log \frac{Z_{n,N}(\mu_n^{(x)}, f)}{Z_{n,N}(\widehat{\mu}_{m,n}^{(x)}, f)}. \tag{4.38}$$

The comparison $\mathrm{e}^{-\|\log f\|_\infty} Z_{n,N}(\nu) \leqslant Z_{n,N}(\nu, f) \leqslant \mathrm{e}^{\|\log f\|_\infty} Z_{n,N}(\nu)$ transfers the negative-moment estimate used in (4.32) *to f. Using* (4.28) *in place of* (4.26) *then introduces an additional factor $(\log|\log\delta|)^{1/6}$ in* (4.33)*. To obtain the analogue of* (4.35) *for $G_{m,n,N}(x,f)$, we estimate $\mathbb{E}[\mathbf{1}_{\mathcal{G}^c}|G_{m,n,N}(x,f)|]$ as in* (4.34)*, using Proposition 4.5 with $p=2$. The same proposition with $p=4$ bounds $\|G_{m,n,N}(x,f)\|_{L^4}$. Interpolation as in* (4.36) *therefore gives*

$$\limsup_{N\to\infty} \sup_x \|G_{m,n,N}(x,f)\|_{L^2} \leqslant C_{\|\log f\|_\infty} |\log\delta|^{-1/40}. \tag{4.39}$$

The bound is uniform over terminal functions with a common bound on $\|\log f\|_\infty$.

Remark 4.8. *Let us record here the dependence of the bound on the function φ. Combining* (4.37) *with* (4.36) *yields*

$$\limsup_{N\to\infty} \mathbb{E}\big|\mathcal{G}^{(N;\delta)}(\varphi)\big|^2 \leqslant \frac{C\|\varphi\|_1^2}{|\log\delta|^{1/20}}. \tag{4.40}$$

In particular, we may apply (4.40) *to $\varphi_\delta(\cdot) = \delta^{-2}\varphi(\cdot/\delta)$, which verifies $\|\varphi_\delta\|_1 = \|\varphi\|_1$.*

5. Extension to joint convergence and non-flat initial condition

In this section, we extend the convergence in Theorem 2.1 to joint convergence at finitely many times and to non-flat initial condition $f_0 = \mathrm{e}^{h_0} \not\equiv \mathbf{1}$. Let us first make the time-reversal more explicit than in Section 2.2. For $0 \leqslant t < T$, $0 < \delta < T - t$, we write

$$\mathscr{H}_{t,T}^{(\delta)}(x, h_0) := \log \mathscr{Z}_{t+\delta,T}^{\vartheta}(g_\delta^{(x)}, \mathrm{e}^{h_0}) - \mathbb{E}\big[\log \mathscr{Z}_{t+\delta,T}^{\vartheta}(g_\delta^{(0)}, \mathbf{1})\big]. \tag{5.1}$$

Here x is the starting point at time t, and h_0 specifies the terminal weight $f_0 = \mathrm{e}^{h_0}$ at time T. Note that disorder is removed on $[t, t+\delta]$. Let $k \in \mathbb{N}$ and let $0 < \delta_i < t_i \leqslant T$. Then, invariance of the law of SHF under time reversal gives

$$\big(\mathscr{H}_{t_i}^{(\delta_i)}(h_0, \cdot)\big)_{i=1}^k \overset{d}{=} \big(\mathscr{H}_{T-t_i,T}^{(\delta_i)}(\cdot, h_0)\big)_{i=1}^k. \tag{5.2}$$

We may similarly consider the discrete counterparts. Let f_N be a discrete approximation of $f_0 = \mathrm{e}^{h_0}$, defined by the cell averages

$$f_N(z) = |\mathtt{C}_N(z)|^{-1} \int_{\mathtt{C}_N(z)} \mathrm{e}^{h_0(y)}\,\mathrm{d}y\,, \tag{5.3}$$

with $\mathtt{C}_N(z)$ the cell defined in (1.7). Let also $x_N := [\sqrt{N/2}\,x]$, so that $x \in \mathtt{C}_N(x_N)$. For $0 \leqslant t_i < T$ and $0 < \delta < T - t_i$, define

$$\mathcal{H}^{(N)}_{t_i,T}(x, h_0) := \log Z_{[Nt_i],[NT]}(x_N, f_N) - \mathbb{E}\log Z_{[Nt_i],[NT]}(x_N, \mathbf{1}), \tag{5.4}$$

and also

$$\mathcal{H}^{(N;\delta)}_{t_i,T}(x, h_0) := \log Z_{[N(t_i+\delta)],[NT]}(q^{(x_N)}_{[\delta N]}, f_N) - \mathbb{E}\log Z_{[N(t_i+\delta)],[NT]}(q^{(x_N)}_{[\delta N]}, \mathbf{1})\,. \tag{5.5}$$

For a test function φ, write $\mathscr{H}^{(\delta)}_{t,T}(\varphi, h_0) = \int_{\mathbb{R}^2} \varphi(x)\mathscr{H}^{(\delta)}_{t,T}(x, h_0)\,\mathrm{d}x$, and similarly for the lattice fields.

We now extend Theorem 2.1 to a joint convergence at different times and different test functions, with common terminal weight $f_0 = \mathrm{e}^{h_0}$. To this end, we extend Proposition 2.2 to a joint convergence with non-flat initial condition (see Lemma 5.2 below) and we also extend Proposition 2.3 to the case of general initial condition (see Lemma 5.3). Together, this proves the following finite-dimensional convergence that extends Theorem 2.1 as follows.

Theorem 5.1 (Critical 2D KPZ Process, finite-dimensional convergence)**.** *Fix $\vartheta \in \mathbb{R}$, $T > 0$, $h_0 \in L^\infty(\mathbb{R}^2) \cap C(\mathbb{R}^2)$, and $k \in \mathbb{N}$. For $1 \leqslant i \leqslant k$, let $0 \leqslant t_i < T$ and $\varphi_i \in C^\infty_c(\mathbb{R}^2)$. We then have the following joint convergence, where the limits $\mathscr{H}^{\vartheta}_{t_i,T}(\varphi_i, h_0)$ are defined on the same probability space as the SHF:*

$$\big(\mathscr{H}^{\vartheta,(\delta)}_{t_i,T}(\varphi_i, h_0)\big)_{i=1}^k \xrightarrow[\delta\downarrow 0]{L^p} \big(\mathscr{H}^{\vartheta}_{t_i,T}(\varphi_i, h_0)\big)_{i=1}^k, \quad \text{for } p \in [1,\infty)$$
$$\big(\mathcal{H}^{\vartheta,(N)}_{t_i,T}(\varphi_i, h_0)\big)_{i=1}^k \xrightarrow[N\to\infty]{d} \big(\mathscr{H}^{\vartheta}_{t_i,T}(\varphi_i, h_0)\big)_{i=1}^k.$$

By time reversal (5.2)*,*

$$\big(\mathscr{H}^{\vartheta}_{t_i,T}(\varphi_i, h_0)\big)_{i=1}^k \overset{d}{=} \big(\mathscr{H}^{\vartheta}_{T-t_i}(h_0, \varphi_i)\big)_{i=1}^k.$$

Once Lemmas 5.2 and 5.3 are at hand, the proof of Theorem 5.1 is analogous to the proof in Section 2.2. The extension to convergence in H^{-a}_{loc} for any $a > 0$ is proven in Section 6 below.

5.1. Extension of Proposition 2.2. The following result extends Proposition 2.2 to joint convergence at different starting times, with a common terminal time T and terminal weight $f_0 = \mathrm{e}^{h_0}$.

Lemma 5.2 (Joint convergence at fixed temporal regularisation)**.** *Fix $\vartheta \in \mathbb{R}$, $T > 0$, $h_0 \in L^\infty(\mathbb{R}^2) \cap C(\mathbb{R}^2)$, and $k \in \mathbb{N}$. For $1 \leqslant i \leqslant k$, let $0 \leqslant s_i < T$, $\delta_i > 0$, and $x_i \in \mathbb{R}^2$. Under the critical tuning* (1.6)*, the SHF integrals $\mathscr{Z}^{\vartheta}_{s_i,T}(g^{(x_i)}_{\delta_i}, \mathrm{e}^{h_0})$ are almost surely positive, and letting f_N be defined as in* (5.3) *we have that*

$$\Big(\log Z^{\beta_N}_{[Ns_i],[NT]}\big(q^{([\sqrt{N/2}\,x_i])}_{[N\delta_i]}, f_N\big)\Big)_{i=1}^k \xrightarrow[N\to\infty]{d} \Big(\log \mathscr{Z}^{\vartheta}_{s_i,T}(g^{(x_i)}_{\delta_i}, \mathrm{e}^{h_0})\Big)_{i=1}^k. \tag{5.6}$$

All absolute moments of positive order are finite and converge as well. Moreover, for $\varphi_1, \dots, \varphi_k \in C^\infty_c(\mathbb{R}^2)$, letting $\varphi^{(N)}_i$ be defined as in (2.10)*, we have*

$$\Big(\sum_z \varphi^{(N)}_i(z)\log Z^{\beta_N}_{[Ns_i],[NT]}\big(q^{(z)}_{[N\delta_i]}, f_N\big)\Big)_{i=1}^k \xrightarrow[N\to\infty]{d} \Big(\int_{\mathbb{R}^2} \varphi_i(x)\log \mathscr{Z}^{\vartheta}_{s_i,T}(g^{(x)}_{\delta_i}, \mathrm{e}^{h_0})\,\mathrm{d}x\Big)_{i=1}^k. \tag{5.7}$$

As a consequence of this lemma, taking $s_i = t_i + \delta_i < T$ with $t_i \geqslant 0$, we have the joint convergence

$$\Big(\mathcal{H}^{(N;\delta_i)}_{t_i,T}(\varphi_i, h_0)\Big)_{i=1}^{k} \xrightarrow[N\to\infty]{d} \Big(\mathscr{H}^{(\delta_i)}_{t_i,T}(\varphi_i, h_0)\Big)_{i=1}^{k}. \tag{5.8}$$

Indeed, this is only a matter of subtracting the centring constants (with flat final weight), whose convergence follows from the moment convergence provided by Lemma 5.2.

Proof of Lemma 5.2. To simplify notation, let us introduce the abbreviation

$$A^{(i)}_N(x; f) \coloneqq Z^{\beta_N}_{[Ns_i],[NT]}\big(q^{([\sqrt{N/2}\,x])}_{[N\delta_i]}, f\big) \quad \text{and} \quad A^{(i)}(x; f) \coloneqq \mathscr{Z}^{\vartheta}_{s_i,T}(g^{(x)}_{\delta_i}, f).$$

We first establish the joint convergence of the partition functions before taking logarithm:

$$\big(A^{(i)}_N(x_i; f_N)\big)_{i=1}^{k} \xrightarrow[N\to\infty]{d} \big(A^{(i)}(x_i; \mathrm{e}^{h_0})\big)_{i=1}^{k}. \tag{5.9}$$

For compactly supported initial test functions and bounded continuous terminal functions, the joint convergence of such averaged partition functions follows from [CSZ23, Theorem 9.1], recalled in Theorem A. To allow Gaussian initial test functions such as $g^{(x)}_{\delta_i}$, we use Step 1 of Section 4.2, in particular (4.8)–(4.9). The same estimates hold with an additional factor $\mathrm{e}^{\|h_0\|_\infty}$.

We next prove the a.s. strict positivity of $A^{(i)}(x_i; \mathrm{e}^{h_0})$ and control their logarithms. For terminal condition $\mathbf{1}$, the moment bounds in (4.5) give

$$\mathbb{E}[A^{(i)}_N(x; \mathbf{1})] = 1, \qquad \mathbb{E}[A^{(i)}_N(x; \mathbf{1})^{-1}] \leqslant \mathbb{E}[A^{(i)}_N(x; \mathbf{1})^2] \leqslant C_{\delta_i, T-s_i}. \tag{5.10}$$

These bounds are uniform in x and sufficiently large N. The constant can be chosen uniformly when δ_i and $T - s_i$ range over compact subsets of $(0, \infty)$, and ϑ ranges over a compact subset of $\mathbb{R}$. Since

$$\mathrm{e}^{-\|h_0\|_\infty} A^{(i)}_N(x; \mathbf{1}) \leqslant A^{(i)}_N(x; f_N) \leqslant \mathrm{e}^{\|h_0\|_\infty} A^{(i)}_N(x; \mathbf{1}),$$

the corresponding negative-first moments are uniformly bounded as well. Using the bound (5.10) and applying weak convergence and Fatou's Lemma, we obtain

$$\mathbb{E}[A^{(i)}(x; \mathrm{e}^{h_0})^{-1}] \leqslant \mathrm{e}^{\|h_0\|_\infty} C_{\delta_i, T-s_i} < \infty.$$

This shows that $A^{(i)}(x; \mathrm{e}^{h_0}) > 0$ almost surely and its logarithm is well-defined. This allows us to pass the weak convergence of the partition functions in (5.9) to the weak convergence of their logarithms in (5.6).

To establish the convergence of the moments of the logarithms, Markov's inequality and (5.10) give

$$\mathbb{P}\big(|\log A^{(i)}_N(x; \mathbf{1})| > u\big) \leqslant (1 + C_{\delta_i, T-s_i})\mathrm{e}^{-u}, \qquad u \geqslant 0. \tag{5.11}$$

Replacing $\mathbf{1}$ by f_N only enlarges the tail bound by a factor of $\mathrm{e}^{\|h_0\|_\infty}$, and the same bounds hold for the limiting SHF integrals. They imply uniform integrability of every positive power of the absolute logarithm, and hence convergence of all absolute moments of positive order.

To prove (5.7), apply Steps 2 and 3 of Section 4.2 jointly to $\varphi_1, \dots, \varphi_k$. The approximation by finite spatial sums is unchanged, since (4.8)–(4.9) hold with an additional factor $\mathrm{e}^{\|h_0\|_\infty}$. The logarithmic tail bound (5.11) removes the truncation uniformly in N. □

5.2. Extension of Proposition 2.3. We now extend the mean square approximation in Proposition 2.3 to the case of a general (non-flat) terminal weight. Recall the definitions (5.4) and (5.5) of the fields $\mathcal{H}^{(N)}_{t,T}$ and $\mathcal{H}^{(N;\delta)}_{t,T}$.

Lemma 5.3. *Fix $h_0 \in L^\infty(\mathbb{R}^2) \cap C(\mathbb{R}^2)$ and let $0 \leqslant t < T$. For $\varphi \in C^\infty_c(\mathbb{R}^2)$ and all sufficiently small $\delta > 0$,*

$$\limsup_{N\to\infty} \big\| \mathcal{H}^{(N)}_{t,T}(\varphi, h_0) - \mathcal{H}^{(N;\delta)}_{t,T}(\varphi, h_0) \big\|_{L^2} \leqslant C_{T-t,\vartheta,\varphi,\|h_0\|_\infty} |\log \delta|^{-1/40} .$$

The bound is locally uniform in $(T-t, \vartheta) \in (0,\infty) \times \mathbb{R}$.

Proof. By time translation and diffusive rescaling, we may reduce to the case $t = 0, T = 1$, replacing δ by $\delta/(T-t)$ and ϑ by $\vartheta + \log(T-t)$. We thus drop the dependence on t, T in the following.

Let $m = [\delta N]$ and $n = [\sqrt{\delta} N]$, and note that $\mathrm{e}^{-\|h_0\|_\infty} \leqslant f_N \leqslant \mathrm{e}^{\|h_0\|_\infty}$. The decomposition at time n in (2.21)–(2.22) gives

$$(\mathcal{H}^{(N)} - \mathcal{H}^{(N;\delta)})(\varphi, h_0) = \mathcal{L}^{(N;\delta)}(\varphi) + \sum_z \varphi^{(N)}(z) \big\{ G_{m,n,N}(z, f_N) - \mathbb{E} G_{m,n,N}(0, \mathbf{1}) \big\}, \quad (5.12)$$

where $\mathcal{L}^{(N;\delta)}$ is defined in (2.28) and $G_{m,n,N}(z, f_N)$ in (4.38). The local term $L_{m,n}$ in (2.25) and its centring do not depend on f_N, so the local error $\mathcal{L}^{(N;\delta)}$ is unchanged and Proposition 2.4 still applies. For the global term in (5.12), we can apply Remark 4.7-(4.39) with $f = f_N$, noting that $\| \log f_N \|_\infty \leqslant \|h_0\|_\infty$. Triangle inequality and $\sum_z |\varphi^{(N)}(z)| \leqslant \|\varphi\|_1$ give the required bound, while the centring constant $\mathbb{E} G_{m,n,N}(0, \mathbf{1})$ is bounded by (4.36). This concludes the proof of the lemma. □

5.3. Proof of Theorem 5.1. The same argument as in (2.17), together with (5.8) applied jointly to two regularisation parameters δ, δ', and Lemma 5.3, shows that $\mathscr{H}^{(\delta)}_{t_i,T}(\varphi_i, h_0)$ is Cauchy in L^2 as $\delta \downarrow 0$, for each i. Denote its limit by $\mathscr{H}_{t_i,T}(\varphi_i, h_0)$. Letting $\delta' \downarrow 0$ in the same estimate gives

$$\| \mathscr{H}^{(\delta)}_{t_i,T}(\varphi_i, h_0) - \mathscr{H}_{t_i,T}(\varphi_i, h_0) \|_{L^2} \leqslant C_{T-t_i,\vartheta,\varphi_i,\|h_0\|_\infty} \left| \log \frac{\delta}{T - t_i} \right|^{-1/40}, \quad (5.13)$$

where the rescaling in Lemma 5.3 replaces δ by $\delta/(T-t_i)$. The bound holds for sufficiently small $\delta/(T-t_i)$, locally uniformly in $(T-t_i, \vartheta) \in (0,\infty) \times \mathbb{R}$.

This shows the convergence in L^2 and we now show that the convergence also holds in L^p for any $p \in [1,\infty)$. The case $p \in [1,2]$ follows from the L^2 convergence so we focus on the case $p > 2$ and some fixed i. Let us set for simplicity $X_\delta = \mathscr{H}^{(\delta)}_{t_i,T}(\varphi_i, h_0)$ and set $\delta_j = (T - t_i) \exp(-2^j)$ for $j \geqslant 1$. By (5.13) we obtain that for any $j \geqslant 1$

$$\sup_{\delta_{j+1} \leqslant \delta \leqslant \delta_j} \|X_\delta - X_{\delta_j}\|_{L^2} \leqslant C 2^{-j/40},$$

On the other hand, by Remark 4.6, after time translation and rescaling we obtain

$$\sup_{\delta_{j+1} \leqslant \delta \leqslant \delta_j} \|X_\delta - X_{\delta_j}\|_{L^{2p}} \leqslant C_p (j+1) ,$$

where the bound (4.30) extends to any initial condition $h_0 \in L^\infty(\mathbb{R}^2)$ because changing the terminal weight from $\mathbf{1}$ to e^{h_0} changes X_δ by at most $\|\varphi_i\|_1 \|h_0\|_\infty$. Therefore, by Hölder's

inequality we get that

$$\begin{aligned}\sup_{\delta_{j+1}\leqslant\delta\leqslant\delta_j}\|X_\delta - X_{\delta_j}\|_{L^p} &\leqslant \sup_{\delta_{j+1}\leqslant\delta\leqslant\delta_j}\|X_\delta - X_{\delta_j}\|_{L^2}^{1/(p-1)}\|X_\delta - X_{\delta_j}\|_{L^{2p}}^{(p-2)/(p-1)}\\ &\leqslant C_p\, 2^{-j/[40(p-1)]}(j+1)^{(p-2)/(p-1)}.\end{aligned} \tag{5.14}$$

This shows in particular that $\|X_{\delta_{j+1}} - X_{\delta_j}\|_{L^p}$ is summable in j, so that X_{δ_j} converges in L^p to its already identified L^2 limit. Applying again (5.14) shows the L^p convergence for arbitrary $\delta \downarrow 0$.

For the lattice fields $\mathcal{H}^{\vartheta,(N)}_{t_i,T}(\varphi_i, h_0)$, we can apply the same bounded Lipschitz distance argument as in the proof of Theorem 2.1, but now on $\mathbb{R}^k$. Letting $N \to \infty$ and then $\delta \downarrow 0$, we obtain

$$\big(\mathcal{H}^{(N)}_{t_i,T}(\varphi_i, h_0)\big)_{i=1}^k \xrightarrow[N\to\infty]{d} \big(\mathscr{H}_{t_i,T}(\varphi_i, h_0)\big)_{i=1}^k. \tag{5.15}$$

The statement on time reversal is a direct consequence of (5.2). □

6. Convergence in local Sobolev spaces

Theorem 5.1 gives joint convergence of the fields $\mathcal{H}^{\vartheta,(N)}$ and $\mathscr{H}^{\vartheta,(\delta)}$ against test functions. We now establish the convergence assertions of Theorem 1.2 for one fixed initial condition $h_0 \in L^\infty(\mathbb{R}^2) \cap C(\mathbb{R}^2)$, jointly at finitely many times and in the local Sobolev space $H^{-a}_{\mathrm{loc}}(\mathbb{R}^2)$ for every $a > 0$.

Let us first recall the norm on the negative Sobolev space H^{-s}:

$$\|f\|^2_{H^{-s}} = (2\pi)^{-2}\int_{\mathbb{R}^2}(1+|\xi|^2)^{-s}|\widehat{f}(\xi)|^2\,\mathrm{d}\xi, \quad \text{with} \quad \widehat{f}(\xi) = \int_{\mathbb{R}^2} \mathrm{e}^{-i\xi\cdot x} f(x)\,\mathrm{d}x. \tag{6.1}$$

The local Sobolev space $H^{-s}_{\mathrm{loc}}(\mathbb{R}^2)$ is defined by requiring $\chi f \in H^{-s}$ for every smooth compactly supported function $\chi \in C^\infty_c(\mathbb{R}^2)$. In what follows, *localising a field* means multiplying it by such a smooth compactly supported function χ. We also denote $\|\mathcal{H}\|^2_{L^2(\mathbb{P};H^{-s})} := \mathbb{E}[\|\mathcal{H}\|^2_{H^{-s}}]$.

Recall the forward fields $\mathcal{H}^{\vartheta,(N)}_t(h_0,\cdot)$ and $\mathscr{H}^{\vartheta,(\delta)}_t(h_0,\cdot)$ from (1.14) and (1.18). The following proposition gives the convergence in Theorem 1.2 for this fixed initial condition.

Proposition 6.1 (Convergence in finite-dimensional distribution)**.** *Fix $\vartheta \in \mathbb{R}$, $h_0 \in L^\infty(\mathbb{R}^2) \cap C(\mathbb{R}^2)$, and the critical tuning* (1.6)*. There exists an SHF-measurable process $(\mathscr{H}^\vartheta_t(h_0,\cdot))_{t>0}$ with values in $\bigcap_{a>0} H^{-a}_{\mathrm{loc}}(\mathbb{R}^2)$ such that for every $0 < t_1 < \cdots < t_k$ and $a > 0$, the following convergences hold in $(H^{-a}_{\mathrm{loc}}(\mathbb{R}^2))^k$:*

$$\begin{aligned}&(\mathcal{H}^{\vartheta,(N)}_{t_i}(h_0,\cdot))_{i=1}^k \xrightarrow[N\to\infty]{d} (\mathscr{H}^\vartheta_{t_i}(h_0,\cdot))_{i=1}^k,\\ &(\mathscr{H}^{\vartheta,(\delta)}_{t_i}(h_0,\cdot))_{i=1}^k \xrightarrow[\delta\downarrow 0]{\mathbb{P}} (\mathscr{H}^\vartheta_{t_i}(h_0,\cdot))_{i=1}^k.\end{aligned} \tag{6.2}$$

Moreover, for every $\chi \in C^\infty_c(\mathbb{R}^2)$ and $s > 0$,

$$\lim_{\delta\downarrow 0}\sum_{i=1}^k \mathbb{E}\big\|\chi\big(\mathscr{H}^{\vartheta,(\delta)}_{t_i}(h_0,\cdot) - \mathscr{H}^\vartheta_{t_i}(h_0,\cdot)\big)\big\|^2_{H^{-s}} = 0. \tag{6.3}$$

The conclusion for the lattice field also holds for deterministic $\vartheta_N \to \vartheta$ in (1.6)*.*

Instead of removing disorder on the final interval $[t-\delta, t]$, as in the definition (1.18) of $\mathscr{H}_t^{\vartheta,(\delta)}(h_0,\cdot)$, we may also spatially smooth the SHF solution at time t. For the same initial condition h_0, and a test function $\varphi \in C_c^\infty(\mathbb{R}^2)$, define

$$\widetilde{\mathscr{H}}_t^{\vartheta,(\delta)}(h_0,\varphi) := \int_{\mathbb{R}^2} \varphi(x)\Big\{\log \mathscr{Z}_{0,t}^{\vartheta}(\mathrm{e}^{h_0}, g_\delta^{(x)}) - \mathbb{E}\log \mathscr{Z}_{0,t}^{\vartheta}(\mathbf{1}, g_\delta^{(0)})\Big\}\,\mathrm{d}x. \tag{6.4}$$

The following corollary shows that the limit is the same as in Proposition 6.1.

Corollary 6.2. *Fix $\vartheta \in \mathbb{R}$ and $h_0 \in L^\infty(\mathbb{R}^2) \cap C(\mathbb{R}^2)$. For every $0 < t_1 < \cdots < t_k$ and $a > 0$,*

$$(\widetilde{\mathscr{H}}_{t_i}^{\vartheta,(\delta)}(h_0,\cdot))_{i=1}^k \xrightarrow[\delta\downarrow 0]{\mathbb{P}} (\mathscr{H}_{t_i}^{\vartheta}(h_0,\cdot))_{i=1}^k \quad \text{in } (H^{-a}_{\mathrm{loc}}(\mathbb{R}^2))^k. \tag{6.5}$$

For every $\chi \in C_c^\infty(\mathbb{R}^2)$ and $s > 0$,

$$\lim_{\delta\downarrow 0} \sum_{i=1}^k \mathbb{E}\big\|\chi\big(\widetilde{\mathscr{H}}_{t_i}^{\vartheta,(\delta)}(h_0,\cdot) - \mathscr{H}_{t_i}^{\vartheta}(h_0,\cdot)\big)\big\|_{H^{-s}}^2 = 0. \tag{6.6}$$

Following Section 5, we prove these results after performing the time reversal (5.2), so that the initial condition e^{h_0} becomes a terminal weight at a fixed terminal time $T > t_k$. For Corollary 6.2, we use the spatial regularisation $\widetilde{\mathscr{H}}_{t,T}^{\vartheta,(\delta)}(x,h_0)$ obtained by replacing $t+\delta$ by t in (5.1):

$$\widetilde{\mathscr{H}}_{t,T}^{\vartheta,(\delta)}(x,h_0) := \log \mathscr{Z}_{t,T}^{\vartheta}\big(g_\delta^{(x)}, e^{h_0}\big) - \mathbb{E}\Big[\log \mathscr{Z}_{t,T}^{\vartheta}\big(g_\delta^{(0)},1\big)\Big].$$

Integrating in x against a test function φ, we write $\mathscr{H}_{t,T}^{\vartheta,(\delta)}(\varphi,h_0)$, and similarly for $\widetilde{\mathscr{H}}$.

The rest of the section is organised as follows: In Section 6.1, we establish tightness in H^{-a}_{loc} for every $a > 0$ (Lemma 6.4), using the spatial covariance bounds for flat data (Lemma 6.3) and comparison with the flat field. Lastly, Section 6.2 identifies the subsequential limits through their integrals against test functions, and proves Proposition 6.1 as well as Corollary 6.2.

6.1. Covariance bounds and tightness. To obtain tightness in H^{-a}_{loc} for every $a > 0$, we first prove spatial covariance bounds for the KPZ fields with $h_0 = 0$. These give uniform Sobolev moment bounds, which can then be extended to more general h_0 by comparison with the flat fields. Tightness then follows by compact embedding.

Fix $T > 0$ and $\vartheta \in \mathbb{R}$. We use the time-reversed fields $\mathscr{H}_{0,T}^{\vartheta,(\delta)}(\cdot,h_0)$ defined in (5.1), and $\widetilde{\mathscr{H}}_{0,T}^{\vartheta,(\delta)}(\cdot,h_0)$ introduced after Corollary 6.2. For the lattice field, we use (5.4) with starting time zero: for $x \in \mathtt{C}_N(z)$,

$$\mathcal{H}^{(N)}(x,h_0) := \log Z_{0,[NT]}(z,f_N) - \mathbb{E}\log Z_{0,[NT]}(0,\mathbf{1}), \tag{6.7}$$

where f_N is defined in (5.3). As before, we suppress the terminal argument when $h_0 = 0$.

Lemma 6.3 (Covariance bounds)**.** *The covariance kernel of the flat field $\widetilde{\mathscr{H}}_{0,T}^{\vartheta,(\delta)}$ with $h_0 = 0$ satisfies the following bounds: for $0 < \delta, \delta' < T/2$ and $x \neq y$,*

$$0 \leqslant \mathrm{Cov}\big(\widetilde{\mathscr{H}}_{0,T}^{\vartheta,(\delta)}(x), \widetilde{\mathscr{H}}_{0,T}^{\vartheta,(\delta')}(y)\big) \leqslant C\Big[1 + \log\Big(1 + \log^+ \frac{1}{|x-y|}\Big)\Big]. \tag{6.8}$$

The same bounds hold for the covariance kernels of $\mathscr{H}_{0,T}^{\vartheta,(\delta)}$, and $\mathcal{H}^{(N)}$ for all sufficiently large N. The constant C is uniform when T and ϑ range over compact subsets of $(0,\infty)$ and $\mathbb{R}$, respectively.

Proof of Lemma 6.3. We prove (6.8) for the spatially and temporally regularised log-SHF fields $\widetilde{\mathscr{H}}^{\vartheta,(\delta)}_{0,T}$ and $\mathscr{H}^{\vartheta,(\delta)}_{0,T}$ via lattice approximations.

Let us fix even integers $0 \leqslant m, \ell < n$, with n/N in a fixed compact subset of $(0,\infty)$, and let us recall that we denote $Z_n(q_m^{(z)}) = Z_{0,n}(q_m^{(z)}, \mathbf{1})$. By time-stationarity, $Z_n(q_m^{(z)})$ has the same law as $Z_{m,m+n}(q_m^{(z)}, \mathbf{1})$, so Proposition 4.1, specifically (4.18), gives that $\Delta(\log Z_n(q_m^{(z)})) \leqslant C$ uniformly in m. Therefore, Lemma 3.4 (in particular (3.9)) gives similarly to (3.12) that

$$0 \leqslant \mathrm{Cov}(\log Z_n(q_m^{(z)}), \log Z_n(q_\ell^{(w)})) \leqslant C + \log \mathbb{E}[Z_n(q_m^{(z)}) Z_n(q_\ell^{(w)})]. \tag{6.9}$$

Applying the first collision decomposition (4.2) with $s = 0$ and averaging the initial positions against $q_m^{(z)}$ and $q_\ell^{(w)}$, we obtain

$$\mathbb{E}[Z_n(q_m^{(z)}) Z_n(q_\ell^{(w)})] = 1 + \sum_{k=1}^{n} q_{m+\ell+2k}(w-z)\sigma_N^2 \mathbb{E}[Z_{k,n}(0)^2]\,. \tag{6.10}$$

By (4.1), $\sigma_N^2 \mathbb{E}[Z_{k,n}(0)^2] \leqslant \sigma_N^2 \mathbb{E}[Z_n(0)^2] \leqslant C$ uniformly in $1 \leqslant k \leqslant n$. For time horizons larger than N, the same bound follows from the last assertion of Claim B.1, with the constant depending on an upper bound for n/N. Together with the Gaussian heat-kernel upper bound (see [LL10, Propositions 2.1.2 and 2.4.6], applied to the aperiodic two-step walk on $\mathbb{Z}^2_{\mathrm{even}}$) this yields

$$\begin{aligned} \mathbb{E}[Z_n(q_m^{(z)}) Z_n(q_\ell^{(w)})] &\leqslant 1 + C \sum_{k=1}^{n} \frac{1}{m+\ell+2k} \exp\left\{ -\frac{c|w-z|^2}{m+\ell+2k} \right\} \\ &\leqslant C \left\{ 1 + \log^+ \frac{n}{1+|z-w|^2} \right\}. \end{aligned} \tag{6.11}$$

To see the last inequality, the terms with $m + \ell + 2k \leqslant |z-w|^2$ have a bounded sum by comparison with the corresponding Gaussian integral, while the remaining terms form a harmonic sum with upper endpoint $m + \ell + 2n < 4n$.

We now pass from these lattice estimates to the spatially regularised SHF field. Fix $0 < \delta, \delta' < T/2$ and $x \neq y$, and take

$$n = [NT], \qquad m = [N\delta], \qquad \ell = [N\delta'], \qquad z = [\sqrt{N/2}\, x], \qquad w = [\sqrt{N/2}\, y].$$

Lemma 5.2 gives joint convergence of the logarithms, as well as convergence of their mean and, by uniform integrability, their mixed second moments. Hence

$$\mathrm{Cov}\big(\log Z_n(q_m^{(z)}), \log Z_n(q_\ell^{(w)})\big) \longrightarrow \mathrm{Cov}\big(\log \mathscr{Z}^{\vartheta}_{0,T}(g_\delta^{(x)}, \mathbf{1}), \log \mathscr{Z}^{\vartheta}_{0,T}(g_{\delta'}^{(y)}, \mathbf{1})\big).$$

Since $n/(1+|z-w|^2) \to 2T/|x-y|^2$, (6.9) and (6.11) give (6.8) for $\widetilde{\mathscr{H}}^{\vartheta,(\delta)}_{0,T}$.

For the temporal regularisation $\mathscr{H}^{\vartheta,(\delta)}_{0,T}$, use instead $\widehat{Z}_{m,n}(z) = Z_{m,n}(q_m^{(z)}, \mathbf{1})$. Lemma 3.4 and Proposition 4.1 give

$$0 \leqslant \mathrm{Cov}\big(\log \widehat{Z}_{m,n}(z), \log \widehat{Z}_{\ell,n}(w)\big) \leqslant C + \log \mathbb{E}[\widehat{Z}_{m,n}(z) \widehat{Z}_{\ell,n}(w)].$$

Suppose $m \leqslant \ell < n/2$. Averaging over the disorder on $(m, \ell]$ replaces $q_m^{(z)}$ by $q_\ell^{(z)}$. Time-stationarity then gives

$$\mathbb{E}[\widehat{Z}_{m,n}(z) \widehat{Z}_{\ell,n}(w)] = \mathbb{E}[Z_{n-\ell}(q_\ell^{(z)}) Z_{n-\ell}(q_\ell^{(w)})],$$

so (6.11), with duration $n - \ell$ and both smoothing times equal to ℓ, bounds this mixed moment. With the same scaling as above, Lemma 5.2 also gives

$$\mathrm{Cov}\big(\log \widehat{Z}_{m,n}(z), \log \widehat{Z}_{\ell,n}(w)\big) \longrightarrow \mathrm{Cov}\big(\log \mathscr{Z}^{\vartheta}_{\delta,T}(g_\delta^{(x)}, \mathbf{1}), \log \mathscr{Z}^{\vartheta}_{\delta',T}(g_{\delta'}^{(y)}, \mathbf{1})\big).$$

This proves (6.8) for $\mathscr{H}_{0,T}^{\vartheta,(\delta)}$ as well.

For the lattice field $\mathcal{H}^{(N)}$, apply (6.9) and (6.11) with $m = \ell = 0$ and $n = [NT]$. If $x \in \mathtt{C}_N(z)$ and $y \in \mathtt{C}_N(w)$, the cell definition (1.7) gives

$$|x-y|^2 \leqslant C\frac{1+|z-w|^2}{N}, \qquad \frac{n}{1+|z-w|^2} \leqslant \frac{Cn}{N|x-y|^2} \quad (x \neq y).$$

Since $n/N \to T$, the discrete covariance bound therefore implies (6.8) for $\mathcal{H}^{(N)}$, including when x and y belong to the same cell. □

Lemma 6.4 (Sobolev bounds and tightness)**.** *Fix $h_0 \in L^\infty(\mathbb{R}^2) \cap C(\mathbb{R}^2)$. For every $a > 0$ and $\chi \in C_c^\infty(\mathbb{R}^2)$,*

$$\max\left\{\begin{array}{l} \sup\limits_{0<\delta<T/2} \mathbb{E}\|\chi\widetilde{\mathscr{H}}_{0,T}^{\vartheta,(\delta)}(\cdot,h_0)\|_{H^{-a}}^2, \\ \sup\limits_{0<\delta<T/2} \mathbb{E}\|\chi\mathscr{H}_{0,T}^{\vartheta,(\delta)}(\cdot,h_0)\|_{H^{-a}}^2, \\ \limsup\limits_{N\to\infty} \mathbb{E}\|\chi\mathcal{H}^{(N)}(\cdot,h_0)\|_{H^{-a}}^2 \end{array}\right\} \leqslant C_{\chi,a,\|h_0\|_\infty}. \tag{6.12}$$

The constant is locally uniform in $(T,\vartheta) \in (0,\infty)\times\mathbb{R}$. All three families are tight in $H^{-a}_{\mathrm{loc}}(\mathbb{R}^2)$.

Proof of Lemma 6.4. We first prove the moment bounds for $h_0 = 0$. We claim that the second moment for the H^{-a} norm of $\chi\mathscr{H}_{0,T}^{\vartheta,(\delta)}$ has the representation

$$\begin{aligned} &\mathbb{E}\Big[\|\chi\mathscr{H}_{0,T}^{\vartheta,(\delta)}\|_{H^{-a}}^2\Big] \\ =&\frac{1}{\Gamma(a)}\int_0^\infty u^{a-1}\mathrm{e}^{-u}\iint g_{2u}(x-y)\chi(x)\chi(y)\,\mathrm{Cov}\big(\mathscr{H}_{0,T}^{\vartheta,(\delta)}(x),\mathscr{H}_{0,T}^{\vartheta,(\delta)}(y)\big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}u\,. \end{aligned} \tag{6.13}$$

Indeed, for $a > 0$, the change of variables $v = (1+|\xi|^2)u$ in $\Gamma(a) = \int_0^\infty v^{a-1}\mathrm{e}^{-v}\,\mathrm{d}v$ gives

$$(1+|\xi|^2)^{-a} = \frac{1}{\Gamma(a)}\int_0^\infty u^{a-1}\mathrm{e}^{-u}\mathrm{e}^{-u|\xi|^2}\,\mathrm{d}u\,. \tag{6.14}$$

Substituting this integral representation into the Sobolev norm for $\|\chi\mathscr{H}_{0,T}^{\vartheta,(\delta)}\|_{H^{-a}}$ (recall (6.1)) gives

$$\begin{aligned} \mathbb{E}\Big[\|\chi\mathscr{H}_{0,T}^{\vartheta,(\delta)}\|_{H^{-a}}^2\Big] &= \frac{1}{(2\pi)^2\Gamma(a)}\int_0^\infty u^{a-1}\mathrm{e}^{-u}\int_{\mathbb{R}^2}\mathrm{e}^{-u|\xi|^2}\mathbb{E}\Big[\big|\widehat{\chi\mathscr{H}}_{0,T}^{\vartheta,(\delta)}(\xi)\big|^2\Big]\,\mathrm{d}\xi\,\mathrm{d}u \\ &= \frac{1}{\Gamma(a)}\int_0^\infty u^{a-1}\mathrm{e}^{-u}\iint \chi(x)\chi(y)\mathbb{E}\big[\mathscr{H}_{0,T}^{\vartheta,(\delta)}(x)\mathscr{H}_{0,T}^{\vartheta,(\delta)}(y)\big] \\ &\qquad\times\left(\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\mathrm{e}^{-u|\xi|^2+i\xi\cdot(x-y)}\,\mathrm{d}\xi\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}u\,, \end{aligned}$$

The first identity uses Tonelli's theorem. For the second, expand the Fourier transform and use Fubini for each fixed $u > 0$ and $\delta > 0$, then integrate in u. The finite logarithmic second moments from Lemma 5.2 justify the fixed-u interchange. Since $\mathrm{e}^{-u|\xi|^2} = \widehat{g}_{2u}(\xi)$ and the field $\mathscr{H}_{0,T}^{\vartheta,(\delta)}(\cdot)$ has mean zero, Fourier inversion identifies the last expression with (6.13).

From (6.13) and the covariance upper bound in (6.8), we obtain

$$\mathbb{E}\Big[\|\chi\mathscr{H}_{0,T}^{\vartheta,(\delta)}\|_{H^{-a}}^2\Big] \leqslant C_{\chi,a}\int_0^\infty u^{a-1}\mathrm{e}^{-u}\left(\int_{\mathbb{R}^2} g_{2u}(z)\Big[1+\log\Big(1+\log^+\frac{1}{|z|}\Big)\Big]\,\mathrm{d}z\right)\mathrm{d}u.$$

We claim that there is a constant $C > 0$ such that for any $u > 0$,

$$\int_{\mathbb{R}^2} g_{2u}(z)\Big[1 + \log\Big(1 + \log^+ \frac{1}{|z|}\Big)\Big]\,\mathrm{d}z \leqslant C[1 + \log(1 + \log^+(1/u))]\,.$$

Indeed, the integral is bounded by a universal constant for $u \geqslant 1$. For $u \in (0,1)$, we use the change of variable $z = \sqrt{u}\,v$ and use that $\log^+(1/(\sqrt{u}|v|)) \leqslant \frac{1}{2}\log(1/u) + \log^+(1/|v|)$, giving the desired bound. Combining these estimates gives

$$\mathbb{E}\Big[\|\chi \mathscr{H}^{\vartheta,(\delta)}_{0,T}\|^2_{H^{-a}}\Big] \leqslant C_{\chi,a}\left\{1 + \int_0^1 u^{a-1}[1 + \log(1 + \log(1/u))]\,\mathrm{d}u\right\} < \infty, \tag{6.15}$$

uniformly in δ. The same calculation applies to the lattice field $\mathcal{H}^{(N)}$ and the spatially mollified field $\widetilde{\mathscr{H}}^{\vartheta,(\delta)}_{0,T}$, since their covariance kernels satisfy the same bound by Lemma 6.3. This proves (6.12) for $h_0 = 0$.

For general h_0, compare the terminal weights using $\mathrm{e}^{-\|h_0\|_\infty} \leqslant \mathrm{e}^{h_0}, f_N \leqslant \mathrm{e}^{\|h_0\|_\infty}$ (see definition (5.3)). Since the centring is defined using the flat terminal condition,

$$\big|\mathscr{H}^{\vartheta,(\delta)}_{0,T}(x, h_0) - \mathscr{H}^{\vartheta,(\delta)}_{0,T}(x)\big| \leqslant \|h_0\|_\infty,$$

and the identical bound holds for $\widetilde{\mathscr{H}}^{\vartheta,(\delta)}_{0,T}$ and $\mathcal{H}^{(N)}$. Using $\|\cdot\|_{H^{-a}} \leqslant \|\cdot\|_2$, we obtain

$$\big\|\chi\big(\mathscr{H}^{\vartheta,(\delta)}_{0,T}(\cdot, h_0) - \mathscr{H}^{\vartheta,(\delta)}_{0,T}\big)\big\|_{H^{-a}} \leqslant \|\chi\|_2 \|h_0\|_\infty,$$

and similarly for the other two fields. Combining this with the flat moment bounds (6.15) proves (6.12).

For tightness, apply (6.12) with $a/2$ in place of a. Markov's inequality gives

$$\sup_{0<\delta<T/2} \mathbb{P}\left(\|\chi \mathscr{H}^{\vartheta,(\delta)}_{0,T}(\cdot, h_0)\|_{H^{-a/2}} > R\right) \leqslant C_{\chi,a,\|h_0\|_\infty} R^{-2}.$$

Distributions supported on $\mathrm{supp}(\chi)$ with $H^{-a/2}$ norm at most R form a relatively compact subset of H^{-a}. Thus the localised fields are tight in H^{-a}. Taking smooth cutoffs equal to one on successively larger balls, and choosing the radii so that the corresponding probability bounds are summable, gives tightness in H^{-a}_{loc}. The same argument applies to the spatially regularised and lattice fields. □

6.2. Proof of Proposition 6.1 and Corollary 6.2. We combine Theorem 5.1 with the tightness in Lemma 6.4. Throughout the proof, fix $h_0 \in L^\infty(\mathbb{R}^2) \cap C(\mathbb{R}^2)$ and a common terminal time $T > 0$, and suppress ϑ from the notation. All time-reversed SHF fields below are constructed from the same SHF with terminal weight e^{h_0} at time T.

Proof of Proposition 6.1. Fix $0 \leqslant u < T$. Theorem 5.1 gives the scalar limits $\mathscr{H}_{u,T}(\varphi, h_0)$, the L^2 approximation (5.13), and the joint lattice convergence (5.15).

We next use the Sobolev bounds of Lemma 6.4 to show that these scalar limits define a random distribution on the same SHF space. Fix $\chi \in C^\infty_c(\mathbb{R}^2)$ and $a > 0$, and set $Y_{\delta,\delta'} = \chi(\mathscr{H}^{(\delta)}_{u,T}(\cdot, h_0) - \mathscr{H}^{(\delta')}_{u,T}(\cdot, h_0))$. For $R > 0$, splitting the Fourier integral and using (6.12) with $a/2$ gives

$$\mathbb{E}\|Y_{\delta,\delta'}\|^2_{H^{-a}} \leqslant \frac{1}{(2\pi)^2}\int_{|\xi|\leqslant R} \mathbb{E}|\widehat{Y}_{\delta,\delta'}(\xi)|^2\,\mathrm{d}\xi + C_{\chi,a,\|h_0\|_\infty}(1 + R^2)^{-a/2}. \tag{6.16}$$

For each ξ, the Fourier transform tends to zero in L^2 by applying (5.13) to the real and imaginary parts of $\chi(x)\mathrm{e}^{-i\xi\cdot x}$. Since $Y_{\delta,\delta'}$ is supported on $\mathrm{supp}\,\chi$, Sobolev duality and (6.12) bound the integrand uniformly on $\{|\xi| \leqslant R\}$, so dominated convergence applies. First letting

$\delta, \delta' \downarrow 0$ and then $R \to \infty$ shows that $\chi \mathscr{H}^{(\delta)}_{u,T}(\cdot, h_0)$ is Cauchy in $L^2(\mathbb{P}; H^{-a})$. The local limits therefore define an SHF-measurable random distribution $\mathscr{H}_{u,T}(\cdot, h_0)$, with the integrals against test functions identified above. This gives convergence in probability in H^{-a}_{loc} and the mean-square convergence in (6.3), jointly at finitely many starting times.

For the lattice fields $\left(\mathcal{H}^{\theta,(N)}_{t_i}(\cdot, h_0)\right)_{i=1}^{k}$, Lemma 6.4 gives tightness in $(H^{-a}_{\mathrm{loc}})^k$. By (5.15), every subsequential weak limit has the same law as the random distributions $(\mathscr{H}_{u_i,T}(\cdot, h_0))_{i=1}^{k}$, which implies weak convergence in $(H^{-a}_{\mathrm{loc}})^k$.

Finally, take $T > t_k$ and $u_i = T - t_i$. Time reversal (5.2), applied jointly to two regularisation parameters, gives the asserted convergence for the forward SHF fields. The lattice convergence follows by the corresponding time reversal.

If $\vartheta_N \to \vartheta$, Lemma 5.2 has the same limiting SHF, while the approximation and tightness bounds are uniform for ϑ_N in a compact interval. The preceding argument therefore proves the last assertion of the proposition as well. □

Remark 6.5. *The argument above also gives joint convergence for finitely many initial conditions $h_0^{(j)}$. The joint SHF convergence in Theorem A allows the proof of Lemma 5.2 to use the terminal weights $\mathrm{e}^{h_0^{(j)}}$ simultaneously, while the approximation estimates apply to each initial condition.*

Proof of Corollary 6.2. Keep the common terminal time T and terminal weight e^{h_0}, and fix $0 < u < T$. For $0 < \delta < \min\{u, T-u\}$, we have the exact identity

$$\widetilde{\mathscr{H}}^{(\delta)}_{u,T}(\cdot, h_0) = \mathscr{H}^{(\delta)}_{u-\delta,T}(\cdot, h_0). \tag{6.17}$$

For $\varphi \in C_c^\infty(\mathbb{R}^2)$, write

$$\begin{aligned} &\widetilde{\mathscr{H}}^{(\delta)}_{u,T}(\varphi, h_0) - \mathscr{H}_{u,T}(\varphi, h_0) \\ &= \big(\mathscr{H}^{(\delta)}_{u-\delta,T}(\varphi, h_0) - \mathscr{H}_{u-\delta,T}(\varphi, h_0)\big) + \big(\mathscr{H}_{u-\delta,T}(\varphi, h_0) - \mathscr{H}_{u,T}(\varphi, h_0)\big). \end{aligned} \tag{6.18}$$

The first term tends to zero in $L^2(\mathbb{P})$ by (5.13), uniformly when u varies in a compact subset of $(0, T)$. For the second, it suffices to show that $u \mapsto \mathscr{H}_{u,T}(\varphi, h_0)$ is continuous in $L^2(\mathbb{P})$.

By [Tsa24, Definition 1.2(a) and Proposition 1.5], the SHF is continuous in its time indices in the vague topology. Truncating the Gaussian starting weight and the bounded continuous terminal weight e^{h_0}, with the heat-kernel first-moment bound as in Step 1 of Section 4.2, gives continuity in probability of $\mathscr{Z}_{u+\varepsilon,T}(g_\varepsilon^{(x)}, \mathrm{e}^{h_0})$. The positivity and locally uniform logarithmic moment bounds in Lemma 5.2 (and its proof) then give continuity of its logarithm in $L^2(\mathbb{P})$. Integrating against φ shows that $u \mapsto \mathscr{H}^{(\varepsilon)}_{u,T}(\varphi, h_0)$ is continuous in $L^2(\mathbb{P})$ on $[0, T-\varepsilon)$. By (5.13), these continuous maps converge locally uniformly in $L^2(\mathbb{P})$ as $\varepsilon \downarrow 0$. Their limit $\mathscr{H}_{u,T}(\varphi, h_0)$ is therefore continuous. Both terms in (6.18) consequently tend to zero in $L^2(\mathbb{P})$.

The Sobolev bounds in Lemma 6.4 and the Fourier argument in (6.16) now give $\chi \widetilde{\mathscr{H}}^{(\delta)}_{u,T}(\cdot, h_0) \to \chi \mathscr{H}_{u,T}(\cdot, h_0)$ in $L^2(\mathbb{P}; H^{-a})$ for every $\chi \in C_c^\infty(\mathbb{R}^2)$ and $a > 0$. This proves both assertions in Corollary 6.2, jointly at finitely many starting times. Finally, take $T > t_k$ and perform the time reversal $u_i = T - t_i$ to conclude the proof. □

7. Geometric block factorisation and moment estimates

This section supplies the three inputs used in the preceding convergence proofs: the one-point comparison in Proposition 4.1, the comparison of partition functions on $(n, N]$

in Proposition 4.4, and the logarithmic moments in Proposition 4.5. We adapt the block decomposition and interval expansion of Gu–Tsai [GT26, Section 2.3 and Proposition 2.1]: the idea is to replace the endpoint distribution on each geometric time interval by a heat kernel, which produces independent partition functions. One key idea of [GT26] is to provide an exact expansion for the error of this replacement.

We first describe the factorisation and then state the main estimates that are needed in this section.

Notation. For kernels, we use the contractions of [GT26, Section 2.2]. The left triangle weights and sums the starting point, whereas the right triangle weights and sums the terminal point:

$$(\nu \triangleleft Z_{a,b})(y) = \sum_x \nu(x) Z_{a,b}(x,y), \qquad (Z_{a,b} \triangleright f)(x) = \sum_y Z_{a,b}(x,y) f(y).$$

In the discrete setting, we can interpret $Z_{a,b}$ as a matrix. Then $\nu \triangleleft Z_{a,b}$ and $Z_{a,b} \triangleright f$ are simply the multiplication of $Z_{a,b}$ with the row vector ν on the left and the column vector f on the right, respectively. We note that $\nu \triangleleft Z_{a,b}$ is an endpoint measure and applying both triangles gives the scalar $\nu \triangleleft Z_{a,b} \triangleright f = Z_{a,b}(\nu, f)$ from (2.3). In particular, total terminal mass is obtained by closing with $\triangleright \mathbf{1}$; if this contraction is absent, the endpoint remains free.

Remark 7.1. *It is also worth stressing that this notation is the analogue of the bra-ket notation in quantum mechanics. In particular $Z \triangleright f \equiv Z_{a,b} \,|\, f\rangle$ and $\nu \triangleleft Z_{a,b} \equiv \langle \nu \,|\, Z_{a,b}$ This analogy is particularly useful to keep in mind when deriving the "block-expansions" in Section 7.2.*

The circle composes successive kernels by summing their common intermediate point: in other words, it corresponds to the matrix multiplication

$$(Z_{a,b} \circ Z_{b,c})(x,z) = \sum_y Z_{a,b}(x,y) Z_{b,c}(y,z) = Z_{a,c}(x,z).$$

The last equality is the Chapman-Kolmogorov equation, and we stress that the two factors use the disorder on the disjoint intervals $(a,b]$ and $(b,c]$.

7.1. The geometric block decomposition and factorisation. We divide $(0,N]$ into geometric time intervals. For $\rho > 1$, let $K = \lfloor \log_\rho N \rfloor + 1$ and define

$$N_0 = 0, \qquad N_i = \lfloor N\rho^{i-K} \rfloor \quad (1 \leqslant i < K), \qquad N_K = N\,. \tag{7.1}$$

The time intervals satisfy $N_1 \leqslant 2\rho$, $N_i \asymp N\rho^{i-K}$, and the successive ratios (except the first one) are comparable to ρ. We refer to the interval $(N_{i-1}, N_i]$ as the *i-th block*.

Remark 7.2. *We choose the geometric grid so that $m = \lfloor \delta N \rfloor$ and $n = \lfloor \sqrt{\delta} N \rfloor$ are grid points, say $m = N_a$ and $n = N_b$, with the same number of blocks between m and n as between n and N, i.e. $b - a = K - b$. More precisely, given $\rho > 1$ large enough (to be fixed later), we write $\delta = \rho^{-2J}$ for a suitable J, namely*

$$J := \frac{|\log \delta|}{2 \log \rho}\,. \tag{7.2}$$

To lighten notation, we will assume that J is an integer. For large N, the indices

$$a = K - 2J, \qquad b = K - J$$

then satisfy

$$N_a = \lfloor N\rho^{-2J} \rfloor = m, \qquad N_b = \lfloor N\rho^{-J} \rfloor = n.$$

Thus $(m,n]$ consists of exactly J blocks, as does $(n,N]$.

(For $\delta>0$ small, if J is not an integer, we can write $\delta=\tilde\rho^{-2\tilde J}$ for $\tilde J=\lfloor J\rfloor\geqslant 1$ and $\tilde\rho=\rho^{J/\tilde J}\in[\rho,\rho^2]$. All the identities still above hold with (ρ,J) replaced by $(\tilde\rho,\tilde J)$, where δ, m and n remain unchanged. The estimates below continue to hold too with these replacements, since $\tilde J$ differs from J by less than one and $\tilde\rho\in[\rho,\rho^2]$; see Remark 7.5.)

All block formulas are first written for a starting point at the origin. The version starting at x is obtained by translating every heat-kernel starting distribution by x. Let

$$\mathsf{Z}_i(x,y)\coloneqq Z_{N_{i-1},N_i}(x,y),\qquad \mathsf{Z}_i(q_{N_{i-1}})\coloneqq \mathsf{Z}_i(q_{N_{i-1}},\mathbf{1})\coloneqq q_{N_{i-1}}\triangleleft\mathsf{Z}_i\triangleright\mathbf{1} \tag{7.3}$$

be respectively the kernel in block i and its total mass under the reference heat-kernel starting distribution. These interval partition functions are independent, because the blocks contain disjoint disorder.

Let us now introduce the heat-kernel replacement, or "rewired" kernel,

$$\mathsf{Z}_i^{\mathrm{rewired}}(x,y)\coloneqq \mathsf{Z}_i\triangleright\mathbf{1}(x)\,q_{N_i}(y)=\mathsf{Z}_i(x,\mathbf{1})\,q_{N_i}(y) \tag{7.4}$$

which retains the total mass of Z_i but replaces its endpoint distribution by q_{N_i}. The superscript rewired refers to this replacement. We also consider the "discrepancy" kernel

$$D_i(x,y)\coloneqq \mathsf{Z}_i(x,y)-\mathsf{Z}_i^{\mathrm{rewired}}(x,y). \tag{7.5}$$

It is easy to see that for any $x\in\mathbb{Z}^2$, it holds that

$$D_i(x,\mathbf{1})=D_i\triangleright\mathbf{1}(x)=\sum_{z\in\mathbb{Z}^2}D_i(x,z)=0. \tag{7.6}$$

We also define normalised versions of the kernels:

$$\breve{\mathsf{Z}}_i(x,y)\coloneqq \mathsf{Z}_i(q_{N_{i-1}})^{-1}\mathsf{Z}_i(x,y),\qquad \breve D_i(x,y)\coloneqq \mathsf{Z}_i(q_{N_{i-1}})^{-1}D_i(x,y). \tag{7.7}$$

With this notation, for every test function f, we have $q_{N_{i-1}}\triangleleft\breve{\mathsf{Z}}_i\triangleright f=q_{N_i}(f)+q_{N_{i-1}}\triangleleft\breve D_i\triangleright f$.

For consecutive blocks $r,\dots,s$, define

$$\mathscr{R}_{r,s}\coloneqq q_{N_{r-1}}\triangleleft\breve{\mathsf{Z}}_r\circ\cdots\circ\breve{\mathsf{Z}}_s\triangleright\mathbf{1}=\frac{q_{N_{r-1}}\triangleleft Z_{N_{r-1},N_s}\triangleright\mathbf{1}}{\prod_{i=r}^s q_{N_{i-1}}\triangleleft\mathsf{Z}_i\triangleright\mathbf{1}}, \tag{7.8}$$

with by convention $\mathscr{R}_{r,r-1}\coloneqq 1$. This quantity compares the partition function on the complete interval $(N_{r-1},N_s]$ with the product of independent partition functions on the different intervals $(N_{i-1},N_i]$, with starting distribution replaced by $q_{N_{i-1}}$.

We then have the following result that estimates $\mathscr{R}_{r,s}$ and ratios of partition function on some good event. Let us introduce, similarly as in [GT26, Eq. (2.24)], the events

$$\Omega_{r,s}\coloneqq\bigcap_{i=r}^s\{|\mathsf{Z}_i(q_{N_{i-1}})-1|\leqslant 1/2\}, \tag{7.9}$$

with by convention $\Omega_{r,r-1}$ the whole probability space. Let us also introduce the following probability law weighted by the product of inverse interval partition functions:

$$\frac{\mathrm{d}\mathbb{Q}_{r,s}}{\mathrm{d}\mathbb{P}}\coloneqq\frac{\prod_{i=r}^s\mathsf{Z}_i(q_{N_{i-1}})^{-1}}{\mathbb{E}\big[\prod_{i=r}^s\mathsf{Z}_i(q_{N_{i-1}})^{-1}\big]}. \tag{7.10}$$

Proposition 7.3 (Ratios estimates)**.** *There is a constant $C=C(\rho)>0$ such that for $r\leqslant s\leqslant K-1$ we have*

$$\mathbb{P}(\Omega_{r,s}^c)\leqslant\frac{C}{K-s}\quad\text{ and }\quad\mathbb{E}\big[\mathbf{1}_{\Omega_{r,s}}|\mathscr{R}_{r,s}-1|^2\big]\leqslant\frac{C}{K-s}. \tag{7.11}$$

One also has

$$\mathbb{Q}_{r,s}(\Omega_{r,s}^c) \leqslant \frac{C}{\sqrt{K-s}} \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}_{r,s}}\big[\mathbf{1}_{\Omega_{r,s}}|\mathscr{R}_{r,s}-1|^2\big] \leqslant \frac{C}{\sqrt{K-s}}. \tag{7.12}$$

The first part of the proposition, *i.e.* (7.11), is already present in [GT26, Proposition 4.1] in a slightly different setting; we provide a proof here in order to explain more precisely the idea, which is needed afterwards. The main novelty here is the control of the event $\Omega_{r,s}$ and of the ratio $\mathscr{R}_{r,s}$ under the weighted measure $\mathbb{Q}_{r,s}$. This will crucially allow us to control the negative moment $\mathbb{E}[\widehat{Z}_{s,n}(x)^{-1}]$ that appears in (4.19) thanks to its "product version" $\prod_{i=a}^{b}\mathbb{E}[Z_{N_{i-1},N_i}(q_{N_{i-1}})^{-1}]$ (see (7.31) for the explicit decoupling identity).

7.2. The block expansion. One idea of Gu–Tsai [GT26] is to expand the product $\mathscr{R}_{r,s} = q_{N_{r-1}} \prec \check{\mathsf{Z}}_r \circ \cdots \circ \check{\mathsf{Z}}_s \succ \mathbf{1}$, using that

$$\check{\mathsf{Z}}_i = \check{D}_i + \frac{\mathsf{Z}_i \succ \mathbf{1} \otimes q_{N_i}\prec}{q_{N_{i-1}} \prec \mathsf{Z}_i \succ \mathbf{1}}\,. \tag{7.13}$$

More explicitly, specifying the coordinates, we can write

$$\check{\mathsf{Z}}_i(x,y) = \check{D}_i(x,y) + \frac{\mathsf{Z}_i \succ \mathbf{1}(x)\, q_{N_i}(y)}{q_{N_{i-1}} \prec \mathsf{Z}_i \succ \mathbf{1}}\,. \tag{7.14}$$

Recalling Remark 7.1, the particular notation in (7.13) is particularly useful in obtaining an expansion for $\mathscr{R}_{r,s}$. In particular, expanding the product in the definition (7.8) of $\mathscr{R}_{r,s}$ and decomposing over intervals of successive correctors $\check{D}_i$, we get the following exact identity, which is the block expansion of [GT26, Proposition 2.1] written for lattice kernels:

$$\mathscr{R}_{r,s} = \sum_{\mathcal{I}\in\mathfrak{I}(r,s)}\prod_{I\in\mathcal{I}}\mathscr{D}_I^{\mathrm{closed}} = 1 + \sum_{\mathcal{I}\in\mathfrak{I}(r,s),\mathcal{I}\neq\varnothing}\prod_{I\in\mathcal{I}}\mathscr{D}_I^{\mathrm{closed}}\,, \tag{7.15}$$

where $\mathfrak{I}(r,s)$ is the set of families of pairwise disjoint non-degenerate integer intervals $[u,v]\subseteq[r,s]$, with $u<v$ (including the empty family) and where we defined

$$\mathscr{D}_{[u,v]}^{\mathrm{closed}} := \mathscr{D}_{u,v}^{\mathrm{closed}} := q_{N_{u-1}} \prec \check{D}_u \circ \cdots \circ \check{D}_{v-1} \circ \check{\mathsf{Z}}_v \succ \mathbf{1}\,. \tag{7.16}$$

Let us mention that we also need later on to perform an expansion for the endpoint distribution $\mu^{(q_{N_{r-1}})}_{N_{r-1},N_s}$, in the same spirit as the above, see (7.48) below.

We need the following estimates on the closed correctors. We let

$$D_{u,v}^{\mathrm{closed}} := q_{N_{u-1}} \prec D_u \circ \cdots \circ D_{v-1} \circ \mathsf{Z}_v \succ \mathbf{1}$$

which is the non-normalised closed correction.

Proposition 7.4 (Compositions of discrepancy kernels)**.** *There are constants C, C_0, independent of N,ρ, such that, for $u<v\leqslant K-1$*

$$\|D_{u,v}^{\mathrm{closed}}\|_{L^2} \leqslant \frac{C(\varepsilon_\rho)^{v-u}}{\sqrt{(K-u)(K-v)}} \qquad \text{with } \varepsilon_\rho := \frac{C_0}{\sqrt{\log\rho}} < 1 \tag{7.17}$$

when $\rho > e^{C_0^2}$. Additionally, for every $u<v\leqslant K-1$,

$$\big\|q_{N_{u-1}} \prec D_u \circ \cdots \circ D_{v-1} \circ Z_{N_{v-1},N} \succ \mathbf{1}\big\|_{L^2} \leqslant \frac{C(\varepsilon_\rho)^{v-u-1}}{\sqrt{K-u}}\,. \tag{7.18}$$

Let us stress here that this result is analogous to what is obtained by Gu and Tsai, see in particular [GT26, Corollary 3.5] (our ρ corresponds to their b^{-2}). Since we were not able to simply adapt their proof to our discrete setting, we have decided to write a complete proof for the sake of completeness; we present it in Appendix A.

Remark 7.5 (Uniformity in the parameters)**.** *We stress that in Propositions 7.3 and 7.4, the bounds obtained are uniform in N and the constants can be chosen uniformly for ϑ in a compact set of $\mathbb{R}$ and more importantly uniformly for $\rho > 1$ large enough e.g. replacing ε_ρ by $C_0/\sqrt{\log \rho}$ in Proposition 7.4.*

7.3. Ratios of partition functions: proof of Proposition 7.3. We prove Proposition 7.3 in two steps: we first prove (7.11), introducing all the necessary tools, and then we turn to the proof of (7.12).

Proof of (7.11)**.** First of all, using the fourth moment estimate from Lemma B.2, we get that

$$\mathbb{P}\big(|\mathsf{Z}_i(q_{N_{i-1}}) - 1| > 1/2\big) \leqslant 16\,\mathbb{E}\big[|\mathsf{Z}_i(q_{N_{i-1}}) - 1|^4\big] \leqslant \frac{C}{(K-i)^2}\,. \tag{7.19}$$

Using a union bound for $\Omega^c_{r,s} := \bigcup_{i=r}^{s}\{|\mathsf{Z}_i(q_{N_{i-1}}) - 1| > 1/2\}$, we obtain that $\mathbb{P}(\Omega^c_{r,s}) \leqslant \frac{C}{K-s}$.

We next show the second estimate, namely

$$\mathbb{E}\big[\mathbf{1}_{\Omega_{r,s}}|\mathscr{R}_{r,s} - 1|^2\big] \leqslant \frac{C}{K-s}\,. \tag{7.20}$$

Recall that $\mathscr{R}_{r,s}$ is defined in (7.8) and recall also the expansion (7.15). To proceed, we will need some auxiliary estimates. First, denote by $\mathcal{B}_i := \{|\mathsf{Z}_i(q_{N_{i-1}}) - 1| \leqslant 1/2\}$ and define the truncated inverse multiplier

$$M_i := \frac{\mathbf{1}_{\mathcal{B}_i}}{\mathsf{Z}_i(q_{N_{i-1}})}$$

and the truncated normalized closed corrector

$$\widetilde{\mathscr{D}}^{\mathrm{closed}}_{u,v} = D^{\mathrm{closed}}_{u,v} \prod_{i=u}^{v} M_i.$$

In particular, for any $r \leqslant u \leqslant v \leqslant s$, we have $\mathbf{1}_{\Omega_{r,s}}\mathscr{D}^{\mathrm{closed}}_{u,v} = \widetilde{\mathscr{D}}^{\mathrm{closed}}_{u,v}\prod_{i\notin[u,v]}\mathbf{1}_{\mathcal{B}_i}$, and, noticing that $M_i \leqslant 2$, Proposition 7.4-(7.17) yields

$$\|\widetilde{\mathscr{D}}^{\mathrm{closed}}_{u,v}\|_{L^2} \leqslant \frac{C(2\varepsilon_\rho)^{v-u}}{\sqrt{(K-u)(K-v)}} \leqslant \frac{C(2\varepsilon_\rho)^{v-u}}{K-v}\,. \tag{7.21}$$

It is not difficult to see that $|M_i - 1| \leqslant 2|\mathsf{Z}_i(q_{N_{i-1}}) - 1|$ a.s., including on $\mathcal{B}_i^c$, so that combined with the second moment estimate from Lemma B.2, it leads to

$$\|M_i - 1\|_{L^2} \leqslant 2\sqrt{\mathrm{Var}\big(\mathsf{Z}_i(q_{N_{i-1}})\big)} \leqslant \frac{C}{\sqrt{K-i}}, \tag{7.22}$$

for any $i < K$. We will use (7.22) to obtain an improved estimate on $|\mathbb{E}\widetilde{\mathscr{D}}^{\mathrm{closed}}_{u,v}|$ compared to that of a simple Cauchy–Schwarz's inequality.

Let us first decompose

$$\mathbb{E}\big[\widetilde{\mathscr{D}}^{\mathrm{closed}}_{u,v}\big] = \mathbb{E}\big[D^{\mathrm{closed}}_{u,v}\,(M_u - 1)\prod_{i=u+1}^{v} M_i\big] + \mathbb{E}\big[D^{\mathrm{closed}}_{u,v}\prod_{i=u+1}^{v} M_i\big] \tag{7.23}$$

Then, denoting by $\mathbb{E}_u$ the expectation with respect to disorder in block u, we notice that

$$\mathbb{E}\big[D^{\mathrm{closed}}_{u,v}\prod_{i=u+1}^{v} M_i\big] = \mathbb{E}\Big[q_{N_{u-1}} \prec \mathbb{E}_u\big[D_u\big] \circ \cdots \circ D_{v-1} \circ Z_v \succ \mathbf{1}\prod_{i=u+1}^{v} M_i\Big] = 0\,.$$

In the above, the last identity follows from the fact that

$$\mathbb{E}_u[D_u(x,y)] = \mathbb{E}_u[\mathsf{Z}_u(x,y) - \mathsf{Z}_u^{\mathrm{rewired}}(x,y)] = q_{N_u-N_{u-1}}(y-x) - q_{N_u}(y)$$

and so $q_{N_{u-1}} \prec \mathbb{E}_u[D_u](y) = 0$.

Going back to (7.23), using the bound $M_i \leqslant 2$ and then Cauchy–Schwarz's inequality, we get that

$$\big|\mathbb{E}\big[\widetilde{\mathscr{D}}^{\mathrm{closed}}_{u,v}\big]\big| \leqslant 2^{v-u}\big\|D^{\mathrm{closed}}_{u,v}\big\|_{L^2}\big\|M_u - 1\big\|_{L^2} \leqslant \frac{C(2\varepsilon_\rho)^{v-u}}{(K-v)^{3/2}}, \tag{7.24}$$

where we have used (7.22) and Proposition 7.4-(7.17) for the last inequality (together with the fact that $K-u \geqslant K-v$).

We now have the ingredients to prove (7.20). To this end, we expand $(\mathscr{R}_{r,s}-1)^2$ using (7.15): we obtain

$$\mathbb{E}\big[\mathbf{1}_{\Omega_{r,s}}|\mathscr{R}_{r,s}-1|^2\big] = \sum_{\substack{\mathcal{I},\mathcal{J}\in\mathfrak{I}(r,s)\\ \mathcal{I},\mathcal{J}\neq\varnothing}} \mathbb{E}\Big[\mathbf{1}_{\mathcal{B}_{\mathcal{I},\mathcal{J}}}\prod_{I\in\mathcal{I}}\widetilde{\mathscr{D}}^{\mathrm{closed}}_I\prod_{J\in\mathcal{J}}\widetilde{\mathscr{D}}^{\mathrm{closed}}_J\Big] \tag{7.25}$$

where $\mathcal{B}_{\mathcal{I},\mathcal{J}}$ accounts for the part of the event $\Omega_{r,s}$ that has not been used in the definition of $\widetilde{\mathscr{D}}_I$, $\widetilde{\mathscr{D}}_J$; in particular $\mathcal{B}_{\mathcal{I},\mathcal{J}}$ is independent of the family $\{\widetilde{\mathscr{D}}_I, \widetilde{\mathscr{D}}_J; I\in\mathcal{I}, J\in\mathcal{J}\}$. Recall that $\mathfrak{I}(r,s)$ is the set of families of pairwise disjoint non-degenerate integer intervals $[u,v]\subseteq[r,s]$, with $u<v$. Even though intervals in $\mathcal{I}$ and intervals in $\mathcal{J}$ are internally disjoint, intervals from the two families can overlap. Products over disjoint connected components are independent, so their expectations factorise.

If a component contains one interval from one family and none from the other, its expectation is estimated via (7.24). For any other connected component $[a,b]$, Cauchy–Schwarz and block independence within each family give

$$\begin{aligned}\mathbb{E}\Big[\Big|\prod_{\substack{I\in\mathcal{I}\\ I\subset[a,b]}}\widetilde{\mathscr{D}}^{\mathrm{closed}}_I\prod_{\substack{J\in\mathcal{J}\\ J\subset[a,b]}}\widetilde{\mathscr{D}}^{\mathrm{closed}}_J\Big|\Big] &\leqslant \prod_{\substack{I\in\mathcal{I}\\ I\subset[a,b]}}\|\widetilde{\mathscr{D}}^{\mathrm{closed}}_I\|_{L^2}\prod_{\substack{J\in\mathcal{J}\\ J\subset[a,b]}}\|\widetilde{\mathscr{D}}^{\mathrm{closed}}_J\|_{L^2}\\ &\leqslant \frac{(2C\varepsilon_\rho)^{b-a}}{(K-b)^2}\,.\end{aligned} \tag{7.26}$$

Here we used (7.21): the sum of the interval lengths over both families is at least $b-a$. Each interval $[u,v]\subset[a,b]$ contributes a factor $(K-v)^{-1}\leqslant(K-b)^{-1}$, and at least two such factors occur. As above, we take ρ large enough that $2C\varepsilon_\rho<1$.

Going back to (7.25), we express the union of the intervals in $\mathcal{I}$ and $\mathcal{J}$ as $\bigcup_{i=1}^m[a_i,b_i]$, where the $[a_i,b_i]$ are its connected components and $r\leqslant a_1<b_1<\cdots<a_m<b_m\leqslant s$. Then, using block independence and employing either (7.24) (for isolated intervals) or (7.26) (for overlapping intervals) we obtain

$$\mathbb{E}\big[\mathbf{1}_{\Omega_{r,s}}|\mathscr{R}_{r,s}-1|^2\big] \leqslant \sum_{m\geqslant1}\sum_{r\leqslant a_1<b_1<\cdots<a_m<b_m\leqslant s}\sum_{\substack{\alpha_i\in\{1/2,1\}\\ \alpha_1+\cdots+\alpha_m\geqslant1}}\prod_{i=1}^m\frac{(8C\varepsilon_\rho)^{b_i-a_i}}{(K-b_i)^{1+\alpha_i}}\,, \tag{7.27}$$

where $\alpha_i := 1/2$ for isolated intervals and $\alpha_i := 1$ for overlapping intervals; the restriction $\alpha_1+\cdots+\alpha_m\geqslant1$ accounts for the fact that there cannot be a single isolated interval (in

other words one must have $\alpha_1 = 1$ when $m = 1$). We have also used that on $[a_i, b_i]$ there are at most $4^{b_i - a_i}$ pairs of internally disjoint interval families, since each family is determined by an even-cardinality subset of the $b_i - a_i + 1$ possible endpoints, giving $2^{b_i - a_i}$ choices.

First, we sum the above over the a_i indices, obtaining

$$\mathbb{E}\big[\mathbf{1}_{\Omega_{r,s}} |\mathscr{R}_{r,s} - 1|^2\big] \leqslant \sum_{m \geqslant 1} (C' \varepsilon_\rho)^m \sum_{r \leqslant b_1 < \cdots < b_m \leqslant s} \sum_{\substack{\alpha_i \in \{1/2, 1\} \\ \alpha_1 + \cdots + \alpha_m \geqslant 1}} \prod_{i=1}^{m} (K - b_i)^{-(1+\alpha_i)}$$
$$\leqslant \sum_{m \geqslant 1} (C' \varepsilon_\rho)^m \sum_{\substack{\alpha_i \in \{1/2, 1\} \\ \alpha_1 + \cdots + \alpha_m \geqslant 1}} \prod_{i=1}^{m} (K - s)^{-\alpha_i} .$$

Now, for $m = 1$ we have that necessarily $\alpha_1 \geqslant 1$, while for $m \geqslant 2$ we use that $\alpha_i \geqslant 1/2$: this gives

$$\mathbb{E}\big[\mathbf{1}_{\Omega_{r,s}} |\mathscr{R}_{r,s} - 1|^2\big] \leqslant \frac{C' \varepsilon_\rho}{K - s} + \sum_{m \geqslant 2} \Big(\frac{2C' \varepsilon_\rho}{\sqrt{K - s}} \Big)^m \leqslant \frac{C' \varepsilon_\rho}{K - s} + \frac{C'' (\varepsilon_\rho)^2}{K - s}$$

where the last inequality holds provided that ε_ρ is small enough. This yields the desired bound (7.20), concluding the proof of (7.11). □

Proof of (7.12)**.** Recall the definition (7.10) of the weighted law $\mathbb{Q}_{r,s}$. Let us start with the first bound in (7.12). Recalling that $\mathcal{B}_i := \{|\mathsf{Z}_i(q_{N_{i-1}}) - 1| \leqslant 1/2\}$, we have that

$$\mathbb{Q}_{r,s}\big(\mathcal{B}_i^c\big) = \frac{\mathbb{E}\big[\mathsf{Z}_i(q_{N_{i-1}})^{-1} \mathbf{1}_{\mathcal{B}_i^c}\big]}{\mathbb{E}\big[\mathsf{Z}_i(q_{N_{i-1}})^{-1}\big]} \leqslant \mathbb{E}\big[\mathsf{Z}_i(q_{N_{i-1}})^{-4}\big]^{1/4} \, \mathbb{P}\big(\mathcal{B}_i^c\big)^{3/4} .$$

The equality follows from (7.10) and block independence, since $\mathcal{B}_i^c$ depends only on the disorder in block i. The inequality follows from Hölder's inequality and the fact that $\mathbb{E}[\mathsf{Z}_i(q_{N_{i-1}})^{-1}] \geqslant \mathbb{E}[\mathsf{Z}_i(q_{N_{i-1}})]^{-1} = 1$. Then, we have that $\mathbb{E}[\mathsf{Z}_i(q_{N_{i-1}})^{-4}] \leqslant C$ thanks to the negative moment estimate of (3.5) from Lemma 3.2, with $p = 4$. Together with (7.19) above, this shows that $\mathbb{Q}_{r,s}(\mathcal{B}_i^c) \leqslant C(K - i)^{-3/2}$. Then, a union bound shows that

$$\mathbb{Q}_{r,s}(\Omega_{r,s}^c) \leqslant \sum_{i=r}^{s} \mathbb{Q}_{r,s}\big(\mathcal{B}_i^c\big) \leqslant \frac{C}{\sqrt{K - s}} ,$$

which proves the first part of (7.12).

We now proceed to check the second inequality in (7.12), namely

$$\mathbb{E}_{\mathbb{Q}_{r,s}}\big[\mathbf{1}_{\Omega_{r,s}} |\mathscr{R}_{r,s} - 1|^2\big] \leqslant \frac{C}{\sqrt{K - s}}. \tag{7.28}$$

The steps are similar to those taken for (7.20), so we will only provide a sketch.

As in (7.25), the starting point is the expansion

$$\mathbb{E}_{\mathbb{Q}_{r,s}}\big[\mathbf{1}_{\Omega_{r,s}} |\mathscr{R}_{r,s} - 1|^2\big] = \sum_{\substack{\mathcal{I}, \mathcal{J} \in \mathfrak{I}(r,s) \\ \mathcal{I}, \mathcal{J} \neq \varnothing}} \mathbb{E}_{\mathbb{Q}_{r,s}}\Big[\mathbf{1}_{\mathcal{B}_{\mathcal{I},\mathcal{J}}} \prod_{I \in \mathcal{I}} \widetilde{\mathscr{D}}_I^{\text{closed}} \prod_{J \in \mathcal{J}} \widetilde{\mathscr{D}}_J^{\text{closed}}\Big]. \tag{7.29}$$

To estimate this quantity we need the analogues of (7.24) and (7.26) for the $\mathbb{Q}$-expectation.

For the analogue of (7.24), we have

$$\mathbb{E}_{\mathbb{Q}_{r,s}}\big[\widetilde{\mathscr{D}}_{u,v}^{\text{closed}}\big] = \frac{\mathbb{E}\Big[D_{u,v}^{\text{closed}} \prod_{i=u}^{v} M_i^2\Big]}{\prod_{i=u}^{v} \mathbb{E}\big[Z_i^{-1}(q_{N_{i-1}})\big]}$$

for $r \leqslant u < v \leqslant s$. The denominator is (as we have seen earlier) bounded below by 1. The numerator can be written as

$$\mathbb{E}\Big[D^{\mathrm{closed}}_{u,v}\,(M_u^2-1)\prod_{i=u+1}^{v} M_i^2\Big] + \mathbb{E}\big[D^{\mathrm{closed}}_{u,v}\prod_{i=u+1}^{v} M_i^2\big]$$

The second term on the right-hand side vanishes, using again the observation that $q_{N_{u-1}} \lhd \mathbb{E}_u[D_u] = 0$. The first term can be estimated as in (7.24) after writing $M_i^2 - 1 = (M_i - 1)(M_i + 1)$ and using that $M_i + 1 \leqslant 3$.

Finally, notice that the analogue of (7.21) holds (*i.e.* we have a bound on the $L^2(\mathbb{Q})$ norm of $\widetilde{\mathscr{D}}^{\mathrm{closed}}_{u,v}$) thanks to Proposition 7.4-(7.17), using again that $M_i \leqslant 2$ together with the fact that $\mathbb{E}\big[Z_i^{-1}(q_{N_{i-1}})\big] \geqslant 1$. Therefore the analogue (7.26) for the $\mathbb{Q}_{r,s}$ expectation follows, with the same line of proof.

All together, with the analogue of (7.24) and (7.26) for the $\mathbb{Q}$-expectation, the expansion (7.29) allows us to prove the analogue of (7.27) for the $\mathbb{Q}$-expectation so one can conclude similarly as above. □

7.4. One-point estimate: proof of Proposition 4.1. We first prove the small-time comparison (4.19), and then deduce the bounded comparison (4.18) by averaging out the disorder after a fixed small time. By (3.11),

$$0 \leqslant \Delta(\log \widehat{Z}_{s,n}(x)) \leqslant \log \frac{\mathbb{E}[\widehat{Z}_{s,n}(x)^2]}{\mathbb{E}[\widehat{Z}_{s,n}(x)^{-1}]}.$$

It therefore suffices to bound the ratio of the second moment to the negative-first moment. For the small-time comparison, set $n = [\sqrt{\delta}N]$; we will show that

$$\log \frac{\mathbb{E}[\widehat{Z}_{s,n}(x)^2]}{\mathbb{E}[\widehat{Z}_{s,n}(x)^{-1}]} \leqslant \frac{C}{|\log\delta|^{1/4}} + o_N(1). \tag{7.30}$$

By translation invariance, we only need to treat the case $x = 0$.

Reducing to second moment estimates. First suppose that $s = N_u$ for some $0 \leqslant u < b$, on the grid chosen in Remark 7.2, so that $n = N_b$ and $K - b = J$. The case $s = n$ is immediate since $\widehat{Z}_{n,n}(0) = 1$. By the block decomposition (7.8), we have

$$\widehat{Z}_{s,n}(0) = \Big(\prod_{i=u+1}^{b} \mathsf{Z}_i(q_{N_{i-1}})\Big)\mathscr{R}_{u+1,b}.$$

We introduce the law weighted by the product of inverse block partition functions, recall (7.10):

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \frac{\big(\prod_{i=u+1}^{b} \mathsf{Z}_i(q_{N_{i-1}})\big)^{-1}}{\mathbb{E}\big[\prod_{i=u+1}^{b} \mathsf{Z}_i(q_{N_{i-1}})^{-1}\big]}.$$

In particular, we have that

$$\mathbb{E}\big[\widehat{Z}_{s,n}(0)^{-1}\big] = \mathbb{E}_{\mathbb{Q}}\big[\mathscr{R}^{-1}_{u+1,b}\big]\mathbb{E}\Big[\prod_{i=u+1}^{b} \mathsf{Z}_i(q_{N_{i-1}})^{-1}\Big] = \mathbb{E}_{\mathbb{Q}}\big[\mathscr{R}^{-1}_{u+1,b}\big]\prod_{i=u+1}^{b}\mathbb{E}\big[\mathsf{Z}_i(q_{N_{i-1}})^{-1}\big], \tag{7.31}$$

where in the second equality we also used that $(\mathsf{Z}_i(q_{N_{i-1}}))_{i\geqslant 1}$ are independent.

We may now use the second part of Proposition 7.3, namely (7.12), to get that

$$\mathbb{Q}(\Omega^c_{u+1,b}) \leqslant C(K-b)^{-1/2}, \qquad \mathbb{E}_{\mathbb{Q}}\big[|\mathscr{R}_{u+1,b}-1|^2\mathbf{1}_{\Omega_{u+1,b}}\big] \leqslant C(K-b)^{-1/2}, \tag{7.32}$$

Using that $\mathscr{R}^{-1} \geqslant 2 - \mathscr{R}$ for $\mathscr{R} > 0$, we get that

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[\mathscr{R}_{u+1,b}^{-1}] &\geqslant \mathbb{E}_{\mathbb{Q}}[\mathscr{R}_{u+1,b}^{-1}\mathbf{1}_{\Omega_{u+1,b}}] \geqslant \mathbb{Q}(\Omega_{u+1,b}) + \mathbb{E}_{\mathbb{Q}}[(1-\mathscr{R}_{u+1,b})\mathbf{1}_{\Omega_{u+1,b}}] \\ &\geqslant 1 - \mathbb{Q}(\Omega_{u+1,b}^c) - \mathbb{E}_{\mathbb{Q}}[|\mathscr{R}_{u+1,b}-1|^2\mathbf{1}_{\Omega_{u+1,b}}]^{1/2} \geqslant 1 - CJ^{-1/4}\,,\end{aligned} \tag{7.33}$$

where we have used Cauchy–Schwarz's inequality for the second inequality and then (7.32), recalling also that $K - b = J$.

On the other hand, we also claim that

$$\mathbb{E}\big[\mathsf{Z}_i(q_{N_{i-1}})^{-1}\big] = \mathbb{E}\big[\mathsf{Z}_i(q_{N_{i-1}})^2\big]\big(1 + O\big((K-i)^{-3/2}\big)\big)\,. \tag{7.34}$$

Indeed, writing $X_i = \mathsf{Z}_i(q_{N_{i-1}}) - 1$ we can use the identity $(1+X_i)^{-1} = 1 - X_i + X_i^2 - \frac{X_i^3}{1+X_i}$ to get that

$$\mathbb{E}\big[\mathsf{Z}_i(q_{N_{i-1}})^{-1}\big] = \mathbb{E}\big[\mathsf{Z}_i(q_{N_{i-1}})^2\big] - \mathbb{E}\bigg[\frac{(\mathsf{Z}_i(q_{N_{i-1}})-1)^3}{\mathsf{Z}_i(q_{N_{i-1}})}\bigg]\,.$$

Now, by Hölder's inequality, we have that

$$\mathbb{E}\bigg[\frac{|\mathsf{Z}_i(q_{N_{i-1}})-1|^3}{\mathsf{Z}_i(q_{N_{i-1}})}\bigg] \leqslant \mathbb{E}\big[|\mathsf{Z}_i(q_{N_{i-1}})-1|^4\big]^{3/4}\mathbb{E}\big[\mathsf{Z}_i(q_{N_{i-1}})^{-4}\big]^{1/4} \leqslant C(K-i)^{-3/2}\,.$$

For the last inequality, we have used Lemma B.2 to control the fourth moment and Lemma 3.2-(3.5) with $p = 4$ to bound $\mathbb{E}[\mathsf{Z}_i(q_{N_{i-1}})^{-4}] \leqslant \mathbb{E}[\mathsf{Z}_i(q_{N_{i-1}})^2]^{10}$, with the second moment being bounded by $\mathbb{E}[\mathsf{Z}_i(q_{N_{i-1}})^2] \leqslant 1 + \frac{C}{K-i} \leqslant 1 + C$, again by Lemma B.2.

Since $\sum_i (K-i)^{-3/2} \leqslant CJ^{-1/2}$, going back to (7.31) and using (7.33) and (7.34), we end up with

$$\mathbb{E}\big[\widehat{Z}_{s,n}(0)^{-1}\big] \geqslant \big(1 + O(J^{-1/4})\big) \prod_{i=u+1}^{b} \mathbb{E}\big[\mathsf{Z}_i(q_{N_{i-1}})^2\big]\,. \tag{7.35}$$

Estimating the second moments. We first prove (7.30) at grid points, i.e. for $s = N_u$ with $0 \leqslant u \leqslant b$. In view of (7.35), it suffices to show that

$$\frac{\mathbb{E}[\widehat{Z}_{s,n}(0)^2]}{\prod_{i=u+1}^{b}\mathbb{E}\big[\mathsf{Z}_i(q_{N_{i-1}})^2\big]} \leqslant 1 + CJ^{-1}\,. \tag{7.36}$$

We rely on the following sharp second moment estimate. Its proof is classical and postponed to Appendix B. To state it in a more natural way, let us introduce, analogously to [BCT25, Eq. (1.12)], the interpolating parameter

$$\vartheta_N(s) := \frac{\pi}{\sigma_N^2} - \pi R_s\,,$$

which controls the distance to the critical window in time-s partition functions (we use a different sign convention from [BCT25], in order to keep the parameter positive in the sub and quasi-critical regime). The factor π is so that we have that $\vartheta_N(N) \to -\vartheta$ as $N \to \infty$. The following estimates apply whenever $\vartheta_N(t) > 0$, in particular for $t \leqslant \epsilon_0 N$ for $\epsilon_0 > 0$ small enough. To be definite, we set $\epsilon_0 := e^{-1-\max\{\vartheta,0\}}$.

Lemma 7.6. *There exists a constant $C > 0$ such that, for all sufficiently large N and all integers $0 \leqslant s \leqslant t \leqslant \epsilon_0 N$,*

$$0 \leqslant \frac{\vartheta_N(s)}{\vartheta_N(t)} - \mathbb{E}[\widehat{Z}_{s,t}(0)^2] \leqslant \frac{C(1 + R_t - R_s)}{\vartheta_N(t)^2}\,,$$

and therefore

$$\mathbb{E}[\widehat{Z}_{s,t}(0)^2] = \frac{\vartheta_N(s)}{\vartheta_N(t)}\left(1 + O\left(\frac{1 + R_t - R_s}{\vartheta_N(s)\vartheta_N(t)}\right)\right). \tag{7.37}$$

If $1 \leqslant s \leqslant t/2$*, the factor* $1 + R_t - R_s$ *in these bounds can be replaced by* $\log(t/s)$*.*

The asymptotics of R_t give, uniformly for integers $0 \leqslant s < t$,

$$\vartheta_N(s) - \vartheta_N(t) = \pi(R_t - R_s) = \log\frac{t}{s \vee 1} + O(1)\,, \tag{7.38}$$

and since $\vartheta_N(N) \to -\vartheta = O(1)$, we obtain in particular

$$\vartheta_N(t) = \log\frac{N}{t \vee 1} + O(1), \qquad 0 \leqslant t \leqslant N.$$

For fixed ρ and $J = K - b$ sufficiently large, this implies

$$\vartheta_N(N_i) \asymp K - i, \qquad 0 \leqslant i \leqslant b. \tag{7.39}$$

In particular, since $n = N_b = [N\rho^{-J}]$, we have $\vartheta_N(n) = \log(N/n) + O(1) = J\log\rho + O(1)$. Thus, for fixed $\rho > 1$ and J sufficiently large, $\vartheta_N(n) \asymp J$.

For every $0 \leqslant s \leqslant n$, we have

$$\vartheta_N(s) = \vartheta_N(n) + \pi(R_n - R_s) \geqslant 1 + R_n - R_s,$$

provided J is large enough. Thus (7.37) and $\vartheta_N(n) \asymp J$ give

$$\mathbb{E}[\widehat{Z}_{s,n}(0)^2] = \frac{\vartheta_N(s)}{\vartheta_N(n)}\big(1 + O(J^{-1})\big), \qquad 0 \leqslant s \leqslant n\,. \tag{7.40}$$

For every $1 \leqslant i \leqslant b$ we can bound $R_{N_i} - R_{N_{i-1}} \leqslant C_\rho$ by definition of N_i (also for $i = 1$, since $N_0 = 0$ and $N_1 \leqslant 2\rho$). Applying Lemma 7.6 and (7.39), we obtain

$$\mathbb{E}[\mathsf{Z}_i(q_{N_{i-1}})^2] = \frac{\vartheta_N(N_{i-1})}{\vartheta_N(N_i)}(1 - \eta_i) \qquad \text{with} \quad 0 \leqslant \eta_i \leqslant \frac{C}{(K-i)^2}.$$

Since $\sum_{i=u+1}^{b}\eta_i \leqslant C\sum_{j=J}^{\infty}\frac{1}{j^2} \leqslant \frac{C'}{J}$, for J sufficiently large we get

$$\prod_{i=u+1}^{b}\mathbb{E}[\mathsf{Z}_i(q_{N_{i-1}})^2] = \frac{\vartheta_N(N_u)}{\vartheta_N(n)}\prod_{i=u+1}^{b}(1-\eta_i) = \frac{\vartheta_N(N_u)}{\vartheta_N(n)}\big(1 + O(J^{-1})\big).$$

Together with (7.40), this yields, for $s = N_u$,

$$\frac{\mathbb{E}[\widehat{Z}_{s,n}(0)^2]}{\prod_{i=u+1}^{b}\mathbb{E}[\mathsf{Z}_i(q_{N_{i-1}})^2]} = 1 + O(J^{-1}).$$

This proves (7.36) and hence (7.30) at every grid point, uniformly in $0 \leqslant u \leqslant b$.

We can now extend the proof to any $s \in (0, n)$ which may not be the starting point of a block. Let s be even, with $s \in (N_{u-1}, N_u)$ for some $1 \leqslant u \leqslant b$. Averaging the disorder on $(s, N_u]$ and applying conditional Jensen gives

$$\mathbb{E}[\widehat{Z}_{s,n}(0)^{-1}] \geqslant \mathbb{E}[\widehat{Z}_{N_u,n}(0)^{-1}].$$

On the other hand, the first-collision formula (4.2) and $\sigma_N^2\mathbb{E}[Z_{n-k}(0)^2] \leqslant \pi/\vartheta_N(n-k) \leqslant \pi/\vartheta_N(n)$, see Claim B.1, imply

$$0 \leqslant \mathbb{E}[\widehat{Z}_{s,n}(0)^2] - \mathbb{E}[\widehat{Z}_{N_u,n}(0)^2] = \sum_{k=s+1}^{N_u} q_{2k}(0)\sigma_N^2\mathbb{E}[Z_{n-k}(0)^2] \leqslant \frac{\pi(R_{N_u} - R_s)}{\vartheta_N(n)} \leqslant \frac{C}{J}.$$

Since $\mathbb{E}[\widehat{Z}_{N_u,n}(0)^2] \geqslant \mathbb{E}[\widehat{Z}_{N_u,n}(0)]^2 = 1$, the ratio of these second moments is at most $1+C/J$. Combining this with the bound already proved at the grid points $s = N_u$ gives (7.30) for every even $0 \leqslant s \leqslant n$. This concludes the proof of (4.19).

Proof of (4.18)**.** Fix $0 < T_0 < T_1 < \infty$, and now let $T_0 N \leqslant n \leqslant T_1 N$. The expansion (4.2), the last statement in Claim B.1, and the bound $q_{2k}(0) \leqslant C/k$ give

$$\begin{aligned}\mathbb{E}[\widehat{Z}_{s,n}(0)^2] &= 1 + \sigma_N^2 \sum_{k=s+1}^{n} q_{2k}(0)\mathbb{E}[Z_{n-k}(0)^2] \\ &\leqslant 1 + C_{\vartheta,T_1} \sum_{k=s+1}^{n} \frac{1}{k} \leqslant C\left\{1 + \log\frac{n}{s \vee 1}\right\}.\end{aligned} \tag{7.41}$$

Fix $0 < \delta = \delta_{T_0} \leqslant \delta_0$ sufficiently small, independently of s, n, N, with $\sqrt{\delta} < T_0$, and put $n_0 = [\sqrt{\delta}N]$. If $s \geqslant n_0$, then $n/s \leqslant T_1/\sqrt{\delta}$, so (7.41) and $\mathbb{E}[\widehat{Z}_{s,n}(0)^{-1}] \geqslant 1$ give the required comparison.

For $s < n_0$, averaging over the disorder on $(n_0, n]$ and applying a conditional Jensen's inequality gives

$$\mathbb{E}[\widehat{Z}_{s,n}(0)^{-1}] \geqslant \mathbb{E}[\widehat{Z}_{s,n_0}(0)^{-1}] \geqslant c\,\mathbb{E}[\widehat{Z}_{s,n_0}(0)^2]$$

where the second inequality follows from the small-time comparison (4.19) proven above. Using the bound (7.40), we obtain

$$\mathbb{E}[\widehat{Z}_{s,n_0}(0)^2] \geqslant \frac{1}{2}\frac{\vartheta_N(s)}{\vartheta_N(n_0)} \geqslant c_{T_0,T_1}\left\{1 + \log\frac{n}{s \vee 1}\right\}, \tag{7.42}$$

where the last inequality holds thanks to (7.38), which shows in particular that

$$\vartheta_N(n_0) = \frac{1}{2}|\log\delta| + O(1) \leqslant C_{T_0}, \quad \vartheta_N(s) = \log\frac{N}{s \vee 1} + O(1) \geqslant c_{T_0,T_1}\left\{1 + \log\frac{n}{s \vee 1}\right\}.$$

Combining (7.42) with (7.41) proves (4.18). This concludes the proof of Proposition 4.1. □

7.5. Comparing the endpoint distributions: proof of Proposition 4.4. The main goal of this section is to prove Proposition 4.4. It relies on estimating quantitatively the difference between the starting distributions

$$\mu_n = \mu_{N_0,N_b}^{(q_{N_0})}, \qquad \widehat{\mu}_{m,n} = \mu_{N_a,N_b}^{(q_{N_a})}.$$

If $\eta = \nu - \nu'$ is a signed measure of total mass zero which is $\mathcal{F}_n$-measurable, we define

$$\|\eta\|_{\mathcal{H}_{n,N}}^2 := \mathbb{E}\big[Z_{n,N}(\eta)^2 \big| \mathcal{F}_n\big]. \tag{7.43}$$

Then, by using the conditional Cauchy–Schwarz inequality, we get

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{G}}|Z_{n,N}(\mu_n) - Z_{n,N}(\widehat{\mu}_{m,n})|\big] \leqslant \mathbb{E}\big[\mathbf{1}_{\mathcal{G}}\|\mu_n - \widehat{\mu}_{m,n}\|_{\mathcal{H}_{n,N}}\big], \tag{7.44}$$

for any $\mathcal{F}_n$-measurable event $\mathcal{G}$. Thus, (4.26) reduces to the control of $\|\mu_n - \widehat{\mu}_{m,n}\|_{\mathcal{H}_{n,N}}$.

To this end, in analogy with Subsection 7.2, we perform an expansion in terms of the endpoint distributions, since

$$\mu_{N_{r-1},N_s}^{(q_{N_{r-1}})} = \frac{q_{N_{r-1}} \lhd Z_{N_{r-1},N_s}}{q_{N_{r-1}} \lhd Z_{N_{r-1},N_s} \rhd \mathbf{1}} = \frac{q_{N_{r-1}} \lhd \mathsf{Z}_r \circ \cdots \circ \mathsf{Z}_s}{q_{N_{r-1}} \lhd Z_{N_{r-1},N_s} \rhd \mathbf{1}}, \tag{7.45}$$

which is a (random) measure. Recall the definition

$$\mathscr{R}_{r,s} := q_{N_{r-1}} \lhd \breve{\mathsf{Z}}_r \circ \cdots \circ \breve{\mathsf{Z}}_s \rhd \mathbf{1} = \frac{q_{N_{r-1}} \lhd Z_{N_{r-1},N_s} \rhd \mathbf{1}}{\prod_{i=r}^{s} q_{N_{i-1}} \lhd \mathsf{Z}_i \rhd \mathbf{1}}. \tag{7.46}$$

Dividing the numerator and denominator in (7.45) by $\prod_{i=r}^{s} q_{N_{i-1}} \triangleleft \mathsf{Z}_i \triangleright \mathbf{1}$ we have

$$\mu^{(q_{N_{r-1}})}_{N_{r-1},N_s} = \frac{1}{\mathscr{R}_{r,s}} q_{N_{r-1}} \triangleleft \check{\mathsf{Z}}_r \circ \cdots \circ \check{\mathsf{Z}}_s .$$

Using

$$\check{\mathsf{Z}}_i = \check{D}_i + \frac{\mathsf{Z}_i \triangleright \mathbf{1} \otimes q_{N_i} \triangleleft}{q_{N_{i-1}} \triangleleft \mathsf{Z}_i \triangleright \mathbf{1}} = \check{D}_i + \check{\mathsf{Z}}_i \triangleright \mathbf{1} \otimes q_{N_i} \triangleleft \tag{7.47}$$

(recall (7.13) for notational clarity), we get that

$$\mu^{(q_{N_{r-1}})}_{N_{r-1},N_s} = q_{N_s} + \frac{1}{\mathscr{R}_{r,s}} \sum_{u=r}^{s} \mathscr{R}_{r,u-1} \mathscr{D}^{\mathrm{open}}_{u,s} , \tag{7.48}$$

where we have introduced the following open corrector (which is a signed measure):

$$\mathscr{D}^{\mathrm{open}}_{u,s} := q_{N_{u-1}} \triangleleft \check{D}_u \circ \cdots \circ \check{D}_s . \tag{7.49}$$

Indeed, we can successively replace operators $\check{\mathsf{Z}}_i$ from $i = s, s-1, ..., r$ by the decomposition (7.47) as (recall also (7.46)):

$$\begin{aligned}
\mu^{(q_{N_{r-1}})}_{N_{r-1},N_s} &= \frac{1}{\mathscr{R}_{r,s}} q_{N_{r-1}} \triangleleft \check{\mathsf{Z}}_r \circ \cdots \circ \check{\mathsf{Z}}_{s-1} \circ \left\{ \check{D}_s + \check{\mathsf{Z}}_s \triangleright \mathbf{1} \times q_{N_s} \triangleleft \right\} \\
&= \frac{1}{\mathscr{R}_{r,s}} q_{N_{r-1}} \triangleleft \check{\mathsf{Z}}_r \circ \cdots \circ \check{\mathsf{Z}}_{s-1} \circ \check{\mathsf{Z}}_s \triangleright \mathbf{1} \times q_{N_s} + \frac{1}{\mathscr{R}_{r,s}} q_{N_{r-1}} \triangleleft \check{\mathsf{Z}}_r \circ \cdots \circ \check{\mathsf{Z}}_{s-1} \circ \check{D}_s \\
&= q_{N_s} + \frac{1}{\mathscr{R}_{r,s}} q_{N_{r-1}} \triangleleft \check{\mathsf{Z}}_r \circ \cdots \circ \left\{ \check{\mathsf{D}}_{s-1} + \check{\mathsf{Z}}_{s-1} \triangleright \mathbf{1} \times q_{N_{s-1}} \triangleleft \right\} \circ \check{D}_s \\
&= q_{N_s} + \frac{\mathscr{R}_{r,s-1}}{\mathscr{R}_{r,s}} q_{N_{s-1}} \triangleleft \check{D}_s + \frac{1}{\mathscr{R}_{r,s}} q_{N_{r-1}} \triangleleft \check{\mathsf{Z}}_r \circ \cdots \circ \check{\mathsf{Z}}_{s-2} \circ \check{D}_{s-1} \circ \check{D}_s
\end{aligned}$$

and continuing these successive replacements we arrive at (7.48).

Let us also introduce the following corrector, which is the leading (linear) contribution of the corrector in (7.48),

$$\Delta\mu^{\mathrm{open}}_{r,s} := \sum_{u=r}^{s} \mathscr{D}^{\mathrm{open}}_{u,s} . \tag{7.50}$$

We therefore rely on the following lemma, whose proof is postponed to the end of this subsection. We state it for arbitrary consecutive blocks, since the proof of Proposition 4.5 will also use it in this generality. Recalling the definition (7.9) of the event $\Omega_{r,s}$, we define

$$\mathcal{G}_{r,s} := \Omega_{r,s} \cap \{ |\mathscr{R}_{r,s} - 1| \leqslant 1/2 \} . \tag{7.51}$$

Lemma 7.7 (Profile approximation and second moments)**.** *Let ℓ_* be large enough and $1 \leqslant r \leqslant s \leqslant K - \ell_*$. Then, recalling the definition* (7.51)*, we have*

$$\mathbb{P}(\mathcal{G}^c_{r,s}) \leqslant \frac{C}{K-s} , \tag{7.52}$$

and also

$$\mathbb{E}\Big[\mathbf{1}_{\mathcal{G}_{r,s}} \| \mu^{(q_{N_{r-1}})}_{N_{r-1},N_s} - q_{N_s} - \Delta\mu^{\mathrm{open}}_{r,s} \|_{\mathcal{H}_{N_s,N}} \Big] \leqslant \frac{C}{K-s} . \tag{7.53}$$

Proof of Proposition 4.4. We will prove the proposition with the choice

$$\mathcal{G} = \mathcal{G}_{N,\delta}(0) := \Omega_{1,b} \cap \{|\mathscr{R}_{1,b} - 1| \leqslant 1/2\} \cap \{|\mathscr{R}_{a+1,b} - 1| \leqslant 1/2\}. \tag{7.54}$$

It is the intersection of the two good events in (7.51), so $\mathbb{P}(\mathcal{G}^c) \leqslant C/J$ by Lemma 7.7, since $K - b = J$. This shows (4.25), recalling that $J \asymp |\log \delta|$, see Remark 7.2.

In view of (7.44), we need to control $\|\mu_n - \widehat{\mu}_{m,n}\|_{\mathcal{H}_{n,N}}$. In the difference of the two sums defined by (7.50), the terms with $u \geqslant a+1$ cancel, giving

$$\Delta\mu^{\mathrm{open}}_{1,b} - \Delta\mu^{\mathrm{open}}_{a+1,b} = \sum_{u=1}^{a} \mathscr{D}^{\mathrm{open}}_{u,b}. \tag{7.55}$$

By (7.7) and (7.49),

$$Z_{N_s,N}(\mathscr{D}^{\mathrm{open}}_{u,s}) = \frac{q_{N_{u-1}} \triangleleft D_u \circ \cdots \circ D_s \circ Z_{N_s,N} \triangleright \mathbf{1}}{\prod_{i=u}^{s} \mathsf{Z}_i(q_{N_{i-1}})}.$$

On $\Omega_{u,s}$, each inverse factor is at most 2, so (7.43) and Proposition 7.4-(7.18), with $v = s+1$, give for $1 \leqslant u \leqslant s \leqslant K-2$

$$\mathbb{E}\big[\mathbf{1}_{\Omega_{u,s}} \|\mathscr{D}^{\mathrm{open}}_{u,s}\|_{\mathcal{H}_{N_s,N}}\big] \leqslant \mathbb{E}\big[\mathbf{1}_{\Omega_{u,s}} \|\mathscr{D}^{\mathrm{open}}_{u,s}\|^2_{\mathcal{H}_{N_s,N}}\big]^{1/2} \leqslant \frac{C(2\varepsilon_\rho)^{s-u}}{\sqrt{K-u}}. \tag{7.56}$$

Since $K - b = J$, $b - a = J$, and $2\varepsilon_\rho < 1$ for large ρ, using (7.55) and the triangle inequality gives

$$\mathbb{E}\big[\mathbf{1}_{\Omega_{1,b}} \|\Delta\mu^{\mathrm{open}}_{1,b} - \Delta\mu^{\mathrm{open}}_{a+1,b}\|^2_{\mathcal{H}_{N_b,N}}\big]^{1/2} \leqslant \sum_{i=J}^{\infty} \frac{C(2\varepsilon_\rho)^i}{\sqrt{J+i}} \leqslant \frac{C(2\varepsilon_\rho)^J}{\sqrt{J}}. \tag{7.57}$$

Therefore, if we apply the triangle inequality

$$\begin{aligned}\|\mu_n - \widehat{\mu}_{m,n}\|_{\mathcal{H}_{n,N}} \leqslant \|\mu_n - q_{N_b} - \Delta\mu^{\mathrm{open}}_{1,b}\|_{\mathcal{H}_{n,N}} + \|\widehat{\mu}_{m,n} - q_{N_b} - \Delta\mu^{\mathrm{open}}_{a+1,b}\|_{\mathcal{H}_{n,N}} \\ + \|\Delta\mu^{\mathrm{open}}_{1,b} - \Delta\mu^{\mathrm{open}}_{a+1,b}\|_{\mathcal{H}_{n,N}},\end{aligned}$$

together with (7.53) (once with $(r,s) = (1,b)$ and once with $(r,s) = (a+1,b)$, recalling that $K - b = J$) and the above (7.57), we finally obtain

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{G}} \|\mu_n - \widehat{\mu}_{m,n}\|_{\mathcal{H}_{n,N}}\big] \leqslant \frac{C}{J} + \frac{C(2\varepsilon_\rho)^J}{\sqrt{J}} \leqslant \frac{C}{J}. \tag{7.58}$$

Recalling (7.44), this proves (4.26), using again that $J \asymp |\log \delta|$.

We next bound the second moment of $Z_{n,N}(\nu)$ as claimed in (4.27) for $\nu = \mu_n, \widehat{\mu}_{m,n}$. Recall (7.48), which provides an expansion of $\mu_n = \mu^{(q_{N_0})}_{N_0,N_b}$ and $\widehat{\mu}_{m,n} = \mu^{(q_{N_a})}_{N_a,N_b}$; let us denote $o = 1$ if $\nu = \mu_n$ and $o = a+1$ if $\nu = \widehat{\mu}_{m,n}$. Then, using the inequality $(x+y)^2 \leqslant 2x^2 + 2y^2$, we get that

$$\mathbb{E}\big[Z_{n,N}(\nu)^2 \mid \mathcal{F}_n\big] \leqslant 2\mathbb{E}\big[Z_{n,N}(q_{N_b})^2 \mid \mathcal{F}_n\big] + \frac{2}{(\mathscr{R}_{o,b})^2}\mathbb{E}\big[Z_{n,N}(\Delta\mu^{\mathrm{weighted}}_{o,b})^2 \mid \mathcal{F}_n\big],$$

where we have defined the signed measure

$$\Delta\mu^{\mathrm{weighted}}_{r,s} := \sum_{u=r}^{s} \mathscr{R}_{r,u-1} \mathscr{D}^{\mathrm{open}}_{u,s}.$$

Since $\mathscr{R}_{o,b} \geqslant 1/2$ on the event $\mathcal{G} \in \mathcal{F}_n$, we thus get that

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{G}} Z_{n,N}(\nu)^2\big] \leqslant 2\mathbb{E}\big[Z_{n,N}(q_{N_b})^2\big] + 8\mathbb{E}\big[\mathbf{1}_{\Omega_{o,b}} \|\Delta\mu^{\mathrm{weighted}}_{o,b}\|^2_{\mathcal{H}_{n,N}}\big],$$

where we have used the definition (7.43) of the semi-norm $\|\cdot\|_{\mathcal{H}_{n,N}}$; note that $\Delta\mu_{o,b}^{\text{weighted}}$ is a signed measure of zero total mass and is $\mathcal{F}_n$-measurable.

Now, we can apply (4.4) to bound

$$\mathbb{E}\big[Z_{n,N}(q_{N_b})^2\big] = \mathbb{E}\big[\widehat{Z}_{n,N}(0)^2\big] \leqslant 1 + C\,\frac{\log(N/n)}{\log 2} \leqslant C\,(K-b)\,.$$

For the remaining term, we may use the bound

$$\mathbb{E}\big[\mathbf{1}_{\Omega_{o,b}}\|\Delta\mu_{o,b}^{\text{weighted}}\|_{\mathcal{H}_{n,N}}^2\big] \leqslant \frac{C}{K-b}\,, \tag{7.59}$$

whose proof is postponed to the proof of Lemma 7.7 below. Recalling that $K-b=J$, we thus have obtained that, for $\nu=\mu_n$ and $\widehat{\mu}_{m,n}$,

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{G}}Z_{n,N}(\nu)^2\big] \leqslant CJ + C/J\,,$$

which concludes the proof of (4.27), recalling that $J \asymp |\log\delta|$.

It remains to prove (4.28) for a bounded deterministic terminal function f. Writing $\Delta\mu = \mu_n - \widehat{\mu}_{m,n}$, we will control the fluctuations around the conditional mean using the bound (7.58) on $\|\Delta\mu\|_{\mathcal{H}_{n,N}}$, defined in (7.43). This need not vanish for a general f, so we will need to compare the endpoint distributions obtained by evolving μ_n and $\widehat{\mu}_{m,n}$ from time n to time N without disorder.

We first compute the conditional mean and variance of $Z_{n,N}(\Delta\mu, f)$. Averaging over the disorder on $(n,N]$ gives the conditional mean, while the first-collision calculation in (4.2) gives the conditional variance:

$$\begin{aligned}\mathbb{E}[Z_{n,N}(\Delta\mu,f)\mid\mathcal{F}_n] &= \sum_y (\Delta\mu * q_{N-n})(y) f(y),\\ \operatorname{Var}(Z_{n,N}(\Delta\mu,f)\mid\mathcal{F}_n) &= \sigma_N^2 \sum_{r=1}^{N-n}\sum_w (\Delta\mu * q_r)(w)^2\,\mathbb{E}[Z_{n+r,N}(w,f)^2].\end{aligned} \tag{7.60}$$

For $f=\mathbf{1}$, the conditional mean vanishes because $\Delta\mu(\mathbf{1})=0$, so the conditional variance equals $\|\Delta\mu\|_{\mathcal{H}_{n,N}}^2$ by (7.43). Since $|Z_{n+r,N}(w,f)| \leqslant \|f\|_\infty Z_{n+r,N}(w,\mathbf{1})$, we have

$$\mathbb{E}[Z_{n+r,N}(w,f)^2] \leqslant \|f\|_\infty^2 \mathbb{E}[Z_{n+r,N}(w,\mathbf{1})^2]\,,$$

which plugged into the variance formula in (7.60) gives

$$\operatorname{Var}(Z_{n,N}(\Delta\mu,f)\mid\mathcal{F}_n) \leqslant \|f\|_\infty^2 \operatorname{Var}(Z_{n,N}(\Delta\mu,\mathbf{1})\mid\mathcal{F}_n) = \|f\|_\infty^2\|\Delta\mu\|_{\mathcal{H}_{n,N}}^2.$$

By the triangle inequality and conditional Cauchy–Schwarz,

$$\mathbb{E}[|Z_{n,N}(\Delta\mu,f)|\mid\mathcal{F}_n] \leqslant \big|\mathbb{E}[Z_{n,N}(\Delta\mu,f)\mid\mathcal{F}_n]\big| + \operatorname{Var}(Z_{n,N}(\Delta\mu,f)\mid\mathcal{F}_n)^{1/2}\,.$$

Using the mean formula above, multiplying by $\mathbf{1}_{\mathcal{G}}$, and applying (7.58), we obtain

$$\mathbb{E}[\mathbf{1}_{\mathcal{G}}|Z_{n,N}(\Delta\mu,f)|] \leqslant \|f\|_\infty \left\{\frac{C}{|\log\delta|} + \mathbb{E}[\mathbf{1}_{\mathcal{G}}\|\Delta\mu * q_{N-n}\|_{\ell^1}]\right\}. \tag{7.61}$$

We now estimate the last term. Fix $R\geqslant 1$ and let $B=\{y\in\mathbb{Z}^2 : |y|\leqslant R\sqrt{N}\}$. Recall from (2.23), with $z=0$, that μ_n is the endpoint distribution at time n, while $\widehat{\mu}_{m,n}$ is the endpoint distribution obtained by switching off the disorder on $(0,m]$. Both are probability measures, so their convolutions with q_{N-n} are nonnegative and have total mass one.

Since $\mu_n * q_{N-n}$ and $\widehat{\mu}_{m,n} * q_{N-n}$ both have total mass one, their difference $\Delta\mu * q_{N-n}$ has total mass zero. Using the triangle inequality on B^c, and then expressing the mass of this difference on B^c as minus its mass on B, we obtain

$$\begin{aligned}\sum_{y\in B^c} |(\Delta\mu * q_{N-n})(y)| &\leqslant (\mu_n * q_{N-n})(B^c) + (\widehat{\mu}_{m,n} * q_{N-n})(B^c)\\ &= 2(\widehat{\mu}_{m,n} * q_{N-n})(B^c) - \sum_{y\in B}(\Delta\mu * q_{N-n})(y)\\ &\leqslant 2(\widehat{\mu}_{m,n} * q_{N-n})(B^c) + \sum_{y\in B}|(\Delta\mu * q_{N-n})(y)|.\end{aligned}$$

Adding the sum over B gives

$$\|\Delta\mu * q_{N-n}\|_{\ell^1} \leqslant 2\sum_{y\in B}|(\Delta\mu * q_{N-n})(y)| + 2(\widehat{\mu}_{m,n} * q_{N-n})(B^c). \tag{7.62}$$

Consider first the sum over B. Since $|B| \leqslant CR^2N$, Cauchy–Schwarz gives

$$\sum_{y\in B}|(\Delta\mu * q_{N-n})(y)| \leqslant CR\sqrt{N}\|\Delta\mu * q_{N-n}\|_{\ell^2}. \tag{7.63}$$

To bound the right hand side, we use the variance identity (7.60) with $f = \mathbf{1}$:

$$\|\Delta\mu\|^2_{\mathcal{H}_{n,N}} = \sum_{r=1}^{N-n} \sigma_N^2\mathbb{E}[Z_{N-n-r}(0)^2]\,\|\Delta\mu * q_r\|^2_{\ell^2}.$$

For a fixed sufficiently small $\varepsilon_0 \in (0, 1/4)$, Claim B.1 gives $\sigma_N^2\mathbb{E}[Z_{[\varepsilon_0 N]}(0)^2] \geqslant c > 0$ for all sufficiently large N. For small δ, we have $n \leqslant N/4$. Thus, whenever $N/4 \leqslant r \leqslant N/2$, monotonicity of the second moment gives $\sigma_N^2\mathbb{E}[Z_{N-n-r}(0)^2] \geqslant c$. Also, $\|\Delta\mu * q_r\|_{\ell^2} \geqslant \|\Delta\mu * q_{N-n}\|_{\ell^2}$, because convolution by q_{N-n-r} is a contraction in ℓ^2. Restricting the sum to these r, of which there are at least $N/4$, yields

$$\|\Delta\mu\|^2_{\mathcal{H}_{n,N}} \geqslant cN\|\Delta\mu * q_{N-n}\|^2_{\ell^2}. \tag{7.64}$$

Combining (7.63) with (7.64) and then applying (7.58), we obtain

$$\mathbb{E}\Big[\mathbf{1}_{\mathcal{G}}\sum_{y\in B}|(\Delta\mu * q_{N-n})(y)|\Big] \leqslant CR\,\mathbb{E}[\mathbf{1}_{\mathcal{G}}\|\Delta\mu\|_{\mathcal{H}_{n,N}}] \leqslant \frac{CR}{|\log\delta|}.$$

We next bound the expectation of the tail $(\widehat{\mu}_{m,n} * q_{N-n})(B^c)$ in (7.62). Introduce the abbreviations

$$Z_1 = \widehat{Z}_{m,n}(0), \qquad Z_2 = Z_{m,n}(q_m, q_{N-n} * \mathbf{1}_{B^c}).$$

By definition and by averaging over the disorder,

$$(\widehat{\mu}_{m,n} * q_{N-n})(B^c) = \frac{Z_2}{Z_1}, \qquad 0 \leqslant Z_2 \leqslant Z_1, \qquad \mathbb{E}[Z_2] = q_N(B^c).$$

By (3.5) and (4.3) with $(s, n) = (m, n)$, we have $\mathbb{E}[Z_1^{-1}] \leqslant \mathbb{E}[Z_1^2] \leqslant C$, uniformly for small δ and sufficiently large N. Applying Cauchy–Schwarz to $Z_2/Z_1 = \sqrt{Z_2}\,(\sqrt{Z_2}/Z_1)$, then using $Z_2 \leqslant Z_1$ and the Gaussian tail bound for q_N, gives

$$\mathbb{E}[(\widehat{\mu}_{m,n} * q_{N-n})(B^c)] = \mathbb{E}\left[\frac{Z_2}{Z_1}\right] \leqslant \left(\mathbb{E}[Z_2]\,\mathbb{E}\left[\frac{Z_2}{Z_1^2}\right]\right)^{1/2} \leqslant (\mathbb{E}[Z_2]\,\mathbb{E}[Z_1^{-1}])^{1/2} \leqslant C\mathrm{e}^{-cR^2}.$$

Substituting the estimates on B and B^c into (7.62), after multiplying by $\mathbf{1}_{\mathcal{G}}$ and taking expectations, we obtain

$$\mathbb{E}[\mathbf{1}_{\mathcal{G}}\|\Delta\mu * q_{N-n}\|_{\ell^1}] \leqslant C\left(\frac{R}{|\log\delta|} + \mathrm{e}^{-cR^2}\right).$$

Taking $R = A\sqrt{\log|\log\delta|}$ for large $A > 0$ and substituting into (7.61) proves (4.28). □

Proof of Lemma 7.7. For the first estimate (7.52), we rely on Proposition 7.3. In particular, (7.11) shows that uniformly for $1 \leqslant r \leqslant s \leqslant K-1$ we have $\mathbb{P}(\Omega_{r,s}^c) \leqslant C/(K-s)$ and also

$$\mathbb{P}(|\mathscr{R}_{r,s} - 1| > 1/2) \leqslant \mathbb{P}(\Omega_{r,s}^c) + 4\mathbb{E}[\mathbf{1}_{\Omega_{r,s}}|\mathscr{R}_{r,s} - 1|^2] \leqslant C/(K-s),$$

by Markov's inequality. Recalling the definition (7.51) of $\mathcal{G}_{r,s}$, this proves (7.52).

Before we prove (7.53), let us show the following auxiliary bound (we also prove (7.59) at the same time):

$$\|\mathbf{1}_{\Omega_{r,s}}\Delta\mu_{r,s}^{\mathrm{open}}\|_{L^2(\mathbb{P},\mathcal{H}_{N_s,N})} := \mathbb{E}\big[\mathbf{1}_{\Omega_{r,s}}\|\Delta\mu_{r,s}^{\mathrm{open}}\|_{\mathcal{H}_{N_s,N}}^2\big]^{1/2} \leqslant \frac{C}{\sqrt{K-s}}. \tag{7.65}$$

Recalling the definition (7.50) of $\Delta\mu_{r,s}^{\mathrm{open}}$, we get by Minkowski's inequality that

$$\|\mathbf{1}_{\Omega_{r,s}}\Delta\mu_{r,s}^{\mathrm{open}}\|_{L^2(\mathbb{P},\mathcal{H}_{N_s,N})} \leqslant C\sum_{u=r}^{s}\|\mathbf{1}_{\Omega_{u,s}}\mathscr{D}_{u,s}^{\mathrm{open}}\|_{L^2(\mathbb{P},\mathcal{H}_{N_s,N})} \leqslant C\sum_{u=r}^{s}\frac{(2\varepsilon_\rho)^{s-u}}{\sqrt{K-u}},$$

where the last inequality follows from (7.56). This proves (7.65). In order to prove (7.59), *i.e.* the same result with $\Delta\mu_{r,s}^{\mathrm{weighted}}$, note that

$$\mathbf{1}_{\Omega_{r,s}}\Delta\mu_{r,s}^{\mathrm{weighted}} = \sum_{u=r}^{s}\mathbf{1}_{\Omega_{r,u-1}}\mathscr{R}_{r,u-1}\mathbf{1}_{\Omega_{u,s}}\mathscr{D}_{u,s}^{\mathrm{open}},$$

with $\mathbf{1}_{\Omega_{r,u-1}}\mathscr{R}_{r,u-1}$ and $\mathbf{1}_{\Omega_{u,s}}\mathscr{D}_{u,s}^{\mathrm{open}}$ depending on disjoint blocks. By independence, we thus have that

$$\big\|\mathbf{1}_{\Omega_{r,s}}\Delta\mu_{r,s}^{\mathrm{weighted}}\big\|_{L^2(\mathbb{P},\mathcal{H}_{N_s,N})} \leqslant C\sum_{u=r}^{s}\|\mathbf{1}_{\Omega_{r,u-1}}\mathscr{R}_{r,u-1}\|_{L^2}\big\|\mathbf{1}_{\Omega_{u,s}}\mathscr{D}_{u,s}^{\mathrm{open}}\big\|_{L^2(\mathbb{P},\mathcal{H}_{N_s,N})}.$$

Now, we have that $\|\mathbf{1}_{\Omega_{r,u-1}}\mathscr{R}_{r,u-1}\|_{L^2} \leqslant C$ by Proposition 7.3, see (7.11), so the same geometric sum as above shows that

$$\|\mathbf{1}_{\Omega_{r,s}}\Delta\mu_{r,s}^{\mathrm{weighted}}\|_{L^2(\mathbb{P},\mathcal{H}_{N_s,N})} = \mathbb{E}\big[\mathbf{1}_{\Omega_{r,s}}\|\Delta\mu_{r,s}^{\mathrm{weighted}}\|_{\mathcal{H}_{N_s,N}}^2\big]^{1/2} \leqslant \frac{C}{\sqrt{K-s}},$$

which is exactly (7.59).

Going back to the proof of (7.53), we use the expansion (7.48) to write the exact identity

$$\mu_{N_{r-1},N_s}^{(q_{N_{r-1}})} - q_{N_s} - \Delta\mu_{r,s}^{\mathrm{open}} = \frac{1}{\mathscr{R}_{r,s}}\sum_{u=r}^{s}(\mathscr{R}_{r,u-1} - 1)\mathscr{D}_{u,s}^{\mathrm{open}} + \frac{1-\mathscr{R}_{r,s}}{\mathscr{R}_{r,s}}\Delta\mu_{r,s}^{\mathrm{open}}. \tag{7.66}$$

Now, on $\mathcal{G}_{r,s}$, the denominators are at least $1/2$. We thus have that

$$\begin{aligned}&\mathbb{E}\big[\mathbf{1}_{\mathcal{G}_{r,s}}\|\mu_{N_{r-1},N_s}^{(q_{N_{r-1}})} - q_{N_s} - \Delta\mu_{r,s}^{\mathrm{open}}\|_{\mathcal{H}_{N_s,N}}\big]\\ &\leqslant 2\sum_{u=r}^{s}\mathbb{E}\big[\mathbf{1}_{\Omega_{r,s}}|\mathscr{R}_{r,u-1} - 1|\,\|\mathscr{D}_{u,s}^{\mathrm{open}}\|_{\mathcal{H}_{N_s,N}}\big] + 2\mathbb{E}\big[\mathbf{1}_{\Omega_{r,s}}|\mathscr{R}_{r,s} - 1|\cdot\|\Delta\mu_{r,s}^{\mathrm{open}}\|_{\mathcal{H}_{N_s,N}}\big].\end{aligned}$$

For $u > r$, using the independence of the blocks, we get

$$\mathbb{E}\big[\mathbf{1}_{\Omega_{r,u-1}}|\mathscr{R}_{r,u-1}-1|\mathbf{1}_{\Omega_{u,s}}\|\mathscr{D}^{\mathrm{open}}_{u,s}\|_{\mathcal{H}_{N_s,N}}\big] \leqslant \frac{C}{\sqrt{K-u}}\,\frac{(2\varepsilon_\rho)^{s-u}}{\sqrt{K-u}}\,,$$

where we have also used that $\mathbb{E}[\mathbf{1}_{\Omega_{r,u-1}}|\mathscr{R}_{r,u-1}-1|] \leqslant C/(K-u+1)^{1/2}$ thanks to Proposition 7.3, see (7.11) and Cauchy–Schwarz's inequality, together with (7.56). Note that the term with $u = r$ vanishes because $\mathscr{R}_{r,r-1} = 1$. On the other hand, using Cauchy–Schwarz, we also have that

$$\mathbb{E}\big[\mathbf{1}_{\Omega_{r,s}}|\mathscr{R}_{r,s}-1|\mathbf{1}_{\Omega_{r,s}}\|\Delta\mu^{\mathrm{open}}_{r,s}\|_{\mathcal{H}_{N_s,N}}\big] \leqslant \frac{C}{\sqrt{K-s}} \times \frac{C}{\sqrt{K-s}}\,,$$

where we have used Proposition 7.3, see (7.11), and (7.65). We thus get that

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{G}_{r,s}}\|\mu^{(q_{N_{r-1}})}_{N_{r-1},N_s} - q_{N_s} - \Delta\mu^{\mathrm{open}}_{r,s}\|_{\mathcal{H}_{N_s,N}}\big] \leqslant \sum_{u=r}^{s}\frac{C(2\varepsilon_\rho)^{s-u}}{K-u} + \frac{C}{K-s}\,,$$

which concludes the proof of (7.53). □

7.6. Logarithmic moment bounds: proof of Proposition 4.5. Recall the two ratios

$$Z_{n,N}(\mu^{(x)}_n) = \frac{Z_N(x)}{Z_n(x)}, \qquad Z_{n,N}(\widehat{\mu}^{(x)}_{m,n}) = \frac{\widehat{Z}_{m,N}(x)}{\widehat{Z}_{m,n}(x)}\,.$$

Our goal is to control the L^p norm of their logarithm. (Again, by stationarity, we may only treat the case $x = 0$ and we drop it from the notation.) For this, we rely on the following log-tail estimate, which we prove below. Recall that $n = N_b$ with $K - b = J$.

Lemma 7.8. *There is a constant $C > 0$ such that, for $\nu = \mu_n$ or $\nu = \widehat{\mu}_{m,n}$,*

$$\mathbb{P}\big(|\log Z_{n,N}(\nu)| > u\big) \leqslant \min\{1, C\sqrt{J}\mathrm{e}^{-u/3}\} \qquad \textit{for all } u \geqslant 0\,.$$

Let us first conclude the proof of Proposition 4.5 thanks to this lemma.

Proof of Proposition 4.5. We start with the integral formula for the p-th moment

$$\mathbb{E}\big[|\log Z_{n,N}(\nu)|^p\big] = p\int_0^\infty u^{p-1}\mathbb{P}\big(|\log Z_{n,N}(\nu)| > u\big)\,\mathrm{d}u\,.$$

We now use Lemma 7.8, and we split the integral into a part with $u \leqslant C\log J$ where we bound the probability by 1 and a part with $u \geqslant C\log J$ where we use the bound $C\sqrt{J}\mathrm{e}^{-u/3}$. This gives

$$\mathbb{E}\big[|\log Z_{n,N}(\nu)|^p\big] \leqslant C_p(\log J)^p + C_p\sqrt{J}(\log J)^{p-1}\mathrm{e}^{-C\log J/3} \leqslant C'_p(\log J)^p\,,$$

where the second inequality is valid when C is large enough, and thus $\|\log Z_{n,N}(\nu)\|_{L^p} \leqslant C_p\log J$. Since $J \asymp |\log\delta|$, this proves (4.29).

For a positive terminal function f, the bounds $\mathrm{e}^{-\|\log f\|_\infty} \leqslant f \leqslant \mathrm{e}^{\|\log f\|_\infty}$ give

$$|\log Z_{n,N}(\nu, f) - \log Z_{n,N}(\nu)| \leqslant \|\log f\|_\infty.$$

The claimed extension follows by the triangle inequality in L^p. □

Proof of Lemma 7.8. First of all, notice that $\mathbb{E}[Z_{n,N}(\nu) \mid \mathcal{F}_n] = 1$. Hence, by taking the exponential and using the conditional mean, we get that for any $u \geqslant 0$,

$$\mathbb{P}\big(\log Z_{n,N}(\nu) > u\big) \leqslant \mathrm{e}^{-u}\,,$$

and we thus need to focus on the left tail of $\log Z_{n,N}(\nu)$. We do this by estimating their negative moments, for which the Gaussian convexity inequalities are central.

The case $\nu = \widehat{\mu}_{m,n}$. Writing $Z_{n,N}(\widehat{\mu}_{m,n}) = \widehat{Z}_{m,N}(0)/\widehat{Z}_{m,n}(0)$ and applying the Cauchy–Schwarz's inequality, we get

$$\begin{aligned}\mathbb{E}\big[Z_{n,N}(\widehat{\mu}_{m,n})^{-1/2}\big] &= \mathbb{E}\big[\widehat{Z}_{m,N}(0)^{-1/2}\widehat{Z}_{m,n}(0)^{1/2}\big] \\ &\leqslant \mathbb{E}[\widehat{Z}_{m,n}(0)]^{1/2}\mathbb{E}[\widehat{Z}_{m,N}(0)^{-1}]^{1/2} \leqslant \mathbb{E}[\widehat{Z}_{m,N}(0)^2]^{1/2},\end{aligned}$$

where we have used (3.5) with $p = 1$ for the last inequality.

Now, recalling that $m = N_a$ with $a = K - 2J$, a rough second moment computation, see e.g. (4.4), gives the following

$$\mathbb{E}[\widehat{Z}_{m,N}(0)^2] \leqslant C\log(N/m) \leqslant CJ\,.$$

We therefore end up with $\mathbb{E}[Z_{n,N}(\widehat{\mu}_{m,n})^{-1/2}] \leqslant C\sqrt{J}$ so that Markov's inequality yields, for $u \geqslant 0$,

$$\mathbb{P}\big(Z_{n,N}(\widehat{\mu}_{m,n}) < \mathrm{e}^{-u}\big) \leqslant C\sqrt{J}\,\mathrm{e}^{-u/2}, \tag{7.67}$$

which is what is claimed in Lemma 7.8.

The case $\nu = \mu_n$. Applying directly the same argument as above shows that

$$\mathbb{E}\big[Z_{n,N}(\mu_n)^{-1/2}\big] \leqslant \mathbb{E}\big[Z_N(0)^2\big]^{1/2} \leqslant C\sqrt{\log N}\,,$$

where the last inequality follows from (4.1) and $\sigma_N^2 \asymp 1/\log N$. Using Markov's inequality, this gives the bound

$$\mathbb{P}\big(Z_{n,N}(\mu_n) < \mathrm{e}^{-u}\big) \leqslant C\sqrt{\log N}\,\mathrm{e}^{-u/2}\,.$$

This is not a satisfying bound, except when u is large. In fact, we will use this bound when $\lceil J\mathrm{e}^{u/3}\rceil \geqslant K \geqslant c\log N$, in which case we obtain

$$\mathbb{P}\big(Z_{n,N}(\mu_n) < \mathrm{e}^{-u}\big) \leqslant C\sqrt{J}\,\mathrm{e}^{-u/3}\,. \tag{7.68}$$

Let us now assume that $u \geqslant 0$ is such that $\lceil J\mathrm{e}^{u/3}\rceil \leqslant K - 1$ and let us introduce the intermediate time $N_1 \leqslant M = N_s \leqslant n = N_b$ with $s := K - \lceil J\mathrm{e}^{u/3}\rceil \in \{1,\dots,b\}$. The reason for this specific choice of s will be clear below, see e.g. (7.70). Then, for any event $\mathcal{G}_M \in \mathcal{F}_M$, applying Cauchy–Schwarz's inequality and (3.5) with $p = 1$, conditionally on $\mathcal{F}_M$, we get analogously as above that

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{G}_M} Z_{n,N}(\mu_n)^{-1/2}\big] = \mathbb{E}\bigg[\mathbf{1}_{\mathcal{G}_M}\Big(\frac{Z_n(0)}{Z_M(0)}\Big)^{1/2}\Big(\frac{Z_N(0)}{Z_M(0)}\Big)^{-1/2}\bigg] \leqslant \mathbb{E}\big[\mathbf{1}_{\mathcal{G}_M}\, Z_{M,N}(\mu_M)^2\big]^{1/2}\,, \tag{7.69}$$

where we have used that $\mathbb{E}[Z_n(0) \mid \mathcal{F}_M] = Z_M(0)$ and noticed that

$$\mathbb{E}\bigg[\Big(\frac{Z_N(0)}{Z_M(0)}\Big)^2 \,\Big|\, \mathcal{F}_M\bigg] = \mathbb{E}\big[Z_{M,N}(\mu_M)^2 \mid \mathcal{F}_M\big].$$

Take $\mathcal{G} = \mathcal{G}_{1,s}$ as defined in (7.51), so that $\mathcal{G} \in \mathcal{F}_M$. Then

$$\mathbb{P}(\mathcal{G}^c) \leqslant \frac{C}{K-s}, \qquad \mathbb{E}\big[\mathbf{1}_{\mathcal{G}} Z_{M,N}(\mu_M)^2\big] \leqslant C(K-s).$$

The first bound follows from (7.52), while the second can be checked repeating the argument which proves (4.27) at time $M = N_s$ (note that $K - s \geqslant J$ is sufficiently large for small δ). We therefore get that, applying Markov's inequality with (7.69)

$$\begin{aligned}\mathbb{P}\big(Z_{n,N}(\mu_n) < \mathrm{e}^{-u}\big) &\leqslant \mathbb{P}(\mathcal{G}^c) + \mathbb{E}\big[\mathbf{1}_{\mathcal{G}} Z_{M,N}(\mu_M)^2\big]^{1/2}\mathrm{e}^{-u/2} \\ &\leqslant \frac{C}{K-s} + C\sqrt{K-s}\,\mathrm{e}^{-u/2}\,.\end{aligned} \tag{7.70}$$

The reason for the choice $K - s := \lceil J\mathrm{e}^{u/3} \rceil$ is made so that both terms on the right-hand side of (7.70) bounded by $O(\sqrt{J}\mathrm{e}^{-u/3})$. We end up with the following bound when $\lceil J\mathrm{e}^{u/3} \rceil < K$:

$$\mathbb{P}\big(Z_{n,N}(\mu_n) < \mathrm{e}^{-u}\big) \leqslant C\sqrt{J}\,\mathrm{e}^{-u/3}. \tag{7.71}$$

Combining with (7.68) for the case $\lceil J\mathrm{e}^{u/3} \rceil \geqslant K$, this concludes the proof. □

8. Small-scale fluctuations and covariance structure

In this section, we prove Theorem 1.4. We recall that we only work here with flat initial condition $h_0 = \mathbf{0}$. By Proposition 1.8,

$$\mathscr{H}_t^{\vartheta}(\mathbf{0}, \varphi_\varepsilon^{(x)}) \overset{\text{dist}}{=} \mathscr{H}_1^{\vartheta + \log t}(\mathbf{0}, \varphi_{\varepsilon/\sqrt{t}}^{(x/\sqrt{t})}),$$

so it suffices to consider only $t = 1$. We use the time-reversed fields $\mathscr{H}^{(\delta)}$ and $\mathscr{H}$ from (2.8) and (2.11), omitting the superscript ϑ as in Section 2.2. The law of $\mathscr{H}$ is that of $\mathscr{H}_1^{\vartheta}(\mathbf{0}, \cdot)$.

8.1. Small-scale Gaussianity. We prove here item (1) in Theorem 1.4. The main idea is to derive the result from Gu and Tsai's log-normality for the SHF [GT26]. Recall that we defined $F_\delta(x) = \log \mathscr{Z}_{\delta,1}^{\vartheta}(g_\delta^{(x)}, \mathbf{1})$ in (3.13), so that

$$\mathscr{H}^{(\delta)}(\varphi) = \int_{\mathbb{R}^2} \varphi(x)\{F_\delta(x) - \mathbb{E}[F_\delta(x)]\}\mathrm{d}x\,.$$

We then have the following log-normality, which differs from [GT26, Theorem 1.3] only by the centring term.

Theorem 8.1. *For any fixed $\vartheta \in \mathbb{R}$, for any $x \in \mathbb{R}^2$ we have that*

$$\frac{F_{\varepsilon^2}(x) - \mathbb{E}[F_{\varepsilon^2}(x)]}{\sqrt{\log\log(1/\varepsilon)}} \xrightarrow[\varepsilon \to 0]{d} \mathcal{N}(0,1)\,. \tag{8.1}$$

Proof. Theorem 1.3 of [GT26] gives a deterministic centring $b_\varepsilon \sim -\frac{1}{2}\log\log(1/\varepsilon)$ such that

$$Y_\varepsilon := \frac{F_{\varepsilon^2}(x) - b_\varepsilon}{\sqrt{\log\log(1/\varepsilon)}} \xrightarrow[\varepsilon \downarrow 0]{d} \mathcal{N}(0,1)\,. \tag{8.2}$$

We now claim that the second moment of Y_ε is uniformly bounded:

$$\sup_{\varepsilon \in (0,1/10)} \mathbb{E}[|Y_\varepsilon|^2] < \infty. \tag{8.3}$$

Once (8.3) is proven, from the weak convergence (8.2), we get $\lim_{\varepsilon\downarrow 0} \mathbb{E}[Y_\varepsilon] = 0$ and thus

$$Y_\varepsilon - \mathbb{E}[Y_\varepsilon] \xrightarrow[\varepsilon \downarrow 0]{d} \mathcal{N}(0,1)$$

which is exactly (8.1).

In what follows, we will prove (8.3). By (3.14), we have

$$\operatorname{Var} Y_\varepsilon = \frac{\operatorname{Var} F_{\varepsilon^2}(x)}{\log\log\frac{1}{\varepsilon}} \leqslant \frac{\log\log\frac{1}{\varepsilon^2} + C_\vartheta}{\log\log\frac{1}{\varepsilon}} \leqslant C < \infty, \quad \forall \varepsilon \in (0, 1/10) \tag{8.4}$$

for some finite $C > 0$. To bound $\mathbb{E}[|Y_\varepsilon|^2] = \operatorname{Var} Y_\varepsilon + |\mathbb{E}[Y_\varepsilon]|^2$, it suffices to bound $|\mathbb{E}[Y_\varepsilon]|$ uniformly for small ε. From (8.4), $\{Y_\epsilon - \mathbb{E}[Y_\varepsilon] : \varepsilon \in (0, 1/10)\}$ is tight. Furthermore, from the weak convergence (8.2), $\{Y_\epsilon : \varepsilon \in (0, 1/10)\}$ is also tight. It follows that $\mathbb{E}[Y_\varepsilon]$ is uniformly bounded for $\varepsilon \in (0, 1/10)$, completing the proof. □

In the rest of this section, we work with a nonnegative $\varphi \in C_c^\infty(\mathbb{R}^2)$. Assume for simplicity that $\int \varphi = 1$. By stationarity, we may also only consider the case $x = 0$ and recall that we defined $\varphi_\varepsilon(z) := \varphi_\varepsilon^{(0)}(z) = \varepsilon^{-2}\varphi(z/\varepsilon)$. Finally, by translation invariance of SHF, we may assume that the barycentre of φ is $\int y\varphi(y)\mathrm{d}y = 0$, that is,

$$\int_{\mathbb{R}^2} y_i \varphi(y)\mathrm{d}y = 0, \quad y = (y_1, y_2),\ i = 1, 2. \tag{8.5}$$

Proof of item (1) **in Theorem 1.4.** Our main task is to prove that there exists a constant C_φ such that, uniformly for $\varepsilon > 0$ small enough,

$$\|\mathscr{H}(\varphi_\varepsilon) - \{F_{\varepsilon^2}(0) - \mathbb{E}F_{\varepsilon^2}(0)\}\|_{L^1} \leqslant C_\varphi. \tag{8.6}$$

Together with Theorem 8.1, this concludes the proof of item (1) in Theorem 1.4.

To prove (8.6), we use the triangle inequality to get that the left-hand side is bounded by

$$\|\mathscr{H}(\varphi_\varepsilon) - \mathscr{H}^{(\varepsilon^2)}(\varphi_\varepsilon)\|_{L^2} + \Big\| \int_{\mathbb{R}^2} \varphi_\varepsilon(x) F_{\varepsilon^2}(x)\mathrm{d}x - F_{\varepsilon^2}(0) \Big\|_{L^1}, \tag{8.7}$$

using also that $\int_{\mathbb{R}^2} \varphi_\varepsilon(x)\mathbb{E}[F_{\varepsilon^2}(x)]\mathrm{d}x = \mathbb{E}[F_{\varepsilon^2}(0)]$ so that the centring terms cancel out.

First term in (8.7). We use the decomposition (2.27), to get that

$$\limsup_{N\to\infty} \|\mathcal{H}^{(N)}(\varphi_\varepsilon) - \mathcal{H}^{(N;\delta)}(\varphi_\varepsilon)\|_{L^2} \leqslant \limsup_{N\to\infty} \|\mathcal{L}^{(N;\delta)}(\varphi_\varepsilon)\|_{L^2} + \limsup_{N\to\infty} \|\mathcal{G}^{(N;\delta)}(\varphi_\varepsilon)\|_{L^2}.$$

Now, we can use Propositions 2.4 and 2.5, or more precisely the bounds (4.24) and (4.40) (see Remarks 4.2 and 4.8), to get that for any δ small enough,

$$\limsup_{N\to\infty} \|\mathcal{H}^{(N)}(\varphi_\varepsilon) - \mathcal{H}^{(N;\delta)}(\varphi_\varepsilon)\|_{L^2}^2 \leqslant \frac{C\mathcal{E}_{\sqrt{\delta}}(\varphi_\varepsilon)}{|\log\delta|} + \frac{C'\|\varphi_\varepsilon\|_1^2}{|\log\delta|^{1/20}},$$

where $\mathcal{E}_{\sqrt{\delta}}(\varphi_\varepsilon)$ is defined in (4.22). In particular, by the triangle inequality (and using that $(a+b)^2 \leqslant 2a^2 + 2b^2$) we get that for any $\delta \leqslant \varepsilon^2$,

$$\limsup_{N\to\infty} \|\mathcal{H}^{(N;\varepsilon^2)}(\varphi_\varepsilon) - \mathcal{H}^{(N;\delta)}(\varphi_\varepsilon)\|_{L^2}^2 \leqslant \frac{C\mathcal{E}_{\varepsilon}(\varphi_\varepsilon)}{|\log\varepsilon|} + \frac{C'\|\varphi_\varepsilon\|_1^2}{|\log\varepsilon|^{1/20}},$$

using also the monotonicity of $\delta \mapsto \mathcal{E}_\delta(\varphi)$. Now, as in Remark 4.2, scaling gives

$$\mathcal{E}_\varepsilon(\varphi_\varepsilon) = \mathcal{E}_{1/\varepsilon}(\varphi) \leqslant C\|\varphi\|_\infty\|\varphi\|_1 + C|\log\varepsilon|\,\|\varphi\|_1^2.$$

By the joint convergence in distribution of Proposition 2.2 and Portmanteau, as in (2.17), we therefore get that, uniformly for $\varepsilon \in (0, \varepsilon_0)$ with ε_0 small enough and $\delta \leqslant \varepsilon^2$

$$\|\mathscr{H}^{(\varepsilon^2)}(\varphi_\varepsilon) - \mathscr{H}^{(\delta)}(\varphi_\varepsilon)\|_{L^2}^2 \leqslant C\|\varphi\|_1\big(\|\varphi\|_\infty + \|\varphi\|_1\big).$$

Letting $\delta \downarrow 0$ and using (2.11), namely $\|\mathscr{H}^{(\delta)}(\varphi_\varepsilon) - \mathscr{H}(\varphi_\varepsilon)\|_{L^2} \to 0$, this shows that the first term in (8.7) satisfies

$$\|\mathscr{H}^{(\varepsilon^2)}(\varphi_\varepsilon) - \mathscr{H}(\varphi_\varepsilon)\|_{L^2} \leqslant C\|\varphi\|_1^{1/2}\big(\|\varphi\|_\infty + \|\varphi\|_1\big)^{1/2}, \tag{8.8}$$

uniformly in $\varepsilon \in (0, \varepsilon_0)$.

Second term in (8.7). First of all, note that

$$F_{\varepsilon^2}(x) - F_{\varepsilon^2}(0) = \log \mathscr{Z}^\vartheta_{\varepsilon^2,1}(g^{(x)}_{\varepsilon^2}, \mathbf{1}) - \log \mathscr{Z}^\vartheta_{\varepsilon^2,1}(g^{(0)}_{\varepsilon^2}, \mathbf{1}) = \log \int_{\mathbb{R}^2} \frac{g_{\varepsilon^2}(x-y)}{g_{\varepsilon^2}(y)} Q_\varepsilon(\mathrm{d}y),$$

where Q_ε is a probability measure on $\mathbb{R}^2$, defined by

$$Q_\varepsilon(\mathrm{d}y) := \frac{g_{\varepsilon^2}(y)\mathscr{Z}^\vartheta_{\varepsilon^2,1}(\mathrm{d}y,\mathbf{1})}{\int_{\mathbb{R}^2} g_{\varepsilon^2}(z)\mathscr{Z}^\vartheta_{\varepsilon^2,1}(\mathrm{d}z,\mathbf{1})}\,.$$

Now, by definition (1.17) of the heat kernel we can write $\frac{g_{\varepsilon^2}(x-y)}{g_{\varepsilon^2}(y)} = \exp(-\frac{|x|^2}{2\varepsilon^2} + \frac{x\cdot y}{\varepsilon^2})$, so that we end up with

$$F_{\varepsilon^2}(x) - F_{\varepsilon^2}(0) = -\frac{|x|^2}{2\varepsilon^2} + \log\int_{\mathbb{R}^2} \mathrm{e}^{\frac{x\cdot y}{\varepsilon^2}} Q_\varepsilon(\mathrm{d}y) \geqslant -\frac{|x|^2}{2\varepsilon^2} + x\cdot\int_{\mathbb{R}^2}\frac{y}{\varepsilon^2}Q_\varepsilon(\mathrm{d}y)\,,$$

where the last inequality follows by Jensen's inequality, since Q_ε is a probability measure. As a result, recalling that φ is non-negative,

$$\begin{aligned}\int_{\mathbb{R}^2}\varphi_\varepsilon(x)[F_{\varepsilon^2}(x) - F_{\varepsilon^2}(0)]\mathrm{d}x &\geqslant -\int_{\mathbb{R}^2}\frac{\varphi_\varepsilon(x)}{2\varepsilon^2}|x|^2\mathrm{d}x + \int_{\mathbb{R}^2}\varphi_\varepsilon(x)x\mathrm{d}x\cdot\int_{\mathbb{R}^2}\frac{y}{\varepsilon^2}Q_\varepsilon(\mathrm{d}y)\\ &= -\frac{1}{2}\int_{\mathbb{R}^2}\varphi(x)|x|^2\mathrm{d}x\,,\end{aligned}$$

where we have used the scaling properties of φ_ε for the first term and that $\int_{\mathbb{R}^2}\varphi_\varepsilon(x)x\mathrm{d}x = 0$ for the second term (recall that φ satisfies (8.5)). In particular, this shows that

$$\Big(\int_{\mathbb{R}^2}\varphi_\varepsilon(x)[F_{\varepsilon^2}(x) - F_{\varepsilon^2}(0)]\mathrm{d}x\Big)^- \leqslant \frac{1}{2}\int_{\mathbb{R}^2}\varphi(x)|x|^2\mathrm{d}x\,, \tag{8.9}$$

where $X^- = \max\{-X, 0\}$ is the negative part of X. Since $\int\varphi_\varepsilon(x)[F_{\varepsilon^2}(x) - F_{\varepsilon^2}(0)]\mathrm{d}x$ has mean zero by the spatial stationarity of $F_{\varepsilon^2}(\cdot)$ and since $\mathbb{E}[|X|] = 2\mathbb{E}[X^-]$ for any mean-zero random variable X, we get from (8.9) that the second term in (8.7) verifies

$$\mathbb{E}\Big[\Big|\int_{\mathbb{R}^2}\varphi_\varepsilon(x)\left(F_{\varepsilon^2}(x) - F_{\varepsilon^2}(0)\right)\mathrm{d}x\Big|\Big] \leqslant \int_{\mathbb{R}^2}\varphi(x)|x|^2\mathrm{d}x\,. \tag{8.10}$$

In the end, plugging (8.10) and (8.8) into (8.7) finishes the proof. □

We conclude this section by estimating the variance of $\mathscr{H}^\vartheta_t(\varphi^{(x)}_\varepsilon)$.

Proposition 8.2 (Variance of shrinking averages)**.** *For every fixed nonnegative smooth compactly supported φ of integral one and satisfying* (8.5)*, for any fixed $\vartheta\in\mathbb{R}$,*

$$\mathrm{Var}\big(\mathscr{H}^\vartheta_1(\varphi_\varepsilon)\big) \sim \log\log(1/\varepsilon) \qquad \text{as } \varepsilon\downarrow 0\,. \tag{8.11}$$

Proof. Notice that since $\varphi\geqslant 0$ with $\int\varphi = 1$, Jensen's inequality yields that

$$\big(\mathscr{H}^{(\varepsilon^2)}(\varphi_\varepsilon)\big)^2 = \Big(\int_{\mathbb{R}^2}\varphi_\varepsilon(x)\{F_{\varepsilon^2}(x) - \mathbb{E}[F_{\varepsilon^2}(x)]\}\mathrm{d}x\Big)^2 \leqslant \int_{\mathbb{R}^2}\varphi_\varepsilon(x)\big(F_{\varepsilon^2}(x) - \mathbb{E}[F_{\varepsilon^2}(x)]\big)^2\mathrm{d}x\,,$$

so that in particular $\mathbb{E}[(\mathscr{H}^{(\varepsilon^2)}(\varphi_\varepsilon))^2] = \mathrm{Var}(\mathscr{H}^{(\varepsilon^2)}(\varphi_\varepsilon)) \leqslant \mathrm{Var}(F_{\varepsilon^2}(0))$, using also the spatial stationarity of F_{ε^2} and $\int\varphi = 1$. Now, using the observation (3.14), we have that $\mathrm{Var}(F_{\varepsilon^2}(0)) \leqslant \log\log(1/\varepsilon) + C_\vartheta$. By (8.8) and the triangle inequality, we thus get the upper bound

$$\mathrm{Var}\big(\mathscr{H}(\varphi_\varepsilon)\big) = \mathbb{E}\big[\mathscr{H}(\varphi_\varepsilon)^2\big] \leqslant \log\log(1/\varepsilon) + O((\log\log 1/\varepsilon)^{1/2})\,.$$

For the reverse bound, the convergence (1.23) and Portmanteau give

$$\liminf_{\varepsilon\downarrow 0}\frac{\mathbb{E}\big[(\mathscr{H}(\varphi_\varepsilon))^2\big]}{\log\log(1/\varepsilon)} \geqslant \mathbb{E}[Z^2] = 1\,,$$

where $Z \sim \mathcal{N}(0,1)$. This proves the lower bound $\mathrm{Var}\big(\mathscr{H}(\varphi_\varepsilon)\big) \geqslant (1+o(1))\log\log(1/\varepsilon)$ for small ε, concluding the proof of Proposition 8.2. □

8.2. No local signed-measure representative. In this section, we prove Proposition 1.5. Again, we focus on the case $t = 1$. Consider a nonnegative $\varphi \in C_c^\infty(B(0,1))$ with $\int \varphi = 1$ and let $\varphi_\varepsilon^{(x)}(y) := \varepsilon^{-2}\varphi((y-x)/\varepsilon)$.

First of all, using the small scale Gaussianity from Theorem 1.4-1, see (1.23), for all $x \in \mathbb{R}^2$ and $k \in \mathbb{N}$ we get that $\lim_{\varepsilon\downarrow 0}\mathbb{P}(|\mathscr{H}(\varphi_\varepsilon^{(x)})| \leqslant k) = 0$. We can thus choose a sequence $\varepsilon_k \downarrow 0$ such that

$$\mathbb{P}\Big(\big|\mathscr{H}(\varphi_{\varepsilon_k}^{(x)})\big| \leqslant k\Big) \leqslant 2^{-k}\,, \qquad \forall x \in \mathbb{R}^2.$$

Thus, applying Fubini–Tonelli, we get that for $B = B(0,1)$,

$$\mathbb{E}\Big[\int_B \sum_{k\geqslant 1} \mathbf{1}_{\{|\mathscr{H}(\varphi_{\varepsilon_k}^{(x)})|\leqslant k\}}\,\mathrm{d}x\Big] \leqslant |B|\sum_{k=1}^{\infty} 2^{-k} < +\infty\,.$$

This shows that almost surely, for almost every $x \in B$, we have $|\mathscr{H}(\varphi_{\varepsilon_k}^{(x)})| > k$ eventually. Put differently, and using a countable union of balls, we get that

$$\mathrm{Leb}\Big(\Big\{x \in \mathbb{R}^2 \,:\, \limsup_{k\to\infty} |\mathscr{H}(\varphi_{\varepsilon_k}^{(x)})| < +\infty\Big\}\Big) = 0 \qquad \mathbb{P}\text{-a.s.} \tag{8.12}$$

Consider now the event that there is some locally finite signed Radon measure on a non-empty open set, namely

$$A := \Big\{\exists\, U = B(z,r)\,, \mu \in \mathcal{M}_{\mathrm{loc}}(U) \text{ s.t. } \mathscr{H}(\varphi) = \int_U \varphi(y)\mu(\mathrm{d}y) \quad \text{for all } \varphi \in C_c^\infty(U)\Big\}\,,$$

where $z \in \mathbb{R}^2$, $r > 0$. Here, $\mathcal{M}_{\mathrm{loc}}(U)$ denotes the set of locally finite signed Radon measures on the open ball $U = B(z,r)$. (Note that the choice of the ball U and of the Radon measure μ may depend on the realisation.)

Consider a realisation in this event A and some $U = B(z,r)$ given by this realization. Now, notice that $\varphi_\varepsilon^{(x)} \in C_c^\infty(B(x,\varepsilon))$ so $\varphi_\varepsilon^{(x)} \in C_c^\infty(U)$ for all $x \in B(z,r/2)$ and $\varepsilon \in (0, r/2)$. Then, for any such x and ε, by definition of A we have that

$$\big|\mathscr{H}(\varphi_\varepsilon^{(x)})\big| = \Big|\int_U \varphi_\varepsilon^{(x)}(y)\mu(\mathrm{d}y)\Big| \leqslant \|\varphi\|_\infty\,\varepsilon^{-2}\,|\mu|(B(x,\varepsilon))\,.$$

Since $|\mu|$ is locally finite on $U = B(z,r)$, by the Lebesgue differentiation theorem, we have that $\limsup_{\varepsilon\downarrow 0}\varepsilon^{-2}\,|\mu|(B(x,\varepsilon)) < \infty$ for almost every $x \in B(z,r/2)$. In other words, we have proven that

$$\text{on the event } A\,, \quad \mathrm{Leb}\Big(\Big\{x \in \mathbb{R}^2 \,:\, \limsup_{\varepsilon\downarrow 0} |\mathscr{H}(\varphi_\varepsilon^{(x)})| < +\infty\Big\}\Big) \geqslant \mathrm{Leb}(B(z,r/2)) > 0\,.$$

(We recall that $z \in \mathbb{R}^2$ and $r > 0$ may depend on the realisation.) In view of (8.12), this proves that $\mathbb{P}(A) = 0$, which is the main statement of Proposition 1.5.

Since a locally integrable function (and in particular a spatially constant distribution) defines a locally finite Radon measure, this proves the remaining assertion. □

8.3. The covariance kernel: existence and continuity. We now prove (1.24) in item (2) of Theorem 1.4. Again, we only deal with $t = 1$ and we omit the dependence on ϑ for simplicity.

Step 1. Existence of the covariance kernel. Let $f, g \in C_c^\infty(\mathbb{R}^2)$ be real test functions. For $0 < \delta < 1$ we have

$$\mathrm{Cov}\big(\mathscr{H}^{(\delta)}(f), \mathscr{H}^{(\delta)}(g)\big) = \int_{\mathbb{R}^2\times\mathbb{R}^2} f(x)g(y)\mathsf{C}_\delta(x-y)\mathrm{d}x\mathrm{d}y\,,$$

where $\mathsf{C}_\delta(z) := \mathrm{Cov}(F_\delta(x), F_\delta(x+z)) \geqslant 0$ (recall the definition (3.13)), using also translation invariance to get that it only depends on z. By the L^2 convergence in Theorem 2.1, we have

$$\mathrm{Cov}\big(\mathscr{H}(f), \mathscr{H}(g)\big) = \lim_{\delta\downarrow 0} \mathrm{Cov}\big(\mathscr{H}^{(\delta)}(f), \mathscr{H}^{(\delta)}(g)\big)\,. \tag{8.13}$$

Now, thanks to Lemma 6.3, we get

$$\sup_{\delta\in(0,\delta_0)} \mathsf{C}_\delta(z) \leqslant C\left[1 + \log\big(1 + \log^+(1/|z|)\big)\right]. \tag{8.14}$$

This shows that, for any ball $B = B(0,R)$ in $\mathbb{R}^2$, the family of functions $(\mathsf{C}_\delta)_{\delta\in(0,\delta_0)}$ is bounded in $L^2(B)$. Thus, by the Banach–Alaoglu theorem, one can find a subsequence along which C_δ converges weakly in $L^2(B)$. By a diagonal extraction in a sequence of growing balls, we can thus find a subsequence $(\delta_k)_{k\geqslant 1}$ and some $\mathsf{C} \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ such that C_{δ_k} converges weakly to C. We thus have that

$$\lim_{k\to\infty} \int_{\mathbb{R}^2\times\mathbb{R}^2} f(x)g(y)\mathsf{C}_{\delta_k}(x-y)\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^2\times\mathbb{R}^2} f(x)g(y)\mathsf{C}(x-y)\mathrm{d}x\mathrm{d}y\,,$$

using the weak convergence applied to $z \mapsto \int f(y+z)g(y)\,\mathrm{d}y$ which is compactly supported.

Because of (8.13), the limit on the right-hand side is equal to $\mathrm{Cov}(\mathscr{H}(f), \mathscr{H}(g))$. Since the covariance structure is fixed, this implies that any weakly convergent subsequence of $(\mathsf{C}_\delta)_{\delta\in(0,\delta_0)}$ must converge to the same limit $\mathsf{C} \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$, showing that we indeed have that $\mathsf{C}_\delta \to \mathsf{C}$ weakly as $\delta \to 0$. This concludes the proof of (1.24).

Note also that C is rotationally invariant thanks to Proposition 1.8, *i.e.* $\mathsf{C}(z)$ depends only on $|z|$, and we get from (8.14) that

$$\mathsf{C}(z) \leqslant C\left[1 + \log\big(1 + \log^+(1/|z|)\big)\right]. \tag{8.15}$$

Step 2. Spectral measure and continuity of the covariance kernel. We now show that $\mathsf{C}(z)$ is continuous outside zero, relying on a Fourier inversion formula. Since C can be interpreted as a tempered distribution (thanks e.g. to (8.15)) and since the covariance distribution is of positive type, the Bochner–Schwartz theorem [RS75, Theorem IX.10] provides a nonnegative tempered (i.e. locally finite and with at most polynomial mass growth at infinity), spectral measure $\nu = \nu_{\mathsf{C}}$ such that for any $\phi \in \mathcal{S}(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2} \phi(\xi)\,\nu(\mathrm{d}\xi) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \widehat{\phi}(z)\mathsf{C}(z)\,\mathrm{d}z\,, \tag{8.16}$$

with the convention (6.1) for the Fourier transform. Because of rotational invariance, we also have that ν is radially symmetric. We now have the following claim that bounds the spectral mass of growing balls (it is proven below).

Proposition 8.3. *We have that* $\nu(\bar{B}(0,R)) \sim \log\log R$ *as* $R \to \infty$.

Because of the radial symmetry, letting ν_{rad} be the radial projection of ν (*i.e.* the push-forward of ν by $\xi \mapsto |\xi|$), the Fourier inversion can then be written as follows:

$$\mathsf{C}(z) = C_0 + \int_{[0,\infty)} \big\{J_0(\rho|z|) - \mathbf{1}_{\{\rho\leqslant 1\}}\big\}\,\nu_{\mathrm{rad}}(\mathrm{d}\rho)\,, \tag{8.17}$$

where J_0 is the Bessel function $J_0(u) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{iu\cos\theta}\,\mathrm{d}\theta$ and $C_0 = \nu_{\mathrm{rad}}([0,1])$.

The integrability of the right-hand side of (8.17) can be justified by using the elementary bounds $|J_0(u)-1| \leqslant Cu^2$ for $u \leqslant 1$ and $|J_0(u)| \leqslant Cu^{-1/2}$ for $u \geqslant 1$, together with the local finiteness of ν and Proposition 8.3, which ensures that

$$\begin{aligned}\int_{(1,\infty)} \rho^{-1/2} \nu_{\mathrm{rad}}(\mathrm{d}\rho) =& \frac{1}{2} \int_{(1,\infty)} a^{-3/2} \nu_{\mathrm{rad}}\big((1,a]\big)\mathrm{d}a \\ \leqslant& \frac{1}{2} \int_{(1,\infty)} a^{-3/2} \nu\big(\bar{B}(0,a)\big)\mathrm{d}a < +\infty\,,\end{aligned} \tag{8.18}$$

where the equality follows from writing $\rho^{-1/2} = \frac{1}{2}\int_{[\rho,\infty)} a^{-3/2}\mathrm{d}a$ and applying Fubini–Tonelli. Thus the integral (8.17) converges absolutely, uniformly for z in a compact set of $\mathbb{R}^2\backslash\{0\}$: since $z \mapsto J_0(\rho|z|)$ is a continuous function of z, this shows that $z \mapsto \mathsf{C}(z)$ is continuous on $\mathbb{R}^2\backslash\{0\}$.

Proof of Proposition 8.3. Consider a nonnegative $\varphi \in C_c^\infty(\mathbb{R}^2)$ which is radially symmetric and of integral 1, so that $\widehat{\varphi}$ is also radially symmetric and verifies $\widehat{\varphi}(0) = 1$ and $|\widehat{\varphi}| \leqslant 1$. Recall that $\varphi_\varepsilon(\cdot) = \varepsilon^{-2}\varphi(\cdot/\varepsilon)$ and notice that $\widehat{\varphi}_\varepsilon(\cdot) = \widehat{\varphi}(\varepsilon\cdot)$.

By the spectral representation (8.16) we have that

$$\mathrm{Cov}(\mathscr{H}(f),\mathscr{H}(g)) = \int_{\mathbb{R}^2\times\mathbb{R}^2} f(x)g(y)\mathsf{C}(x-y)\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^2} \widehat{f}(\xi)\,\overline{\widehat{g}(\xi)}\,\nu(\mathrm{d}\xi)\,,$$

so that taking $f = g = \varphi_{1/r}$ we obtain

$$\int_{\mathbb{R}^2} |\widehat{\varphi}(\xi/r)|^2\,\nu(\mathrm{d}\xi) = \mathrm{Var}\big(\mathscr{H}(\varphi_{1/r})\big) \sim \log\log r \qquad \text{as } r \to \infty\,, \tag{8.19}$$

where the last asymptotics follows from Proposition 8.2.

Upper bound. Let $\eta \in (0,1)$ be fixed. Since $\widehat{\varphi}(0) = 1$ and is continuous, we get that $|\widehat{\varphi}(\xi)|^2 \geqslant 1-\eta$ for all $|\xi| \leqslant c_\eta$. We thus obtain that

$$\int_{\mathbb{R}^2} |\widehat{\varphi}(c_\eta\xi/R)|^2\,\nu(\mathrm{d}\xi) \geqslant (1-\eta)\nu(\bar{B}(0,R))\,.$$

With (8.19) applied with $r = R/c_\eta$, this shows

$$\nu(\bar{B}(0,R)) \leqslant (1+o(1))(1-\eta)^{-1}\log\log(R/c_\eta)\,. \tag{8.20}$$

Since $\eta > 0$ is arbitrary, this shows that $\nu(\bar{B}(0,R)) \leqslant (1+o(1))\log\log R$ as $R \to \infty$, showing the upper bound in Proposition 8.3.

Lower bound. Let us fix some $\eta \in (0,1)$. Since $|\widehat{\varphi}| \leqslant 1$, we get that

$$\int_{\mathbb{R}^2} \Big|\widehat{\varphi}\Big(\frac{\xi}{\eta R}\Big)\Big|^2 \nu(\mathrm{d}\xi) \leqslant \nu(\bar{B}(0,R)) + \int_{|\xi|>R} \Big|\widehat{\varphi}\Big(\frac{\xi}{\eta R}\Big)\Big|^2 \nu(\mathrm{d}\xi)\,. \tag{8.21}$$

Since φ is smooth and compactly supported, $|\widehat{\varphi}(\xi)|$ decreases faster than any polynomial at $+\infty$. Let us simply use that $|\widehat{\varphi}(\xi)|^2 \leqslant C|\xi|^{-1}$ for all $|\xi| \geqslant 1$, so that

$$\int_{|\xi|>R} \Big|\widehat{\varphi}\Big(\frac{\xi}{\eta R}\Big)\Big|^2 \nu(\mathrm{d}\xi) \leqslant C\int_{(R,\infty)} (\rho/\eta R)^{-1}\nu_{\mathrm{rad}}(\mathrm{d}\rho) \leqslant C\eta R\int_{(R,\infty)} a^{-2}\nu(\bar{B}(0,a))\mathrm{d}a\,,$$

where for the last inequality we have used that $\rho^{-1} = \int_{[\rho,\infty)} a^{-2}\mathrm{d}a$ and applied Fubini–Tonelli, as in (8.18). Now, using (8.20), we get that $\nu(\bar{B}(0,a)) \leqslant C(1+\log\log a)$ for all $a > R$ with R large enough, so we end up with

$$\int_{|\xi|>R} \Big|\widehat{\varphi}\Big(\frac{\xi}{\eta R}\Big)\Big|^2 \nu(\mathrm{d}\xi) \leqslant C\eta R\int_{(R,\infty)} a^{-2}(1+\log\log a)\mathrm{d}a \leqslant C'\eta\log\log R$$

for all R sufficiently large. Going back to (8.21) and using (8.19) with $r = \eta R$, we thus get that as $R \to \infty$,

$$\nu(\bar{B}(0,R)) \geqslant (1+o(1))\log\log(\eta R) - C\eta \log\log R\,.$$

This proves the lower bound in Proposition 8.3, since $\eta > 0$ is arbitrary. □

Step 3. Divergence of the covariance kernel near the diagonal. In this section, we prove the sharp asymptotics of $\mathsf{C}(z)$ as $|z| \downarrow 0$, namely (1.25). It relies on the inversion formula (8.17): using the notation $R = 1/|z|$ to match with the notation of the previous step, we need to show that

$$\int_{[0,\infty)} \big\{J_0(\rho/R) - \mathbf{1}_{\{\rho\leqslant 1\}}\big\}\nu_{\mathrm{rad}}(\mathrm{d}\rho) \sim \log\log R \qquad \text{as } R\to\infty\,.$$

Note that $\int_{[0,1]}(J_0(\rho/R) - \mathbf{1}_{\{\rho\leqslant 1\}})\nu_{\mathrm{rad}}(\mathrm{d}\rho)$ remains bounded by a constant, uniformly for $R > 1$, so we only need to show that

$$\int_{(1,\infty)} J_0(\rho/R)\nu_{\mathrm{rad}}(\mathrm{d}\rho) \sim \log\log R \qquad \text{as } R\to\infty\,. \tag{8.22}$$

To show this, we use that $|J_0(u) - 1| \leqslant Cu^2$ for $u \leqslant 1$ and $|J_0(u)| \leqslant Cu^{-1/2}$ for $u \geqslant 1$, to get that

$$\begin{aligned}\Big|\int_{(1,\infty)} J_0(\rho/R)\nu_{\mathrm{rad}}(\mathrm{d}\rho) - \nu_{\mathrm{rad}}((1,R])\Big| \\ \leqslant C\int_{(0,R]}(\rho/R)^2\nu_{\mathrm{rad}}(\mathrm{d}\rho) + C\int_{(R,\infty)}(\rho/R)^{-1/2}\nu_{\mathrm{rad}}(\mathrm{d}\rho)\,.\end{aligned} \tag{8.23}$$

Since by Proposition 8.3 we have that

$$\nu_{\mathrm{rad}}((1,R]) = \nu(\bar{B}(0,R)) - \nu(\bar{B}(0,1)) \sim \log\log R \qquad \text{as } R\to\infty\,,$$

it only remains to show that the two terms on the right-hand side of (8.23) are $o(\log\log R)$. *First term in* (8.23). Using that $\rho^2 = 2\int_{(0,\rho]} a\mathrm{d}a$ and Fubini–Tonelli, we get similarly as in (8.18) that

$$\int_{(0,R]}(\rho/R)^2\nu_{\mathrm{rad}}(\mathrm{d}\rho) = \frac{2}{R^2}\int_{(0,R]} a\,\nu_{\mathrm{rad}}((a,R])\mathrm{d}a = \nu(\bar{B}(0,R)) - \frac{2}{R^2}\int_{(0,R]} a\,\nu(\bar{B}(0,a))\mathrm{d}a\,.$$

Since $a \mapsto \nu(\bar{B}(0,a))$ is slowly varying[6] thanks to Proposition 8.3, thanks to [BGT89, Proposition 1.5.8] we get that $\int_{(0,R]} a\,\nu(\bar{B}(0,a))\mathrm{d}a \sim \frac{1}{2}R^2\nu(\bar{B}(0,R))$ as $R\to\infty$. We thus obtain that

$$\int_{(0,R]}(\rho/R)^2\nu_{\mathrm{rad}}(\mathrm{d}\rho) = \nu(\bar{B}(0,R)) - (1+o(1))\nu(\bar{B}(0,R)) = o(\log\log R) \quad \text{as } R\to\infty\,.$$

Second term in (8.23). We proceed similarly as above. Using that $\rho^{-1/2} = \frac{1}{2}\int_{[\rho,\infty)} a^{-3/2}\mathrm{d}a$ and applying Fubini–Tonelli as in (8.18), we get

$$\begin{aligned}\int_{(R,\infty)}(\rho/R)^{-1/2}\nu_{\mathrm{rad}}(\mathrm{d}\rho) &= \frac{R^{1/2}}{2}\int_{(R,\infty)} a^{-3/2}\nu_{\mathrm{rad}}((R,a])\mathrm{d}a \\ &= \frac{R^{1/2}}{2}\int_{(R,\infty)} a^{-3/2}\nu(\bar{B}(0,a))\mathrm{d}a - \nu(\bar{B}(0,R))\,.\end{aligned}$$

[6]A function f is said to be slowly varying if for any $\alpha > 0$ we have that $\lim_{x\to\infty} f(\alpha x)/f(x) = 1$.

Again, since $a \mapsto \nu(\bar{B}(0,a))$ is slowly varying, we get that $\int_{(R,\infty)} a^{-3/2}\nu(\bar{B}(0,a))\mathrm{d}a \sim 2R^{-1/2}\nu(\bar{B}(0,R))$ as $R \to \infty$, see [BGT89, Proposition 1.5.10]. This yields

$$\int_{(R,\infty)} (\rho/R)^{-1/2}\nu_{\mathrm{rad}}(\mathrm{d}\rho) = (1+o(1))\nu(\bar{B}(0,R)) - \nu(\bar{B}(0,R)) = o(\log\log R) \quad \text{as } R \to \infty\,.$$

We have thus shown that both terms on the right-hand side of (8.23) are $o(\log\log R)$. This concludes the proof of (8.22). □

9. Non-Gaussianity

In this section, we prove Theorem 1.6, which implies that the critical 2D KPZ process $\mathscr{H}_t^\vartheta$ is non-Gaussian. Let us fix $t > 0$ and $\vartheta \in \mathbb{R}$. Fix also a nonzero nonnegative $\psi \in C_c^\infty(\mathbb{R}^2)$ such that $\int\psi = 1$; for the general case, one simply needs to divide ψ by its integral. The proof will be by contradiction: in a nutshell, we show that if $\mathscr{H}_t^\vartheta(\psi)$ were Gaussian, this would contradict the singularity of the SHF.

9.1. Main steps of the proof. Let us first outline the key steps before giving the details in the rest of this section.

Step 1. (Gaussian) Rigidity with respect to the initial condition. By Corollary 6.2, we have

$$\widetilde{\mathscr{H}}_t^{(r)}(\mathbf{0},\psi) := \int \psi(x)\Big(\log \mathscr{Z}_t^\vartheta(g_r^{(x)},\mathbf{1}) - \mathbb{E}\big[\log \mathscr{Z}_t^\vartheta(g_r^{(x)},\mathbf{1})\big]\Big)\,\mathrm{d}x \xrightarrow[r\downarrow 0]{d} \mathscr{H}_t^\vartheta(\mathbf{0},\psi), \tag{9.1}$$

where $g_r^{(x)}$ is the heat kernel at time r centred at x and $\mathbf{0}$ denotes the flat initial condition.

A first crucial observation is that if we replace $\mathbf{1}$ in $\mathscr{Z}_t^\vartheta(g_r^{(x)},\mathbf{1})$ by a bounded Lipschitz function f with $0 < c \leqslant f \leqslant C < \infty$, we have

$$\log \mathscr{Z}_t^\vartheta(g_r^{(x)},f) - \log \mathscr{Z}_t^\vartheta(g_r^{(x)},\mathbf{1}) = \log \mu_{t,r}^x(f), \tag{9.2}$$

where

$$\mu_{t,r}^x(\mathrm{d}y) := \frac{\mathscr{Z}_t^\vartheta(g_r^{(x)},\mathrm{d}y)}{\mathscr{Z}_t^\vartheta(g_r^{(x)},\mathbf{1})} \tag{9.3}$$

can be interpreted as the endpoint distribution at time t of a continuum polymer with partition function $\mathscr{Z}_t^\vartheta(g_r^{(x)},\mathbf{1})$. Since $c \leqslant f \leqslant C$, the quantity in (9.2) lies in $[\log c, \log C]$, uniformly in r and x.

Replacing $\mathbf{1}$ in the left hand side of (9.1) by $f = \mathrm{e}^h$ for some bounded Lipschitz h, then setting

$$\overline{\mathscr{H}}_t^{(r)}(h,\psi) := \int \psi(x)\Big(\log \mathscr{Z}_t^\vartheta(g_r^{(x)},f) - \mathbb{E}\big[\log \mathscr{Z}_t^\vartheta(g_r^{(x)},f)\big]\Big)\,\mathrm{d}x\,, \tag{9.4}$$

we have that

$$\big|\overline{\mathscr{H}}_t^{(r)}(h,\psi) - \overline{\mathscr{H}}_t^{(r)}(\mathbf{0},\psi)\big| \leqslant \log(C/c). \tag{9.5}$$

Let us stress that contrary to the definitions (1.18) or (6.4), the centring in (9.4) depends on the initial condition $h = \log f$. This way, $\overline{\mathscr{H}}_t^{(r)}(h,\psi)$ is a *centred* random variable; note that it coincides with $\widetilde{\mathscr{H}}_t^{(r)}(h,\psi)$ when $h = \mathbf{0}$.

Remark 9.1. *More generally, let us consider two bounded Lipschitz initial conditions h_0 and h_0'. Replacing $f, \mathbf{1}$ by $\mathrm{e}^{h_0}, \mathrm{e}^{h_0'}$ in (9.2) yields that, for any $x \in \mathbb{R}^2$,*

$$\inf\{h_0 - h_0'\} \leqslant \log \mathscr{Z}_t^\vartheta(g_r^{(x)},\mathrm{e}^{h_0}) - \log \mathscr{Z}_t^\vartheta(g_r^{(x)},\mathrm{e}^{h_0'}) \leqslant \sup\{h_0 - h_0'\}\,.$$

When centring both terms by a common $c_t^{\vartheta,(r)} := \mathbb{E}[\log \mathscr{Z}_t^\vartheta(\mathbf{1}, g_r^{(x)})]$*, recalling the definition* (1.18)*, this yields that for any* $\delta > 0$

$$\big|\mathscr{H}_t^{\vartheta,(\delta)}(h_0,\psi) - \mathscr{H}_t^{\vartheta,(\delta)}(h_0',\psi)\big| \leqslant \|\psi\|_1 \, \|h_0 - h_0'\|_\infty.$$

A crucial observation from Section 3.4 is that both sides of (9.1) are of Ehrhard type, see in particular Proposition 3.8. The same is true for (9.4) and its limit as $r \downarrow 0$, and the inequality (9.5) also remains valid in the limit. This is recorded in the following lemma, proven in Section 9.2 below. For simplicity of notation, we let

$$X_r := \overline{\mathscr{H}}_t^{(r)}(\mathbf{0},\psi) \qquad \text{and} \qquad Y_r := \overline{\mathscr{H}}_t^{(r)}(h,\psi)\,.$$

Lemma 9.2. *For every bounded Lipschitz function* f *with* $0 < c \leqslant f \leqslant C < \infty$*, the laws of* (X_r, Y_r) *are tight as* $r \downarrow 0$*. Every joint subsequential limit* (X, Y) *is centred and satisfies*

$$X \overset{\mathrm{d}}{=} \mathscr{H}_t^\vartheta(\mathbf{0},\psi), \qquad |Y - X| \leqslant \log(C/c) \quad \textit{almost surely.}$$

Both Y *and* $X + Y$ *are of Ehrhard type.*

If we now assume that $\mathscr{H}_t^\vartheta(\mathbf{0},\psi)$ is Gaussian, then Lemma 9.2 allows us to apply the following Gaussian rigidity result. A related rigidity result for convex functions of Gaussian vectors is given in [FMP25, Theorem 3.4].

Lemma 9.3 (Gaussian rigidity)**.** *Let* X, Y *be* centred *real random variables with* $X \sim \mathcal{N}(0,\sigma^2)$*,* $\sigma \geqslant 0$*, and* $|Y - X| \leqslant K < \infty$ *almost surely. If* Y *and* $X + Y$ *are of Ehrhard type, then* $Y = X$ *almost surely.*

Under the Gaussianity assumption, Lemma 9.3 implies that $Y = X$ almost surely for every joint subsequential limit (X, Y) in Lemma 9.2. We next deduce from this equality an unlikely consequence for the polymer endpoint distributions.

Step 2. Consequence of Gaussian rigidity on the endpoint distribution. Applying Lemmas 9.2 and 9.3 to every subsequential weak limit of (X_r, Y_r) gives that $Y_r - X_r \to 0$ in probability. Recalling (9.2), this leads to the following consequence (details are given in Section 9.3). If $\mathscr{H}_t^\vartheta(\mathbf{0},\psi)$ is Gaussian or a constant, then, for every bounded Lipschitz function f with $0 < c \leqslant f \leqslant C < \infty$ and every $1 \leqslant p < \infty$,

$$\int \psi(x)\Big(\log \mu_{t,r}^x(f) - \mathbb{E}\big[\log \mu_{t,r}^x(f)\big]\Big)\,\mathrm{d}x \xrightarrow[r\downarrow 0]{} 0 \qquad \text{in } L^p(\mathbb{P}). \tag{9.6}$$

The freedom to choose f in (9.6) suggests that limits of the averaged polymer endpoint distribution at time t are deterministic: this is formalized by Lemma 9.8 below. Define, for $r, t > 0$,

$$\mu_{t,r}^\psi := \int \psi(x)\mu_{t,r}^x\,\mathrm{d}x \in \mathcal{M}_1(\mathbb{R}^2), \tag{9.7}$$

where $\mathcal{M}_1(\mathbb{R}^2)$ denotes the space of probability measures on $\mathbb{R}^2$, equipped with weak convergence. By choosing $f = \mathbf{1} + \varepsilon\varphi$ and then differentiating w.r.t. ε, we can deduce from (9.6) the following result.

Lemma 9.4 (Deterministic endpoint limits)**.** *If* (9.6) *holds for every bounded positive Lipschitz function* f *bounded away from zero, then for any* $\varphi \in C_c(\mathbb{R}^2)$

$$\mathbb{E}\Big[\big|\mu_{t,r}^\psi(\varphi) - \mathbb{E}\big[\mu_{t,r}^\psi(\varphi)\big]\big|\Big] \longrightarrow 0 \qquad \textit{as } r \downarrow 0\,. \tag{9.8}$$

Additionally, every subsequential limit in distribution of $\mu_{t,r}^\psi$ *(for the vague topology) is a deterministic finite positive measure of mass at most one.*

The next lemma shows that the deterministic limit points arising from Lemma 9.4 cannot be the zero measure.

Lemma 9.5 (Nonzero deterministic limits)**.** *If $\mathscr{H}_t^\vartheta(\mathbf{0},\psi)$ is Gaussian, the laws of the family of random measures $(\mu^\psi_{t,r})_{r\in(0,1)}$ (equipped with the vague topology) are tight as $r\downarrow 0$. Every subsequential weak limit μ_t is a deterministic finite positive measure with $0<\mu_t(\mathbb{R}^2)\leqslant 1$.*

The proofs of Lemmas 9.6 and 9.5 are deferred to Section 9.3.

Step 3. Contradiction and conclusion of the proof. Finally we prove two contradicting statements for such a deterministic limiting measure μ obtained from Lemma 9.5.

Lemma 9.6 (Comparison with the later SHF)**.** *Assume that $\mathscr{H}_t^\vartheta(\mathbf{0},\psi)$ is Gaussian and let μ_t be a weak limit of $(\mu^\psi_{t,r})_{r\in(0,1)}$ as in Lemma 9.5. With positive probability, μ_t and $\mathscr{Z}^\vartheta_{2t/3,t}(\mathbf{1},\mathrm{d}y)$ are not mutually singular.*

The key idea behind the proof is to integrate out disorder on the time interval $[t/3,2t/3]$ for the SHF that defines $\mu^\psi_{t,r}$.

On the other hand, the almost sure singularity of the SHF with Lebesgue initial condition, see [CSZ25, Theorem 1.1], together with $\mathbb{E}\big[\mathscr{Z}^\vartheta_{2t/3,t}(\mathbf{1},\mathrm{d}y)\big]=\mathrm{d}y$, gives the opposite relation.

Lemma 9.7 (Singularity with deterministic measures)**.** *For every deterministic finite positive measure μ on $\mathbb{R}^2$, we have that $\mu\perp\mathscr{Z}^\vartheta_{2t/3,t}(\mathbf{1},\mathrm{d}y)$ a.s., that is, μ is singular with respect to $\mathscr{Z}^\vartheta_{2t/3,t}(\mathbf{1},\mathrm{d}y)$.*

The proofs of these two lemmas are postponed to Section 9.4. Let us now show how these ingredients complete the argument.

Proof of Theorem 1.6. Suppose that $\mathscr{H}_t^\vartheta(\mathbf{0},\psi)$ is Gaussian, including a constant one. Lemma 9.5 gives the existence of a nonzero deterministic measure μ_t arising as a weak subsequential limit of $\mu^\psi_{t,r}$ as $r\downarrow 0$. By Lemma 9.6, this measure μ_t and $\mathscr{Z}^\vartheta_{2t/3,t}(\mathbf{1},\mathrm{d}y)$ are not mutually singular with positive probability. On the other hand, since μ_t is deterministic, Lemma 9.7 says that they are mutually singular almost surely. This is a contradiction. □

9.2. Ehrhard type and Gaussian rigidity: Lemmas 9.2 and 9.3. We now prove the "Gaussian rigidity" results that are used in Step 1.

Proof of Lemma 9.2. The convergence of Corollary 6.2, *i.e.* $X_r\xrightarrow{\mathrm{d}}X$, and the bound (9.5) give joint tightness of (X_r,Y_r). Every joint subsequential limit (X,Y) is such that $X\overset{\mathrm{d}}{=}\mathscr{H}_t^\vartheta(\mathbf{0},\psi)$ and satisfies $|Y-X|\leqslant\log(C/c)$. The limit X is centred by Corollary 6.2, and the bounded differences Y_r-X_r have mean zero, which passes to the limit, so Y is centred as well.

For fixed $r>0$, the variables X_r, Y_r and X_r+Y_r are of Ehrhard type, as explained in Section 3.4; see also [LX26, Lemma 3.1 and Section 4]. Indeed, their lattice approximations are nonnegative linear combinations of centred log partition functions, which are convex in the same Gaussian disorder. The weak closure in Lemma 3.6 therefore shows that Y and $X+Y$ are of Ehrhard type. □

Proof of Lemma 9.3. Suppose first that $\sigma>0$. By Lemma 3.6, Y has the law of $h(G)$, where G is standard Gaussian and h is a finite, nondecreasing and convex function. Our goal is to show that h must be affine.

The bound $|Y - X| \leqslant K$ implies that Y has the same Gaussian tails as X, i.e.,

$$\lim_{z\to\infty} \frac{\log \mathbb{P}(Y > z)}{z^2} = \lim_{z\to\infty} \frac{\log \mathbb{P}(Y < -z)}{z^2} = -\frac{1}{2\sigma^2}. \tag{9.9}$$

Let $az + b$ be a supporting line of h, so $h(z) \geqslant az + b$ for all z, with $a \geqslant 0$ by monotonicity. If $a > \sigma$, then

$$\mathbb{P}(Y > z) = \mathbb{P}(h(G) > z) \geqslant \mathbb{P}(aG + b > z),$$

and hence $\liminf_{z\to\infty} z^{-2} \log \mathbb{P}(Y > z) \geqslant -1/(2a^2) > -1/(2\sigma^2)$, contradicting (9.9). Similarly, if $0 \leqslant a < \sigma$, the bound $\mathbb{P}(Y < -z) \leqslant \mathbb{P}(aG + b < -z)$ contradicts (9.9). Therefore every supporting line of h has slope σ so this proves that h is affine, namely $h(z) = \sigma z + b$. The fact that $Y = \sigma G + b$ is centred means that $Y \sim \mathcal{N}(0, \sigma^2)$.

Apply the same argument to $X + Y$, which is of Ehrhard type and differs from $2X$ by at most K. It follows that $X + Y \sim \mathcal{N}(0, 4\sigma^2)$, and hence

$$\mathbb{E}\big[(X - Y)^2\big] = 2\mathbb{E}[X^2] + 2\mathbb{E}[Y^2] - \mathbb{E}\big[(X + Y)^2\big] = 0.$$

This concludes the proof. When $\sigma = 0$, it is easy to see that $X = Y = 0$. □

9.3. About the polymer endpoint distribution: Lemmas 9.4 and 9.5. We now turn to the proof of the estimates in Step 2, which highlights the consequence of Gaussian rigidity on the polymer endpoint distribution.

Proof of (9.6)**.** By Lemma 9.2, every sequence $r_n \downarrow 0$ has a subsequence along which (X_{r_n}, Y_{r_n}) converges in law to a centred pair (X, Y) satisfying the hypotheses of Lemma 9.3, since X is Gaussian by assumption. Thus $Y = X$ almost surely, so $Y_r - X_r \to 0$ in probability. The bound (9.5) upgrades this to convergence in every finite L^p, proving (9.6). □

Proof of Lemma 9.4. Without loss of generality, we may take $\varphi \in C_c^\infty(\mathbb{R}^2)$ with $\|\varphi\|_\infty \leqslant 1$ and $0 < \varepsilon \leqslant 1/2$. Since $\mu^x_{t,r}$ is a probability measure we have $\mu^x_{t,r}(\mathbf{1} + \varepsilon\varphi) = 1 + \varepsilon\mu^x_{t,r}(\varphi)$. Then, using that $|\varepsilon^{-1}\log(1 + \varepsilon a) - a| \leqslant \varepsilon/[2(1 - \varepsilon)^2]$ for any $|a| \leqslant 1$ we otbain that

$$\Big|\frac{1}{\varepsilon}\log \mu^x_{t,r}(\mathbf{1} + \varepsilon\varphi) - \mu^x_{t,r}(\varphi)\Big| \leqslant \frac{\varepsilon}{2(1 - \varepsilon)^2}$$

and hence

$$\Big|\frac{1}{\varepsilon}\int \psi(x)\log \mu^x_{t,r}(\mathbf{1} + \varepsilon\varphi)\,\mathrm{d}x - \mu^\psi_{t,r}(\varphi)\Big| \leqslant \frac{\varepsilon}{2(1 - \varepsilon)^2}. \tag{9.10}$$

Centring both terms by their mean at most doubles this error. Thus, applying (9.6) with $f = \mathbf{1} + \varepsilon\varphi$ and $p = 1$, while ε remains fixed, we get that

$$\limsup_{r\downarrow 0} \mathbb{E}\Big[\big|\mu^\psi_{t,r}(\varphi) - \mathbb{E}\big[\mu^\psi_{t,r}(\varphi)\big]\big|\Big] \leqslant \frac{\varepsilon}{(1 - \varepsilon)^2}.$$

Letting $\varepsilon \downarrow 0$ yields (9.8).

We now turn to the second part of the lemma. Every subsequential limit μ_t is a random measure (on the space of locally finite measures equipped with the vague topology) with mass at most one. This boundedness allows us to pass to the limit in (9.8), giving that

$$\mu_t(\varphi) = \mathbb{E}\big[\mu_t(\varphi)\big] \ a.s., \qquad \forall\, \varphi \in C_c^\infty(\mathbb{R}^2)\,.$$

Applying this to a countable measure-determining family of $\varphi \in C_c^\infty(\mathbb{R}^2)$ then shows that $\mu_t(\cdot) = \mathbb{E}[\mu_t(\cdot)]$ a.s., proving that μ_t is deterministic. □

Proof of Lemma 9.5. Tightness for the vague topology follows from the fact that $\mu^{\psi}_{t,r}$ has mass one almost surely. Let μ_t be an arbitrary deterministic subsequential limit as $r \downarrow 0$, and fix any nonzero $\varphi_0 \in C_c^\infty(\mathbb{R}^2)$ with $0 \leqslant \varphi_0 \leqslant 1$. It suffices to prove that for some $c > 0$ we have that

$$\mathbb{E}\big[\mu(\varphi_0)\big] \geqslant c\,. \tag{9.11}$$

Recall the definition (9.3) of $\mu^x_{t,r}$. By Jensen's inequality we have that

$$\log \mathbb{E}\big[\mu^x_{t,r}(\varphi_0)\big] \geqslant \mathbb{E}\big[\log \mathscr{Z}^{\vartheta}_t(g_r^{(x)}, \varphi_0)\big] - \mathbb{E}\big[\log \mathscr{Z}^{\vartheta}_t(g_r^{(x)}, \mathbf{1})\big]\,.$$

Recalling that $\mu^{\psi}_{t,r}(\varphi_0) \coloneqq \int \psi(x)\mu^x_{t,r}(\varphi_0)\,\mathrm{d}x$, it suffices to prove the lower bound

$$\mathbb{E}\big[\log \mathscr{Z}^{\vartheta}_t(g_r^{(x)}, \varphi_0)\big] - \mathbb{E}\big[\log \mathscr{Z}^{\vartheta}_t(g_r^{(x)}, \mathbf{1})\big] \geqslant -C > -\infty\,, \tag{9.12}$$

for some C that is uniform in $x \in \operatorname{supp}(\psi)$ and r small. Indeed, integrating w.r.t. $\psi(x)\,\mathrm{d}x$ and passing to the subsequential limit $r \downarrow 0$ gives (9.11) with $c = \mathrm{e}^{-C}$.

The desired bound (9.12) follows easily from the next two bounds:

$$\Big|\mathbb{E}\big[\log \mathscr{Z}^{\vartheta}_t(g_r^{(x)}, \mathbf{1})\big] + \frac{1}{2}\log\log\Big(\frac{1}{r}\Big)\Big| \leqslant C < \infty, \qquad 0 < r < r_0, \quad x \in \mathbb{R}^2, \tag{9.13}$$

and

$$\mathbb{E}\big[\log \mathscr{Z}^{\vartheta}_t(g_r^{(x)}, \varphi_0)\big] \geqslant -\frac{1}{2}\log\log\Big(\frac{1}{r}\Big) - C, \tag{9.14}$$

where C is does not depend on $x \in \operatorname{supp}(\psi)$ and r is small. Indeed, the two terms $\frac{1}{2}\log\log(1/r)$ cancel in their difference in (9.12).

It remains to establish (9.13) and (9.14). The first estimate (9.13) is the continuum analogue of Lemma 4.3 and follows the same proof: it requires negative-first and second moment bounds, that can be found in [LX26, Proposition 5.3 with $p = -1$, and (3.14)]. For the second estimate (9.14), we apply (3.3) to $\log\big(\mathscr{Z}^{\vartheta}_t(g_r^{(x)}, \varphi_0)/\mathbb{E}[\mathscr{Z}^{\vartheta}_t(g_r^{(x)}, \varphi_0)]\big)$ to get that

$$\mathbb{E}\big[\log \mathscr{Z}^{\vartheta}_t(g_r^{(x)}, \varphi_0)\big] \geqslant -\frac{1}{2}\log \mathbb{E}\big[\mathscr{Z}^{\vartheta}_t(g_r^{(x)}, \varphi_0)^2\big] + 2\log \mathbb{E}\big[\mathscr{Z}^{\vartheta}_t(g_r^{(x)}, \varphi_0)\big].$$

Since $0 \leqslant \varphi_0 \leqslant 1$, we have

$$\mathbb{E}\big[\mathscr{Z}^{\vartheta}_t(g_r^{(x)}, \varphi_0)\big] = (g_{t+r} * \varphi_0)(x) \geqslant c > 0, \qquad \mathbb{E}\big[\mathscr{Z}^{\vartheta}_t(g_r^{(x)}, \varphi_0)^2\big] \leqslant C\log(1/r),$$

uniformly for $x \in \operatorname{supp}\psi$ and small r, which plugged in the above proves (9.14). This concludes the proof. □

9.4. Continuity vs. singularity with respect to the SHF: Lemmas 9.6 and 9.7. We now prove the two inputs in Step 3.

Proof of Lemma 9.6. Fix a subsequence $r_n \downarrow 0$ along which μ^{ψ}_{t,r_n} converges in distribution to some deterministic measure μ_t. Our goal is to show that, with positive probability, μ_t has a non-trivial part that is absoluetly continuous w.r.t. $\mathscr{Z}^{\vartheta}_{2t/3,t}(\mathbf{1}, \mathrm{d}y)$.

Integrating out disorder on $[t/3, 2t/3]$. Let $\mathcal{G} \coloneqq \mathcal{F}_{[0,t/3]} \vee \mathcal{F}_{[2t/3,t]}$, where $\mathcal{F}_{[a,b]}$ denotes the σ-algebra generated by the SHF increments $\mathscr{Z}^{\vartheta}_{u,v}$ with $[u,v] \subset [a,b]$. Note that conditioning on $\mathcal{G}$ simply integrates out the SHF on $[t/3, 2t/3]$.

It follows from Lemma 9.4 that for any $\varphi \in C_c(\mathbb{R}^2)$ the convergence $\mu^{\psi}_{t,r_n}(\varphi) \to \mu_t(\varphi)$ also holds in $L^1(\mathbb{P})$, because the limiting measure μ_t is deterministic and the measures have bounded mass. By Jensen's inequality, we also have that the following conditional expectation converges to the same deterministic limit:

$$\mathbb{E}\big[\mu^{\psi}_{t,r_n}(\varphi) \mid \mathcal{G}\big] \xrightarrow[n\to\infty]{} \mu_t(\varphi) \qquad \text{in } L^1(\mathbb{P}).$$

Along a further subsequence if necessary, this implies that almost surely, the conditional probability distribution $\mathbb{E}[\mu^{\psi}_{t,r_n}(\cdot) \mid \mathcal{G}]$ converges vaguely to μ_t, that is

$$\mathbb{E}\big[\mu^{\psi}_{t,r_n}(\cdot) \mid \mathcal{G}\big] \xrightarrow[n\to\infty]{v} \mu_t \qquad \mathbb{P}\text{-a.s.} \tag{9.15}$$

Extracting an absolutely continuous part. Recalling the definition (9.3) of $\mu^x_{t,r}(\mathrm{d}y)$, we can write

$$\mathbb{E}\left[\mu^x_{t,r}(\mathrm{d}y) \mid \mathcal{G}\right] = \mathbb{E}\bigg[\frac{1}{R_1}\,\frac{\mathscr{Z}^{\vartheta}_t(g_r^{(x)},\mathrm{d}y)}{\mathscr{Z}^{\vartheta}_{t/3}(g_r^{(x)},\mathbf{1})}\;\bigg|\;\mathcal{G}\bigg], \tag{9.16}$$

where R_1 is a correcting factor given by

$$R_1 = R_1(r,x) \coloneqq \frac{\mathscr{Z}^{\vartheta}_t(g_r^{(x)},\mathbf{1})}{\mathscr{Z}^{\vartheta}_{t/3}(g_r^{(x)},\mathbf{1})}.$$

We prove below that

$$\mathbb{E}\bigg[\frac{\mathscr{Z}^{\vartheta}_t(g_r^{(x)},\mathrm{d}y)}{\mathscr{Z}^{\vartheta}_{t/3}(g_r^{(x)},\mathbf{1})}\;\bigg|\;\mathcal{G}\bigg] \leqslant \frac{3}{2\pi t}\,\mathscr{Z}^{\vartheta}_{2t/3,t}(\mathbf{1},\mathrm{d}y)\,. \tag{9.17}$$

To deal with the prefactor $1/R_1$ inside (9.16), we let $a > 0$ some parameter (chosen sufficiently small below) and we introduce

$$\eta_{t,r}(\mathrm{d}y) \coloneqq \mathbb{E}\bigg[\int \psi(x)\mathbf{1}_{\{R_1(r,x)\geqslant a\}}\mu^x_{t,r}(\mathrm{d}y)\,\mathrm{d}x\;\bigg|\;\mathcal{G}\bigg], \tag{9.18}$$

so that almost surely

$$\eta_r(\mathrm{d}y) \leqslant \frac{1}{a}\int \psi(x)\mathbb{E}\bigg[\frac{\mathscr{Z}^{\vartheta}_t(g_r^{(x)},\mathrm{d}y)}{\mathscr{Z}^{\vartheta}_{t/3}(g_r^{(x)},\mathbf{1})}\;\bigg|\;\mathcal{G}\bigg]\mathrm{d}x \leqslant \frac{3}{2\pi t a}\,\mathscr{Z}^{\vartheta}_{2t/3,t}(\mathbf{1},\mathrm{d}y)\,, \tag{9.19}$$

where the last inequality follows from (9.17). We also show below that that for some nonzero $\varphi_0 \in C^\infty_c(\mathbb{R}^2)$ with $0 \leqslant \varphi_0 \leqslant 1$, the parameter a can be chosen sufficiently small such that, for some $c > 0$,

$$\mathbb{E}\big[\eta_{t,r}(\varphi_0)\big] \geqslant c \quad \text{uniformly in } r \text{ small.} \tag{9.20}$$

The two estimates (9.19)-(9.20) allow us to conclude the proof. Indeed, by (9.19), we have that $\eta_{t,r}$ and any of its weak limit η_t are absolutely continuous with respect to $\mathscr{Z}^{\vartheta}_{2t/3,t}(\mathbf{1},\mathrm{d}y)$. On the other hand, (9.20) ensures that any weak limit η_t is non-zero with positive probability. Additionally, we have $\eta_{t,r} \leqslant \mathbb{E}[\mu^{\psi}_{t,r} \mid \mathcal{G}]$ as follows directly from (9.18), and this inequality also remains true by taking weak limits, giving $\eta_t \leqslant \mu_t$ thanks to (9.15). In other words, we have found some measure $\eta_t \leqslant \mu_t$, which is non-zero and absolutely continuous w.r.t. $\mathscr{Z}^{\vartheta}_{2t/3,t}(\mathbf{1},\mathrm{d}y)$ with positive probability.

This concludes the argument and it thus only remains to show (9.17) and (9.20).

Proof of (9.17). We start by noticing that

$$\mathbb{E}\bigg[\frac{\mathscr{Z}^{\vartheta}_t(g_r^{(x)},\mathrm{d}y)}{\mathscr{Z}^{\vartheta}_{t/3}(g_r^{(x)},\mathbf{1})}\;\bigg|\;\mathcal{G}\bigg] = \int (\mu^x_{t/3,r} * g_{t/3})(z)\;\mathscr{Z}^{\vartheta}_{2t/3,t}(\mathrm{d}z,\mathrm{d}y),$$

where $\mu^x_{t/3,r}(\mathrm{d}w) = \mathscr{Z}^{\vartheta}_{t/3}(g_r^{(x)},\mathrm{d}w)/\mathscr{Z}^{\vartheta}_{t/3}(g_r^{(x)},\mathbf{1})$ as in (9.3) and the heat kernel $g_{t/3}$ arises from integrating out the SHF on the time interval $[t/3, 2t/3]$. For a proof of such an identity,

see e.g. [CM26, Proposition 2.4]. Since $\mu^x_{t/3,r}$ is a probability measure and $g_{t/3}(y) \leqslant 3/(2\pi t)$ uniformly in y, we have the uniform bound

$$(\mu^x_{t/3,r} * g_{t/3})(z) = \int g_{t/3}(z-w)\, \mu^x_{t/3,r}(\mathrm{d}w) \leqslant \frac{3}{2\pi t}.$$

This proves (9.17).

Proof of (9.20). By the definition of $\eta_{t,r}$ in (9.18), we have

$$\mathbb{E}\big[\eta_r(\varphi_0)\big] = \int \psi(x)\mathbb{E}\big[\mu^x_{t,r}(\varphi_0)\mathbf{1}_{\{R_1(r,x)\geqslant a\}}\big]\,\mathrm{d}x. \tag{9.21}$$

We want a positive lower bound on the expectation inside this integral, uniformly for $x \in \operatorname{supp}\psi$ and r small. valid uniformly for r small. For fixed x, let us set

$$R_2 := \frac{\mathscr{Z}^\vartheta_t(g^{(x)}_r, \varphi_0)}{\mathscr{Z}^\vartheta_{t/3}(g^{(x)}_r, \mathbf{1})},$$

so in particular $R_2/R_1 = \mu^x_{t,r}(\varphi_0)$. We thus have

$$\mathbb{E}\big[\mu^x_{t,r}(\varphi_0)\mathbf{1}_{\{R_1\geqslant a\}}\big] = \mathbb{E}\Big[\frac{R_2}{R_1}\Big] - \mathbb{E}\Big[\frac{R_2}{R_1}\mathbf{1}_{\{R_1<a\}}\Big] \geqslant \mathbb{E}\Big[\frac{R_2}{R_1}\Big] - \mathbb{P}(R_1 < a)\,,$$

using also that $R_2 \leqslant R_1$. First, notice that by Jensen's inequality we have that

$$\mathbb{E}\Big[\frac{R_2}{R_1}\Big] \geqslant \exp\big(\mathbb{E}[\log R_2] - \mathbb{E}[\log R_1]\big) \geqslant \mathrm{e}^{-C},$$

where we have used that $\mathbb{E}[\log R_1] \leqslant \log \mathbb{E}[R_1] = 0$ since $\mathbb{E}[R_1] = 1$ and also that $\mathbb{E}[\log R_2] \geqslant -C$ uniformly in r, by applying (9.12) at time $t/3$ and (9.13)–(9.14) at time t. On the other hand, by Markov's inequality we have for $a \in (0,1)$

$$\mathbb{P}(R_1 < a) \leqslant \frac{\mathbb{E}[(\log R_1)^-]}{\log(1/a)} \leqslant \frac{\mathbb{E}[R_1] - \mathbb{E}[\log R_1]}{\log(1/a)} \leqslant \frac{1+C}{\log(1/a)}$$

where we have used first that $(\log b)^- \leqslant b - \log b$ for all $b > 0$ and then, as above, that $\mathbb{E}[R_1] = 1$ and $\mathbb{E}[\log R_1] \geqslant -C$ (by applying (9.13) at time $t/3$ and t).

All together, we have that uniformly for r small

$$\mathbb{E}\big[\mu^x_{t,r}(\varphi_0)\mathbf{1}_{\{R_1\geqslant a\}}\big] \geqslant \mathrm{e}^{-C} - \frac{1+C}{\log(1/a)} \geqslant \frac{1}{2}\mathrm{e}^{-C}\,,$$

the last inequality being valid for a small enough. Plugged in (9.21), this concludes the proof of (9.20). □

Proof of Lemma 9.7. Let μ be a non-zero deterministic measure and let us decompose μ into $\mu = \mu^{\mathrm{a}} + \mu^{\mathrm{s}}$, where μ^{a}, μ^{s} are the absolutely continuous and singular parts w.r.t. the Lebesgue measure. We show that both parts are a.s. singular w.r.t. $\mathscr{Z}^\vartheta_{2t/3,t}(\mathbf{1}, \mathrm{d}y)$.

Indeed, since $\mathscr{Z}^\vartheta_{2t/3,t}(\mathbf{1}, \mathrm{d}y)$ is a.s. singular w.r.t. Lebesgue measure by [CSZ25, Theorem 1.1], μ^{a} must be singular w.r.t. $\mathscr{Z}^\vartheta_{2t/3,t}(\mathbf{1}, \mathrm{d}y)$. On the other hand, let B be a Lebesgue set with $\mathrm{Leb}(B) = 0$ and $\mu^{\mathrm{s}}(B^c) = 0$. We have $\mathbb{E}[\mathscr{Z}^\vartheta_{2t/3,t}(\mathbf{1}, \mathbf{1}_B)] = \mathrm{Leb}(B) = 0$ so that $\mathscr{Z}^\vartheta_{2t/3,t}(\mathbf{1}, \mathbf{1}_B) = 0$ a.s., and thus μ^{s} is singular w.r.t. $\mathscr{Z}^\vartheta_{2t/3,t}(\mathbf{1}, \mathrm{d}y)$. □

Appendix A. On the block expansion: proof of Proposition 7.4

The goal of this section is to prove Proposition 7.4. The result is similar to that of [GT26, Corollary 3.5], but it is not straightforward to adapt their proof to our setting. We present here a complete proof for the sake of completeness.

A.1. Reminder of chaos expansions, second moment and renewal estimates. We gather in this section a number of auxiliary results on local limit theorems, second moment estimate and asymptotics.

Chaos expansion. The estimates in Proposition 7.4, will be done through the following chaos expansion of the partition functions. More precisely, denote by

$$\xi_N(n,z) := e^{\beta_N \omega(n,z) - \beta_N^2/2} - 1$$

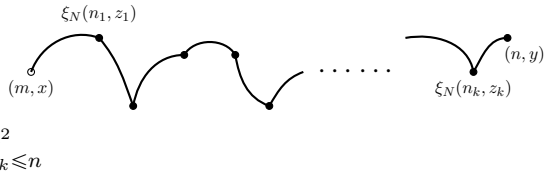
which verifies $\mathbb{E}[\xi_N(n,z)] = 0$ and

$$\mathrm{Var}[\xi_N(n,z)] = e^{\beta_N^2} - 1 =: \sigma_N^2 \,.$$

Expanding, essentially, the exponential Gibbs factor in the partition function, we have the following expansion for the point-to-point partition function $Z^{\beta_N}_{m,n}(x,y)$ (we refer to [CSZ26, Section 2] for details):

$$Z^{\beta_N}_{m,n}(x,y) = \sum_{k=0}^{\infty} \sum_{\substack{z_1,\dots,z_k \in \mathbb{Z}^2 \\ m<n_1<\cdots<n_k\leqslant n}} \Big\{ \prod_{i=1}^{k} q_{n_i-n_{i-1}}(z_i - z_{i-1}) \xi_N(n_i,z_i) \Big\} q_{n-n_k}(y-z_k)$$

$$= q_{n-m}(y-x) + \sum_{k=1}^{\infty} \sum_{\substack{z_1,\dots,z_k \in \mathbb{Z}^2 \\ m<n_1<\cdots<n_k\leqslant n}}$$



, (A.1)

using the convention $(n_0,z_0) = (m,x)$ and $q_0(x) = \mathbf{1}_{\{x=0\}}$; the term $k=0$ simply corresponds to $q_{n-m}(y-x)$. In the graphical representation, the laces represent the transition kernels $q_{n_i-n_{i-1}}(z_i - z_{i-1})$, the solid dots represent the disorder variables $\xi_N(n_i,z_i)$. (Note that disorder is excluded at time m and included at time n.)

Second moments. Using the orthogonality of disorder weights ξ_N, we can use the chaos expansion to obtain formulas for the second moments of the variance of the partition functions. More precisely, let us introduce the following collision renewal density: set $U_N(n,x) = \mathbf{1}_{\{x=0\}}$ if $n=0$ and

$$U_N(n,x) = \sum_{k=1}^{\infty} \sum_{\substack{z_1,\dots,z_{k-1} \in \mathbb{Z}^2 \\ 0<n_1<\dots<n_{k-1}<n}} \prod_{j=1}^{k} \sigma_N^2 \, q_{n_j-n_{j-1}}(z_j - z_{j-1})^2 \tag{A.2}$$

if $n \geqslant 1$, with again the convention $(n_0,z_0) = (0,0)$, $(n_k,z_k) := (n,x)$. We also denote

$$\overline{U}_N(n,x) := \sigma_N^2 U_N(n,x) \qquad \overline{U}_N(n) := \sum_{x\in\mathbb{Z}^2} \overline{U}_N(n,x) \,. \tag{A.3}$$

By orthogonality of the chaos expansion, for $n \geqslant 1$ we have $\mathbb{E}[Z_{0,n}(0,x)^2] = \frac{1+\sigma_N^2}{\sigma_N^2} U_N(n,x)$.

Let us also introduce

$$G_\vartheta(t) = \int_0^\infty \frac{e^{(\vartheta-\gamma)s}\, s\, t^{s-1}}{\Gamma(s+1)}\,\mathrm{d}s\,, \qquad t \in (0,1]\,,\ \vartheta \in \mathbb{R}\,, \tag{A.4}$$

with γ the Euler-Mascheroni constant. The function G_ϑ is related to the Dickman subordinator [CSZ19a] for $t \in [0,1]$. Note also that $G_\vartheta(t) = \mathrm{e}^{\vartheta-\gamma}\nu'(t\,\mathrm{e}^{\vartheta-\gamma})$, where $\nu(T) \coloneqq \int_0^\infty \frac{T^s}{\Gamma(s+1)}\mathrm{d}s$ is the so-called Volterra function [CM26].

Proposition A.1. *For any fixed $\delta > 0$ and uniformly in $\delta N \leqslant n \leqslant N$, as $N \to \infty$, we have*

$$\overline{U}_N(n) = \frac{\pi}{N}\, G_\vartheta\big(\tfrac{n}{N}\big)(1 + o(1)). \tag{A.5}$$

Furthermore, there exists $C > 0$ such that uniformly for any $1 \leqslant n \leqslant N$, we have the upper bound

$$\overline{U}_N(n) \leqslant \frac{C}{N} G_\vartheta(\frac{n}{N}). \tag{A.6}$$

Additionally, there exists a constant $C = C_\vartheta > 0$ such that for any $1 \leqslant n \leqslant N$,

$$\overline{U}_N(n,x) \leqslant C\overline{U}_N(n)\hat{g}_n(x)\,, \tag{A.7}$$

where $\hat{g}_n$ is a $\mathbb{Z}^2$ version of a heat kernel:

$$\hat{g}_n(x) \coloneqq \frac{1}{n}\mathrm{e}^{-\mathsf{c}_0|x|^2/n}\mathbf{1}_{\{n+x_1+x_2\in 2\mathbb{Z}\}}\,, \tag{A.8}$$

with a constant $\mathsf{c}_0 \leqslant 1/2$. By convention, we set $\hat{g}_0(x) = \mathbf{1}_{\{x=0\}}$ in the case $n = 0$.

The proof of (A.5) and (A.6) are both given in [CSZ19a, Theorem 2.3]. The bound (A.7) does not seem to appear exactly in this form in the literature (even though it is close to [CSZ23, Lemma 3.5]): we provide a proof in Subsection A.3.

In the following, we choose the constant c_0 in the definition of $\hat{g}_n$ small enough such that Lemma A.2 below holds, and also small enough such that $q_n(x) \leqslant C\hat{g}_n(x)$, with a constant C uniform in n, x. We stress that $\hat{g}_n$ may not be of mass one, but has a bounded mass $\sum_{x\in\mathbb{Z}^2}\hat{g}_n(x) \leqslant C$, uniformly in n.

A.2. Proof of Proposition 7.4. We first give the details for (7.17) and then we adapt the argument to (7.18). Recall that

$$D_i(x,y) = \mathsf{Z}_i(x,y) - \mathsf{Z}_i(x,\mathbf{1})q_{N_i}(y) = Z_{N_{i-1},N_i}(x,y) - Z_{N_{i-1},N_i}(x,\mathbf{1})q_{N_i}(y).$$

Our goal is to compute the second moment of $D^{\mathrm{closed}}_{u,v} = D^{\mathrm{closed}}_{u,v} \coloneqq q_{N_{u-1}} \prec D_u \circ \cdots \circ D_{v-1} \circ \mathsf{Z}_v \succ \mathbf{1}$ using a chaos expansion for kernels D_i and their compositions. To better illustrate the scheme, we first present the estimate in the case of two consecutive blocks, *i.e.* we estimate $\|D^{\mathrm{closed}}_{u,u+1}\|_{L^2}$. We then show how to adapt the idea of the proof to an arbitrary number of blocks, providing an estimate on $\|D^{\mathrm{closed}}_{u,v}\|_{L^2}$.

Proof of (7.17) **in the case of one D-block.** For a technical reason we first present the case $u \geqslant 2$; we explain in Remark A.3 how the case $u = 1$ is treated with similar estimates. By definition of $\|D^{\mathrm{closed}}_{u,u+1}\|_{L^2}$ and independence, we have that

$$\mathbb{E}\big[(D^{\mathrm{closed}}_{u,u+1})^2\big] = \mathbb{E}\big[(q_{N_{u-1}} \prec D_u \circ \mathsf{Z}_{u+1} \succ \mathbf{1})^2\big] = q^{\otimes 2}_{N_{u-1}} \prec \mathbb{E}[D_u^{\otimes 2}] \circ \mathbb{E}[\mathsf{Z}^{\otimes 2}_{u+1}] \succ \mathbf{1}^{\otimes 2}.$$

By the chaos expansion (A.1), recalling (A.3), we can check that

$$\mathbb{E}\big[(D^{\mathrm{closed}}_{u,u+1})^2\big] = \sum_{\substack{N_{u-1}<a_1\leqslant b_1\leqslant N_u\\ N_u<a_2\leqslant b_2\leqslant N_{u+1}}} \sum_{x_1,y_1,x_2\in\mathbb{Z}^2} q_{a_1}(x_1)^2\,\overline{U}_N(b_1-a_1,y_1-x_1) \times \big(q_{a_2-b_1}(x_2-y_1)-q_{a_2}(x_2)\big)^2\,\overline{U}_N(b_2-a_2)\,. \tag{A.9}$$

Using the upper bound in (A.7) and also $q_{a_1}(x_1)\leqslant C\hat{g}_{a_1}(x_1)$ as noted after Proposition A.1, we can then first focus on controlling the spatial sum. For $a_1\leqslant b_1<a_2\leqslant b_2$ fixed

$$\sum_{x_1,y_1,x_2\in\mathbb{Z}^2} \hat{g}_{a_1}(x_1)^2\hat{g}_{b_1-a_1}(y_1-x_1)\big(q_{a_2-b_1}(x_2-y_1)-q_{a_2}(x_2)\big)^2\,. \tag{A.10}$$

For this, we may use the following lemma, whose proof is postponed to Subsection A.3.

Lemma A.2. *There is a constant $C>0$ such that, if the constant c_0 is small enough in the definition* (A.8) *of $\hat{g}_n$, then for any k,ℓ*

$$\sum_{x\in\mathbb{Z}^2}\hat{g}_k(x)\hat{g}_\ell(y-x)\leqslant C\hat{g}_{k+\ell}(y) \tag{A.11}$$

and for any $k<\ell$

$$\sum_{y\in\mathbb{Z}^2}\hat{g}_k(y)\big[q_{\ell-k}(w-y)-q_\ell(w)\big]^2\leqslant C\,\frac{k}{\ell(\ell-k)}\hat{g}_\ell(w)\,. \tag{A.12}$$

With this lemma at hand, in (A.10) we can first bound $\hat{g}_{a_1}(x_1)^2\leqslant\frac{1}{a_1}\hat{g}_{a_1}(x_1)$ then sum over x_1 (thanks to (A.11)), then sum over y_1 (thanks to (A.12)) and finally sum over x_2 (with $\sum_w\hat{g}_s(w)\leqslant C$): we end up with

$$\sum_{x_1,y_1,x_2\in\mathbb{Z}^2} q_{a_1}(x_1)^2\hat{g}_{b_1-a_1}(y_1-x_1)\big(q_{a_2-b_1}(x_2-y_1)-q_{a_2}(x_2)\big)^2\leqslant C\frac{b_1}{a_1a_2(a_2-b_1)}\,. \tag{A.13}$$

Going back to (A.9), this gives

$$\mathbb{E}\big[(D^{\mathrm{closed}}_{u,u+1})^2\big]\leqslant C\sum_{\substack{N_{u-1}<a_1\leqslant b_1\leqslant N_u\\ N_u<a_2\leqslant b_2\leqslant N_{u+1}}}\frac{b_1}{a_1}\,\overline{U}_N(b_1-a_1)\,\frac{1}{a_2(a_2-b_1)}\,\overline{U}_N(b_2-a_2)\,.$$

Then, thanks to Proposition A.1-(A.6) and a Riemann sum approximation, employing the shorthand notation $\bar{N}_u=N_u/N$, we obtain that

$$\mathbb{E}\big[(D^{\mathrm{closed}}_{u,u+1})^2\big]\leqslant C\iint_{\substack{\bar{N}_{u-1}<s_1\leqslant t_1<\bar{N}_u\\ \bar{N}_u<s_2\leqslant t_2<\bar{N}_{u+1}}}\frac{t_1}{s_1}G_\vartheta(t_1-s_1)\,\frac{1}{s_2(s_2-t_1)}\,G_\vartheta(t_2-s_2)\,\mathrm{d}\vec{s}\,\mathrm{d}\vec{t}\,. \tag{A.14}$$

It remains to estimate this time integral. (Note that the integral is infinite in the case $u=1$, we refer to Remark A.3 for an explanation on how to adapt the estimate in that case.)

First, using that $\int_0^t G_\vartheta(s)\,\mathrm{d}s\leqslant c(\log\frac{1}{t})^{-1}$ as $t\downarrow 0$, see [CSZ19a, Proposition 1.6], we get that $\int_{s_m\leqslant t_m\leqslant\bar{N}_{i_m}}G_\vartheta(t_m-s_m)\mathrm{d}t_m\leqslant C/\log(\bar{N}_{i_m}^{-1})$. Thus, we have that

$$\iint_{\bar{N}_u<s_2\leqslant t_2\leqslant\bar{N}_{u+1}} G_\vartheta(t_2-s_2)\frac{t_1\,\mathrm{d}t_2\,\mathrm{d}s_2}{s_2(s_2-t_1)}\leqslant\frac{C}{\log(\bar{N}_{u+1}^{-1})}\int_{\bar{N}_u}^{\bar{N}_{u+1}}\frac{t_1\,\mathrm{d}s_2}{s_2(s_2-t_1)}\,. \tag{A.15}$$

For the remaining integral over s_2, we have that

$$\int_{\bar N_u}^{\bar N_{u+1}} \frac{t_1\,\mathrm{d}s_2}{s_2(s_2-t_1)} = \int_{\bar N_u}^{\bar N_{u+1}} \Big(\frac{1}{s_2-t_1}-\frac{1}{s_2}\Big) = \int_{\bar N_u - t_1}^{\bar N_{u+1}-t_1} \frac{\mathrm{d}s_2}{s_2} - \int_{\bar N_u}^{\bar N_{u+1}} \frac{\mathrm{d}s_2}{s_2} \leqslant \int_{\bar N_u - t_1}^{\bar N_u} \frac{\mathrm{d}s_2}{s_2}\,.$$

We can then distinguish the cases $t_1 \leqslant \bar N_u/2$ and $t_1 > \bar N_u/2$. In the former case, the integral is at most $2t_1/\bar N_u$ by a uniform bound on the integrand, while in the latter we can use the exact value of the integral, which is $\log(\frac{\bar N_u}{\bar N_u - t_1})$. It is not difficult to check that the two cases can be combined to obtain the bound

$$\int_{\bar N_u}^{\bar N_{u+1}} \frac{t_1\,\mathrm{d}s_2}{s_2(s_2-t_1)} \leqslant C\,\frac{t_1}{\bar N_u}\log\Big(\frac{2\bar N_u}{\bar N_u - t_1}\Big)\,. \tag{A.16}$$

Going back to (A.14), and using that $t_1 \leqslant \bar N_u$, it gives that

$$\mathbb{E}\big[(D^{\mathrm{closed}}_{u,u+1})^2\big] \leqslant \frac{C}{\log(\bar N_{u+1}^{-1})} \iint\limits_{\bar N_{u-1}<s_1\leqslant t_1<\bar N_u} \frac{1}{s_1} G_\vartheta(t_1-s_1)\log\Big(\frac{2\bar N_u}{\bar N_u - t_1}\Big)\mathrm{d}s_1\mathrm{d}t_1\,.$$

Now, one can prove (see Subsection A.3 below) that for any $0 < 2\mathsf{S} \leqslant \mathsf{T} < 1$,

$$\iint\limits_{\mathsf{S}<s\leqslant t<\mathsf{T}} \frac{1}{s} G_\vartheta(t-s)\log\Big(\frac{2\mathsf{T}}{\mathsf{T}-t}\Big)\mathrm{d}s\mathrm{d}t \leqslant \frac{C}{|\log \mathsf{T}|}\log(\mathsf{T}/\mathsf{S})\,. \tag{A.17}$$

All together, we end up with

$$\mathbb{E}\big[(D^{\mathrm{closed}}_{u,u+1})^2\big] \leqslant \frac{C}{\log(\bar N_{u+1}^{-1})}\frac{C\log(\bar N_u/\bar N_{u-1})}{\log(\bar N_u^{-1})} \leqslant \frac{C/\log\rho}{(K-u)(K-u-1)}\,,$$

where we used the definition (7.1) of N_i and $\bar N_i = N_i/N$, which give

$$\log\frac{\bar N_u}{\bar N_{u-1}} \leqslant C\log\rho, \qquad \log\frac{1}{\bar N_i} \geqslant c(K-i)\log\rho, \qquad 1 \leqslant i \leqslant K-1.$$

This completes the proof of (7.17) when there is only one block of D_u in the expansion.

Remark A.3. *In the case $u = 1$, the integral in* (A.14) *diverge at $s_1 = 0$. In that case, in the Riemann sum approximation one can in fact keep the restriction $s_1 > 1/(2N)$ in the integral comparison, so that*

$$\mathbb{E}\big[(D^{\mathrm{closed}}_{1,2})^2\big] \leqslant C \iint\limits_{\substack{1/(2N)<s_1\leqslant t_1<\bar N_1\\ \bar N_1<s_2\leqslant t_2<\bar N_2}} \frac{t_1}{s_1} G_\vartheta(t_1-s_1)\,\frac{1}{s_2(s_2-t_1)}\,G_\vartheta(t_2-s_2)\,\mathrm{d}\vec{s}\,\mathrm{d}\vec{t}.$$

Then the proof proceeds identically, and the required bound follows from (A.17) *applied with $\mathsf{S} = 1/(2N)$ and $\mathsf{T} = \bar N_1$, since $\log(\mathsf{T}/\mathsf{S}) = \log(2N_1) \leqslant C\log\rho$.*

Proof of (7.17) **in the general case of several D-blocks ($v \geqslant u+2$).** Let us denote by $i_1 < i_2 < \cdots < i_m$ (with $m \leqslant v-u+1$) the indices of the blocks where disorder terms are sampled in the chaos expansion of D_i or of $Z_v \rhd \mathbf{1}$. We note that necessarily $i_1 = u$ and $i_m = v$. Indeed, if we denote by $\widetilde D^{\mathrm{closed}}_{u,v}$ the contribution to $D^{\mathrm{closed}}_{u,v}$ by restricting the chaos expansion of $Z_v \rhd \mathbf{1}$ to just the constant 1, then

$$\widetilde D^{\mathrm{closed}}_{u,v} := q_{N_{u-1}} \lhd D_u \circ \cdots \circ D_{v-1} \rhd \mathbf{1} = 0,$$

since $D_{v-1} \triangleright \mathbf{1} = 0$ by (7.6). Similarly, if there is no disorder sampled in the chaos expansion of D_u and we denote this restriction by $\widetilde{D}_u$, then

$$q_{N_{u-1}} \triangleleft \widetilde{D}_u(y) = \sum_{x \in \mathbb{Z}^2} q_{N_{u-1}}(x) \big(q_{N_u - N_{u-1}}(y-x) - q_{N_u}(y) \big) = 0 \,.$$

Using the chaos expansion, we can thus write the second moment as

$$\begin{aligned} \mathbb{E}\big[(D^{\mathrm{closed}}_{u,v})^2\big] = \sum_{m=2}^{v-u+1} \sum_{u=i_1<\cdots<i_m=v} \sum_{\substack{N_{i_j-1}<a_j\leqslant b_j\leqslant N_{i_j} \\ x_1,y_1,\dots,x_m,y_m\in\mathbb{Z}^2}} & q_{a_1}(x_1)^2 \overline{U}_N(b_1-a_1, y_1-x_1) \\ & \times \prod_{j=2}^{m} \big[q_{a_j-b_{j-1}}(x_j - y_{j-1}) - q_{a_j}(x_j) \big]^2 \overline{U}_N(b_j-a_j, y_j-x_j). \end{aligned} \tag{A.18}$$

We now follow the same steps as in the simpler case treated above. We use the bound $\overline{U}_N(b_j-a_j, y_j-x_j) \leqslant C\overline{U}_N(b_j-a_j)\hat{g}_{b_j-a_j}(y_j-x_j)$ from Proposition A.1 and we then first focus, for fixed $(a_i,b_i)_{i=1}^m$, on the sum over the spatial coordinates

$$\sum_{x_1,y_1,\dots,x_m,y_m\in\mathbb{Z}^2} q_{a_1}(x_1)^2 \prod_{j=1}^{m} \hat{g}_{b_j-a_j}(y_j-x_j) \prod_{j=2}^{m} \big[q_{a_j-b_{j-1}}(x_j-y_{j-1}) - q_{a_j}(x_j) \big]^2 .$$

Then, bounding $q_{a_1}(x_1)^2 \leqslant \frac{C}{a_1}\hat{g}_{a_1}(x_1)$ and applying Lemma A.2 to sum successively over $x_1, y_1, \dots, x_m, y_m$ (*i.e.* we sum the space indices in their "chronological order" and we use (A.11) when summing over x_j and (A.12) when summing over y_i), we obtain similarly to (A.13) that the above is bounded by

$$C^m \frac{1}{a_1} \prod_{j=2}^{m} \frac{b_{j-1}}{a_j(a_j-b_{j-1})} \,.$$

Plugged inside (A.18) and using Proposition A.1-(A.6) to deal with $\overline{U}_N(b_j-a_j)$, a Riemann sum approximation then yields, analogously to (A.14), that $\mathbb{E}\big[(D^{\mathrm{closed}}_{u,v})^2\big]$ is bounded by

$$\sum_{m=2}^{v-u+1} C^m \sum_{u=i_1<\cdots<i_m=v} \iint\limits_{\substack{\bar{N}_{i_j-1}<s_j\leqslant t_j\leqslant \bar{N}_{i_j}: \\ 1\leqslant j\leqslant m}} \frac{1}{s_1} \prod_{j=1}^{m} G_\vartheta(t_j-s_j) \prod_{j=2}^{m} \frac{t_{j-1}}{s_j(s_j-t_{j-1})} \mathrm{d}\vec{s}\,\mathrm{d}\vec{t}, \tag{A.19}$$

where we recall the shorthand notation $\bar{N}_u = N_u/N$. Again, in the case $u=1$, the above integral is in fact infinite: one simply needs to replace $\bar{N}_0$ by $1/(2N)$ in the Riemann sum approximation and proceed as in Remark A.3.

We now integrate the time variables backwards, fixing for the moment m and the indices $u = i_1 < \cdots < i_m = v$. First, for the last integral, similarly to (A.15)–(A.16), we get that

$$\iint\limits_{\bar{N}_{i_m-1}<s_m\leqslant t_m\leqslant \bar{N}_{i_m}} G_\vartheta(t_m-s_m) \frac{t_{m-1}\,\mathrm{d}t_m\,\mathrm{d}s_m}{s_m(s_m-t_{m-1})} \leqslant \frac{C}{\log(\bar{N}_{i_m}^{-1})} \times \frac{t_{m-1}}{\bar{N}_{i_m-1}} \mathsf{W}_{\bar{N}_{i_m-1}}(t_{m-1}) \,, \tag{A.20}$$

where we have defined the following "backward weight"

$$\mathsf{W}_{\mathsf{T}}(\tau) := \log \frac{2\mathsf{T}}{\mathsf{T}-\tau}, \qquad \text{for } \mathsf{T} > 0, \tag{A.21}$$

The reason for expressing the bound (A.20) in this way is that it then allows us to use the following estimate iteratively (its proof is postponed to Subsection A.3).

Lemma A.4. *There is a constant $C > 0$ such that, if $1 \geqslant \mathsf{T} \geqslant 4\mathsf{S}$ and $\tau \leqslant \mathsf{S}$,*

$$\iint\limits_{\mathsf{S}<s<t<\mathsf{T}} G_\vartheta(t-s)\frac{\tau}{s(s-\tau)}\,\mathsf{W}_{\mathsf{T}}(t)\mathrm{d}s\,\mathrm{d}t \leqslant \frac{C}{|\log \mathsf{T}|}\,\frac{\tau}{\mathsf{S}}\,\mathsf{W}_{\mathsf{S}}(\tau)\,, \tag{A.22}$$

Before we start the iteration, let us first make some useful observations. First of all, since $t_{j-1} \leqslant \bar{N}_{i_{j-1}}$, we have

$$\frac{t_{j-1}}{\bar{N}_{i_j-1}} \leqslant \frac{\bar{N}_{i_{j-1}}}{\bar{N}_{i_j-1}} \leqslant \Big(\frac{C}{\rho}\Big)^{i_j-i_{j-1}-1}. \tag{A.23}$$

Second, since for any fixed τ the function $\mathsf{T} \mapsto \mathsf{W}_{\mathsf{T}}(\tau)$ is non-increasing for $\mathsf{T} \geqslant \tau$, we also have for $t_{j-1} \leqslant \bar{N}_{i_{j-1}} \leqslant \bar{N}_{i_j-1}$,

$$\mathsf{W}_{\bar{N}_{i_j-1}}(t_{j-1}) \leqslant \mathsf{W}_{\bar{N}_{i_{j-1}}}(t_{j-1}). \tag{A.24}$$

To start the iteration, we first apply (A.23) and (A.24) to (A.20), to obtain

$$\iint\limits_{\bar{N}_{i_m-1}<s_m\leqslant t_m\leqslant \bar{N}_{i_m}} G_\vartheta(t_m-s_m)\frac{t_{m-1}\,\mathrm{d}t_m\,\mathrm{d}s_m}{s_m(s_m-t_{m-1})} \;\leqslant\; \frac{C}{\log \bar{N}_{i_m}^{-1}}\Big(\frac{C}{\rho}\Big)^{i_m-i_{m-1}-1}\mathsf{W}_{\bar{N}_{i_{m-1}}}(t_{m-1}),$$

which when plugged into (A.19) gives

$$\begin{aligned}\mathbb{E}\big[(D^{\mathrm{closed}}_{u,v})^2\big] \leqslant \sum_{m=2}^{v-u+1} \sum_{u=i_1<\cdots<i_m=v} \iint\limits_{\substack{\bar{N}_{i_j-1}<s_j\leqslant t_j\leqslant \bar{N}_{i_j}:\\ 1\leqslant j\leqslant m-1}} \frac{C^m}{s_1}\prod_{j=1}^{m-1} G_\vartheta(t_j-s_j) \\ \times \prod_{j=2}^{m-1}\frac{t_{j-1}}{s_j(s_j-t_{j-1})}\frac{C}{\log(\bar{N}_{i_m}^{-1})}\Big(\frac{C}{\rho}\Big)^{i_m-i_{m-1}-1}\mathsf{W}_{\bar{N}_{i_{m-1}}}(t_{m-1})\,\mathrm{d}\vec{s}\mathrm{d}\vec{t}.\end{aligned}$$

We then run the recursion, applying Lemma A.4 for $j = m-1, \dots, 2$, with $\mathsf{S} = \bar{N}_{i_j-1}$, $\mathsf{T} = \bar{N}_{i_j}$ and $\tau = t_{j-1}$ and using (A.23)–(A.24); note that the assumption $\mathsf{T} \geqslant 4\mathsf{S}$ holds whenever ρ is sufficiently large. We then obtain

$$\begin{aligned}\mathbb{E}\big[(D^{\mathrm{closed}}_{u,v})^2\big] \leqslant \sum_{m=2}^{v-u+1} \sum_{u=i_1<\cdots<i_m=v} C^m\Big(\frac{C}{\rho}\Big)^{\sum_{j=2}^m(i_j-i_{j-1}-1)}\prod_{j=2}^{m}\frac{1}{\log(\bar{N}_{i_j}^{-1})} \\ \times \iint\limits_{\bar{N}_{u-1}<s_1\leqslant t_1\leqslant \bar{N}_u}\frac{G_\vartheta(t_1-s_1)}{s_1}\,\mathsf{W}_{\bar{N}_u}(t_1)\,\mathrm{d}s_1\,\mathrm{d}t_1\,,\end{aligned}$$

with the last integral bounded by $C\log(\bar{N}_u/\bar{N}_{u-1})/\log(\bar{N}_u^{-1}) \leqslant C\log\rho/\log(\bar{N}_u^{-1})$, thanks to (A.17).

Using also that $\sum_{j=2}^m(i_j - i_{j-1} - 1) = v - u + 1 - m$, we obtain that

$$\mathbb{E}\big[(D^{\mathrm{closed}}_{u,v})^2\big] \leqslant \log\rho \sum_{m=2}^{v-u+1} C^m\Big(\frac{C}{\rho}\Big)^{v-u+1-m}\sum_{u=i_1<\cdots<i_m=v}\prod_{j=1}^{m}\frac{C}{\log\bar{N}_{i_j}^{-1}}\,.$$

Now, we have that $\log(\bar{N}_u^{-1}) \geqslant C(K-u)\log\rho$ and $\log(\bar{N}_v^{-1}) \geqslant C(K-v)\log\rho$. For the indices $j = 2, \dots, m-1$, since $i_j \leqslant v \leqslant K-1$ we simply use that $\log\bar{N}_{i_j}^{-1} \geqslant c(K-i_j)\log\rho \geqslant$

$c \log \rho$. On the other hand the number of choices for the indices $i_2 < \cdots < i_{m-1}$ is $\binom{v-u-1}{m-2}$. Putting everything together, we obtain

$$\mathbb{E}\big[(D^{\mathrm{closed}}_{u,v})^2\big] \leqslant \frac{C}{(K-u)(K-v)\log\rho} \sum_{m=2}^{v-u+1} \binom{v-u-1}{m-2} \Big(\frac{C}{\log\rho}\Big)^{m-2} \Big(\frac{C}{\rho}\Big)^{v-u+1-m}$$
$$= \frac{C}{(K-u)(K-v)\log\rho} \Big(\frac{C}{\log\rho} + \frac{C}{\rho}\Big)^{v-u-1} \leqslant \frac{C(C'/\log\rho)^{v-u}}{(K-u)(K-v)}.$$

This concludes the proof of (7.17) by taking the square root and possibly enlarging ρ to ensure that $\varepsilon_\rho^2 := C'/\log\rho < 1$.

Proof of (7.18)**.** The proof to (7.18) is almost identical to that of (7.17) and follows the same steps. Let us briefly describe how they adapt here.

The second moment of $q_{N_{u-1}} \triangleleft D_u \circ \cdots \circ D_{v-1} \circ Z_{N_{v-1},N} \triangleright \mathbf{1}$ can be decomposed similarly as in (A.18), except that the last indices a_m, b_m are in the time-interval $N_{v-1} < a_m \leqslant b_m \leqslant N$.

Again, we can decouple the time and space variable using the bound from Proposition A.1, then sum over the space variables as above. We end up with the same bound as in (A.19), but with the last time indices verifying $\bar{N}_{v-1} < s_m \leqslant t_m \leqslant 1$. The bound (A.20) is then replaced by

$$\iint_{\bar{N}_{v-1}<s_m\leqslant t_m\leqslant 1} G_\vartheta(t_m - s_m) \frac{t_{m-1}\,\mathrm{d}t_m\,\mathrm{d}s_m}{s_m(s_m - t_{m-1})} \leqslant C\frac{t_{m-1}}{\bar{N}_{i_m-1}} \mathsf{W}_{\bar{N}_{i_{m-1}}}(t_{m-1})$$
$$\leqslant C\Big(\frac{C}{\rho}\Big)^{i_m - i_{m-1}-1} \mathsf{W}_{\bar{N}_{i_{m-1}}}(t_{m-1})\,,$$

using again (A.23)–(A.24) (here by convention $i_m = v$).

We can then apply the same iteration as above: thanks to Lemma A.4 (and (A.23)–(A.24)), we get that the second moment of $q_{N_{u-1}} \triangleleft D_u \circ \cdots \circ D_{v-1} \circ Z_{N_{v-1},N} \triangleright \mathbf{1}$ is bounded by

$$C\log\rho \sum_{m=2}^{v-u+1} C^m \Big(\frac{C}{\rho}\Big)^{v-u+1-m} \sum_{u=i_1<\cdots<i_m=v} \prod_{j=1}^{m-1} \frac{C}{\log \bar{N}_{i_j}^{-1}}$$
$$\leqslant \frac{C}{K-u} \sum_{m=2}^{v-u+1} \binom{v-u-1}{m-2} \Big(\frac{C}{\log\rho}\Big)^{m-2} \Big(\frac{C}{\rho}\Big)^{v-u+1-m} \leqslant \frac{C}{K-u}\Big(\frac{C}{\log\rho} + \frac{C}{\rho}\Big)^{v-u-1}.$$

This shows (7.18).

A.3. Some technical estimates. In this section, we prove the bound (A.7) in Proposition A.1, and also Lemmas A.2 and A.4.

Proof of (A.7) **in Proposition A.1.** For any fixed $k \geqslant 1$ and $m_1, \ldots, m_k \geqslant 1$ with $m_1 + \cdots + m_k = n$, let $(\zeta_i)_{1\leqslant i\leqslant k}$ be independent random variables with distribution $\mathrm{P}(\zeta_j = z) = q_{m_j}(z)^2/q_{2m_j}(0)$. Then, the proof of (A.7) reduces to showing that

$$\mathrm{P}(\zeta_1 + \cdots + \zeta_k = x) \leqslant \frac{C}{n}\,\mathrm{e}^{-c|x|^2/n}\,, \tag{A.25}$$

for some absolute constants $c, C > 0$ which do not depend on $k, n_1, \ldots, n_k$ and x. Indeed, plugged in the definition (A.2) and summing over the space variables (and setting $m_i =$

$n_i - n_{i-1}$), we can rewrite it as

$$U_N(n,x) = \sum_{k=1}^{\infty} \sum_{0<n_1<\ldots<n_{k-1}<n} \prod_{j=1}^{k} \sigma_N^2\, q_{2(n_j-n_{j-1})}(0) \mathrm{P}(\zeta_1 + \cdots + \zeta_k = x)$$
$$\leqslant \frac{C}{n}\, \mathrm{e}^{-c|x|^2/n} \sum_{k=1}^{\infty} \sum_{0<n_1<\ldots<n_{k-1}<n} \prod_{j=1}^{k} \sigma_N^2\, q_{2(n_j-n_{j-1})}(0) = \frac{C}{n}\, \mathrm{e}^{-c|x|^2/n} U_N(n)\,,$$

which is the desired inequality. We are thus reduced to proving (A.25).

Using that the coordinates $S_n^1 + S_n^2$ and $S_n^1 - S_n^2$ are independent one-dimensional simple random walks, we can further reduce (A.25) to a one-dimensional problem. In particular $q_m(z) = p_m(z^1 + z^2)p_m(z^1 - z^2)$, where $p_m(x)$ is the transition kernel of the simple random walk on $\mathbb{Z}$. Letting $X_i = \zeta_i^1 + \zeta_i^2$ and $Y_i = \zeta_i^1 - \zeta_i^2$, we thus get that X_i, Y_i are independent, with law $\mathrm{P}(X_i = w) = p_{m_i}(w)^2/p_{2m_i}(0)$. We are thus reduced to proving the following one-dimensional local limit theorem:

$$\mathrm{P}(X_1 + \cdots X_k = w) \leqslant \frac{C}{\sqrt{n}}\, \mathrm{e}^{-c|w|^2/n}\,. \tag{A.26}$$

First of all, notice that from [CSZ23, Equation (3.49)] we get that for any $\lambda \geqslant 0$

$$\mathrm{E}[\mathrm{e}^{\lambda X_i}] \leqslant \mathrm{e}^{c\lambda^2 m_i}\,.$$

(The proof is for ζ_i in place of X_i but can be applied to X_i by using Cauchy–Schwarz's inequality and adjusting the constant.) Using Chernoff's bound, this thus gives, say for $w \geqslant 0$,

$$\mathrm{P}(X_1 + \cdots X_k \geqslant w) \leqslant \inf_{\lambda \geqslant 0} \mathrm{e}^{-\lambda w + c\lambda^2 n} = \mathrm{e}^{-w^2/(4cn)}$$

recalling first that $m_1 + \cdots + m_k = n$ and then choosing $\lambda = w/(2cn)$ for the second inequality. We can turn this into a local bound as in (A.26), by observing that the law of $X_1 + \cdots + X_k$ is unimodal (on $\{w \in \mathbb{Z}, w = n \mod 2\}$) as the convolution of unimodal laws. From this we deduce that, for $w \geqslant 4$,

$$\mathrm{P}(X_1 + \cdots X_k \in [w/2, w]) \geqslant cw\mathrm{P}(X_1 + \cdots + X_k = w)\,,$$

since there are at least cw elements in $[w/2, w]$ with the same parity as n. Combined with the tail bound above, this yields

$$\mathrm{P}(X_1 + \cdots X_k = w) \leqslant \frac{C}{|w|}\mathrm{e}^{-c|w|^2/n}\,,$$

which gives the bound (A.26) in the case $|w| \geqslant \sqrt{n}$.

To deal with the case $|w| \leqslant \sqrt{n}$, we make the observation that since $p_{m_i}(w) = 2^{-m_i}\binom{m_i}{(m_i-w)/2}$ whenever w has the same parity as m_i, the random variable $W_i := \frac{1}{2}(m_i - X_i)$ has a hypergeometric distribution of parameters $m_i, 2m_i$. In particular W_i has a (discrete) log-concave distribution, and so does $W_1 + \cdots + W_k$. Thus [BMM22, Theorem 1.1] gives that

$$\sup_w \mathrm{P}(W_1 + \cdots + W_k = w) \leqslant \Big(1 + 2\sum_{i=1}^{k} \mathrm{Var}(W_i)\Big)^{-1/2} \leqslant \big(1 + n/4\big)^{-1/2}\,,$$

since $\mathrm{Var}(W_i) = \frac{m_i^2}{4(2m_i-1)} \geqslant \frac{1}{8}m_i$. We thus conclude that

$$\sup_w \mathrm{P}(X_1 + \cdots + X_k = w) \leqslant \frac{C}{\sqrt{n}}\,,$$

which shows (A.26) in the case $|w| \leqslant \sqrt{n}$. □

Proof of Lemma A.2. The proof of the convolution bound (A.11) is a simple consequence of the following classical inequality for $a, b \geqslant 1$

$$\sum_y \mathrm{e}^{-|y|^2/a} \mathrm{e}^{-|x-y|^2/b} \leqslant \frac{Cab}{a+b} \mathrm{e}^{-|x|^2/(a+b)}\,, \tag{A.27}$$

which is proven by a completion of the square. We thus focus on the proof of (A.12).

First of all, if $k < \ell \leqslant 3k$, we first use that $(a+b)^2 \leqslant 2a^2 + 2b^2$ and we bound $q_{\ell-k}(w-y)^2 \leqslant \frac{C}{\ell-k} q_{\ell-k}(w-y)$ and similarly for $q_\ell(w)^2$ to get that

$$\sum_y \hat{g}_k(y)[q_{\ell-k}(w-y) - q_\ell(w)]^2 \leqslant \frac{C}{\ell-k} \sum_y \hat{g}_k(y) q_{\ell-k}(w-y) + \frac{C}{\ell} q_\ell(w)\,,$$

using also that $\sum_y \hat{g}_k(y) \leqslant C$ for the second term. Now, we may bound $q_{\ell-k}(w-y) \leqslant C\hat{g}_{\ell-k}(w-y)$ and use (A.11) for the first term and simply $q_\ell(w) \leqslant C\hat{g}_\ell(w)$ for the second term: we obtain that

$$\sum_y \hat{g}_k(y)[q_{\ell-k}(w-y) - q_\ell(w)]^2 \leqslant \frac{C}{\ell-k} \hat{g}_\ell(w) \leqslant C' \frac{k}{\ell(\ell-k)} \hat{g}_\ell(w)\,.$$

In the case where $\ell > 3k$, using $q_\ell = q_k * q_{\ell-k}$ and Jensen's inequality, we get that

$$[q_{\ell-k}(w-y) - q_\ell(w)]^2 \leqslant \sum_z q_k(z)[q_{\ell-k}(w-y) - q_{\ell-k}(w-z)]^2\,.$$

We now may use the following Gaussian gradient bound (for z, z' in the same parity class)

$$|q_n(z) - q_n(z')|^2 \leqslant \frac{C|z-z'|^2}{n^3} \Big[\mathrm{e}^{-c_1|z|^2/n} + \mathrm{e}^{-c_1|z'|^2/n}\Big]\,,$$

which is e.g. a direct consequence from the gradient bound [HS93, Theorem 5.1, Equation (14)] (applied to the two-step random walk on $\mathbb{Z}^2_{\mathrm{even}}$). Bounding also $q_k(z) \leqslant C\hat{g}_k(z)$, this gives that

$$\sum_y \hat{g}_k(y)[q_{\ell-k}(w-y) - q_\ell(w)]^2 \leqslant \frac{C}{(\ell-k)^3} \sum_{y,z} \hat{g}_k(y)\hat{g}_k(z)\,|z-y|^2 \Big[\mathrm{e}^{-c_1 \frac{|w-y|^2}{\ell-k}} + \mathrm{e}^{-c_1 \frac{|w-z|^2}{\ell-k}}\Big]\,.$$

By symmetry, we only have to treat one of the two terms in the sum, say the first one (with $\mathrm{e}^{-c|w-y|^2/(\ell-k)}$). Bounding $|z-y|^2 \leqslant 2|z|^2 + 2|y|^2$ and using that $\sum_z \hat{g}_k(z)|z|^2 \leqslant Ck$, we get that the above is bounded by

$$\frac{C}{(\ell-k)^3} \sum_y \hat{g}_k(y)(k+|y|^2)\,\mathrm{e}^{-c_1 \frac{|w-y|^2}{\ell-k}} \leqslant \frac{C'}{(\ell-k)^3} \sum_y \mathrm{e}^{-\frac{\mathsf{c}_0}{2}\frac{|y|^2}{k}} \mathrm{e}^{-c_1 \frac{|w-y|^2}{\ell-k}}\,,$$

where we have used that $(k+|y|^2)\hat{g}_k(y) \leqslant C\mathrm{e}^{-\mathsf{c}_0|y|^2/2k}$ (recall the definition (A.8) of $\hat{g}_k$). Applying (A.27) with $a = 2k/\mathsf{c}_0$ and $b = (\ell-k)/c_1$, we obtain

$$\sum_y \mathrm{e}^{-\frac{\mathsf{c}_0}{2}\frac{|y|^2}{k}} \mathrm{e}^{-c_1 \frac{|w-y|^2}{\ell-k}} \leqslant C \frac{k(\ell-k)}{\ell} \mathrm{e}^{-\mathsf{c}_0 \frac{|w|^2}{\ell}}$$

where we have chosen $\mathsf{c}_0 \leqslant c_1/2$ so that $\frac{2k}{\mathsf{c}_0} + \frac{\ell-k}{c_1} \leqslant \frac{\ell}{\mathsf{c}_0}$, uniformly for $\ell > 3k$. All together, this proves that

$$\sum_y \hat{g}_k(y)[q_{\ell-k}(w-y) - q_\ell(w)]^2 \leqslant \frac{Ck}{(\ell-k)^2} \hat{g}_\ell(w) \leqslant \frac{C'k}{\ell(\ell-k)} \hat{g}_\ell(w)\,,$$

for $\ell > 3k$. □

Proof of Lemma A.4 and (A.17)**.** We use the standard estimates which are easy consequences of [CSZ19a, Proposition 1.6]:

$$G_\vartheta(r) \leqslant \frac{C}{r(1+\log(1/r))^2}, \qquad \int_0^h G_\vartheta(r)\,\mathrm{d}r \leqslant \frac{C}{1+\log(1/h)}, \qquad 0<h\leqslant 1. \tag{A.28}$$

Let us first show Lemma A.4 and then turn to (A.17). Recall that $\mathsf{W}_\mathsf{T}(t) = \log(\frac{2\mathsf{T}}{\mathsf{T}-t})$ is defined in (A.21).

For Lemma A.4, we first estimate the integral over t in the left-hand side of (A.22):

$$\int_s^\mathsf{T} G_\vartheta(t-s)\mathsf{W}_\mathsf{T}(t)\mathrm{d}t = \int_0^{\mathsf{T}-s} G_\vartheta(r)\log\frac{2\mathsf{T}}{\mathsf{T}-s-r}\mathrm{d}r,$$

where we have substituted $r := t-s$. We split this integral at $r = \frac{\mathsf{T}-s}{2}$. For $r\in(0,\frac{\mathsf{T}-s}{2}]$, we have

$$\log\frac{2T}{\mathsf{T}-s-r} = \log\frac{2T}{\mathsf{T}-s} + \log\frac{\mathsf{T}-s}{\mathsf{T}-s-r} \leqslant \mathsf{W}_\mathsf{T}(s) + \log 2 \leqslant 2\mathsf{W}_\mathsf{T}(s).$$

Hence together with (A.28) we have

$$\int_0^{\frac{\mathsf{T}-s}{2}} G_\vartheta(r)\log\frac{2\mathsf{T}}{\mathsf{T}-s-r}\mathrm{d}r \leqslant 2\mathsf{W}_\mathsf{T}(s)\int_0^{\frac{\mathsf{T}-s}{2}} G_\vartheta(r)\mathrm{d}r \leqslant \frac{C}{1+\log\frac{1}{\mathsf{T}-s}}\mathsf{W}_\mathsf{T}(s).$$

For $r\in(\frac{\mathsf{T}-s}{2},\mathsf{T}-s)$, we observe that

$$\begin{aligned}\int_{\frac{\mathsf{T}-s}{2}}^{\mathsf{T}-s} G_\vartheta(r)\,\log\frac{2\mathsf{T}}{\mathsf{T}-s-r}\mathrm{d}r &\leqslant \frac{C}{(\mathsf{T}-s)\big(1+\log\frac{1}{\mathsf{T}-s}\big)^2}\int_{\frac{\mathsf{T}-s}{2}}^{\mathsf{T}-s}\log\frac{2\mathsf{T}}{\mathsf{T}-s-r}\mathrm{d}r\\ &= \frac{C(\mathsf{T}-s)}{(\mathsf{T}-s)\big(1+\log\frac{1}{\mathsf{T}-s}\big)^2}\Big(1+\log\frac{4\mathsf{T}}{\mathsf{T}-s}\Big) \leqslant \frac{C}{\big(1+\log\frac{1}{\mathsf{T}-s}\big)^2}\mathsf{W}_\mathsf{T}(s).\end{aligned}$$

Combining the last two estimates, we thus obtain:

$$\int_s^\mathsf{T} G_\vartheta(t-s)\mathsf{W}_\mathsf{T}(t)\mathrm{d}t \leqslant \frac{C}{1+\log\frac{1}{\mathsf{T}-s}}\mathsf{W}_\mathsf{T}(s) \leqslant \frac{C}{|\log\mathsf{T}|}\mathsf{W}_\mathsf{T}(s). \tag{A.29}$$

Plugged into the left-hand side of (A.22), this gives

$$\iint\limits_{\mathsf{S}<s<t<\mathsf{T}} G_\vartheta(t-s)\frac{\tau\,\mathsf{W}_\mathsf{T}(t)}{s(s-\tau)}\,\mathrm{d}s\,\mathrm{d}t \leqslant \frac{C}{|\log\mathsf{T}|}\int_\mathsf{S}^\mathsf{T}\frac{\tau}{s(s-\tau)}\mathsf{W}_\mathsf{T}(s)\mathrm{d}s. \tag{A.30}$$

We now split the above integral at $s=\frac{\mathsf{T}}{2}$; note that this is a valid integration interval as we assumed $4\mathsf{S}\leqslant\mathsf{T}$. For $s\in(\mathsf{S},\frac{\mathsf{T}}{2}]$, notice that $\mathsf{W}_\mathsf{T}(s) = \log\frac{2\mathsf{T}}{\mathsf{T}-s}\leqslant\log 4$. Hence

$$\begin{aligned}\int_\mathsf{S}^{\frac{\mathsf{T}}{2}}\frac{\tau}{s(s-\tau)}\mathsf{W}_\mathsf{T}(s)\mathrm{d}s &\leqslant C\int_\mathsf{S}^{\frac{\mathsf{T}}{2}}\frac{\tau}{s(s-\tau)}\mathrm{d}s\\ &\leqslant C\int_\mathsf{S}^\infty\Big(\frac{1}{s-\tau}-\frac{1}{s}\Big)\mathrm{d}s = C\log\frac{1}{1-\tau/\mathsf{S}} \leqslant C\frac{\tau}{\mathsf{S}}\log\frac{2}{1-\tau/\mathsf{S}},\end{aligned}$$

where the last inequality follows from straightforward calculations (treating the case $\tau/\mathsf{S}\in(0,\frac12]$ and $\tau/\mathsf{S}\in(\frac12,1)$ separately).

It now remains to treat the integral over $s \in (\frac{\mathsf{T}}{2}, \mathsf{T})$. Notice that in this region, $\frac{\tau}{s(s-\tau)} \leqslant \frac{8\tau}{\mathsf{T}^2}$. Therefore, we have that

$$\int_{\frac{\mathsf{T}}{2}}^{\mathsf{T}} \frac{\tau}{s(s-\tau)} \mathsf{W}_{\mathsf{T}}(s) \mathrm{d}s \leqslant \frac{C\tau}{\mathsf{T}^2} \int_{\frac{\mathsf{T}}{2}}^{\mathsf{T}} \mathsf{W}_{\mathsf{T}}(s) \mathrm{d}s = \frac{C\tau}{\mathsf{T}^2} \int_{\frac{\mathsf{T}}{2}}^{\mathsf{T}} \log \frac{2\mathsf{T}}{\mathsf{T}-s} \mathrm{d}s$$
$$= \frac{C\tau}{\mathsf{T}} \int_0^{\frac{1}{2}} \log \frac{2}{u} \mathrm{d}u \leqslant \frac{C\tau}{\mathsf{S}} \leqslant \frac{C\tau}{\mathsf{S}} \mathsf{W}_{\mathsf{S}}(\tau),$$

where we have applied the change of variable $u := 1 - \frac{s}{\mathsf{T}}$ above, and in the last inequality we observe that $\mathsf{W}_{\mathsf{S}}(\tau) \geqslant \log 2$ for all $\tau \in (0, \mathsf{S})$. Combined with the previous estimate this shows that

$$\int_{\mathsf{S}}^{\mathsf{T}} \frac{\tau}{s(s-\tau)} \mathsf{W}_{\mathsf{T}}(s) \leqslant \frac{C\tau}{\mathsf{S}} \mathsf{W}_{\mathsf{S}}(\tau)$$

so that plugged into (A.30) this finishes the proof of (A.22).

We now turn to (A.17); recall that $\log(\frac{2\mathsf{T}}{\mathsf{T}-t}) = \mathsf{W}_{\mathsf{T}}(t)$. We again first estimate the integral over t in the left-hand side thanks to (A.29). Plugged in (A.17), this yields

$$\iint_{\mathsf{S}<s\leqslant t<\mathsf{T}} \frac{1}{s} G_\vartheta(t-s) \mathsf{W}_{\mathsf{T}}(t) \mathrm{d}s \mathrm{d}t \leqslant \frac{C}{|\log \mathsf{T}|} \int_{\mathsf{S}}^{\mathsf{T}} \frac{1}{s} \mathsf{W}_{\mathsf{T}}(s) \mathrm{d}s.$$

Similar to the above, we split the integral at $s = \frac{\mathsf{T}}{2}$:

$$\int_{\mathsf{S}}^{\mathsf{T}/2} \frac{1}{s} \mathsf{W}_{\mathsf{T}}(s) \mathrm{d}s \leqslant \log 4 \int_{\mathsf{S}}^{\mathsf{T}/2} \frac{\mathrm{d}s}{s} \leqslant C \log \frac{\mathsf{T}}{\mathsf{S}},$$

where we have used $\mathsf{W}_{\mathsf{T}}(s) \leqslant \log 4$ for all $s \in (\mathsf{S}, \mathsf{T}/2)$. On the other hand,

$$\int_{\mathsf{T}/2}^{\mathsf{T}} \frac{1}{s} \mathsf{W}_{\mathsf{T}}(s) \mathrm{d}s \leqslant \frac{2}{\mathsf{T}} \int_{\mathsf{T}/2}^{\mathsf{T}} \mathsf{W}_{\mathsf{T}}(s) \mathrm{d}s = \frac{2}{\mathsf{T}} \int_0^{\mathsf{T}/2} \log \frac{2\mathsf{T}}{u} \mathrm{d}u \leqslant C,$$

where we have applied the change of variable $u := \mathsf{T} - s$. Assembling the above inequalities therefore yield the desired bound in (A.17). □

Appendix B. Moment estimates: proof of Lemma 7.6

Let us recall that we have set

$$R_s := \sum_{j=1}^{s} q_{2j}(0), \qquad \vartheta_N(s) := \frac{\pi}{\sigma_N^2} - \pi R_s \,. \tag{B.1}$$

Here $\vartheta_N(s)$ is the remaining distance from the critical renewal value. First, we have the following estimate (we use the same notation as in [BCT25, BTZ26]).

Claim B.1. *There exists a constant $C > 0$ such that, for every integer $s \geqslant 0$ with $\vartheta_N(s) > 0$, i.e. $\sigma_N^2 R_s < 1$,*

$$0 \leqslant \frac{\pi}{\vartheta_N(s)} - \sigma_N^2 \mathbb{E}[Z_s(0)^2] \leqslant \frac{C}{\vartheta_N(s)^2}. \tag{B.2}$$

Moreover, given any $t_0 > 0$, there is $C_{\vartheta,t_0} > 0$ such that, for all large N and all integers $1 \leqslant s \leqslant t_0 N$,

$$\sigma_N^2 \mathbb{E}[Z_s(0)^2] \leqslant \frac{C_{\vartheta,t_0}}{\log(1 + N/s)}.$$

Proof. The second bound follows from [CSZ19a, Propositions A.2 and 1.6]. We prove (B.2).

We assume $s \geqslant 1$, since for $s = 0$ the difference in (B.2) vanishes. The chaos expansion gives

$$\sigma_N^2 \mathbb{E}[Z_s(0)^2] = \sigma_N^2 \Big\{ 1 + \sum_{k=1}^{\infty} \sigma_N^{2k} \sum_{\substack{n_1,\dots,n_k \geqslant 1 \\ n_1+\cdots+n_k \leqslant s}} \prod_{j=1}^{k} q_{2n_j}(0) \Big\} .$$

On the other hand, since $\sigma_N^2 R_s < 1$, recalling the definition of ϑ_N, we can write

$$\frac{\pi}{\vartheta_N(s)} = \sigma_N^2 \sum_{k=0}^{\infty} (\sigma_N^2 R_s)^k .$$

Comparing these two expansions, we obtain

$$0 \leqslant \frac{\pi}{\vartheta_N(s)} - \sigma_N^2 \mathbb{E}[Z_s(0)^2] = \sigma_N^2 \sum_{k=1}^{\infty} \sigma_N^{2k} \sum_{n_1,\dots,n_k=1}^{s} \mathbf{1}_{\{n_1+\cdots+n_k > s\}} \prod_{j=1}^{k} q_{2n_j}(0)$$

Bounding $\mathbf{1}_{\{n_1+\cdots+n_k>s\}} \leqslant (n_1 + \cdots + n_k)/s$, the right hand side at most

$$\begin{aligned} \sigma_N^2 \sum_{k=1}^{\infty} \sigma_N^{2k} \frac{k}{s} \Big(\sum_{n=1}^{s} n\, q_{2n}(0) \Big) \Big(\sum_{n=1}^{s} q_{2n}(0) \Big)^{k-1} &\leqslant C \sigma_N^4 \sum_{k=1}^{\infty} k\, (\sigma_N^2\, R_s)^{k-1} \\ &\leqslant \frac{C\, \sigma_N^4}{(1-\sigma_N^2 R_s)^2} \leqslant \frac{C}{\vartheta_N(s)^2} , \end{aligned}$$

where we used $q_{2n}(0) \leqslant C/n$ and $1 - \sigma_N^2 R_s = \sigma_N^2 \vartheta_N(s)/\pi$. □

We first prove Lemma 7.6 and we then derive Lemma B.2 which gives some important moment estimates.

Proof of Lemma 7.6. The first-collision formula (4.2) gives

$$\mathbb{E}[\widehat{Z}_{s,t}(0)^2] = 1 + \sigma_N^2 \sum_{u=s+1}^{t} q_{2u}(0)\, \mathbb{E}[Z_{t-u}(0)^2]. \tag{B.3}$$

Since $\vartheta_N(s) = \frac{\pi}{\sigma_N^2} - \pi R_s = \vartheta_N(t) + \pi(R_t - R_s)$, we can write

$$\frac{\vartheta_N(s)}{\vartheta_N(t)} = 1 + \frac{\pi}{\vartheta_N(t)} \sum_{u=s+1}^{t} q_{2u}(0) ,$$

therefore

$$0 \leqslant \frac{\vartheta_N(s)}{\vartheta_N(t)} - \mathbb{E}[\widehat{Z}_{s,t}(0)^2] = \sum_{u=s+1}^{t} q_{2u}(0) \Big(\frac{\pi}{\vartheta_N(t)} - \sigma_N^2 \mathbb{E}[Z_{t-u}(0)^2] \Big) .$$

It is convenient to replace $\frac{\pi}{\vartheta_N(t)}$ by $\frac{\pi}{\vartheta_N(t-u)}$, in order to apply Claim B.1. To this purpose, we note that

$$\frac{1}{\vartheta_N(t)} - \frac{1}{\vartheta_N(t-u)} = \frac{\pi(R_t - R_{t-u})}{\vartheta_N(t)\vartheta_N(t-u)} \leqslant \frac{\pi(R_t - R_{t-u})}{\vartheta_N(t)^2} , \tag{B.4}$$

because $\vartheta_N(t-u) \geqslant \vartheta_N(t) > 0$, therefore Claim B.1 for $s = t-u$ yields

$$\begin{aligned}\sum_{u=s+1}^{t} q_{2u}(0)\Big(\frac{\pi}{\vartheta_N(t)} - \sigma_N^2 \mathbb{E}[Z_{t-u}(0)^2]\Big) &\leqslant \sum_{u=s+1}^{t} q_{2u}(0)\frac{\pi^2(R_t - R_{t-u}) + C}{\vartheta_N(t)^2} \\ &\leqslant \frac{C}{\vartheta_N(t)^2}\Big\{ \sum_{u=s+1}^{t} q_{2u}(0) \sum_{v=t-u+1}^{t} q_{2v}(0) + R_t - R_s \Big\}.\end{aligned} \tag{B.5}$$

Using $q_{2n}(0) \leqslant C/n$, we see that the double sum is uniformly bounded by a multiple of

$$\sum_{u=1}^{t} \frac{1}{u} \sum_{v=t-u+1}^{t} \frac{1}{v} = \sum_{\substack{u,v\in\{1,\dots,t\}\\ u+v>t}} \frac{1}{uv} \leqslant \sum_{\substack{u,v\in\{1,\dots,t\}\\ u+v>t}} \frac{1}{t}\frac{u+v}{uv} = \frac{2}{t}\sum_{u=1}^{t}\frac{1}{u}\sum_{v=t-u+1}^{t} 1 = 2\,.$$

This proves the first bound of the lemma. Dividing by $\vartheta_N(s)/\vartheta_N(t)$ gives (7.37). Finally, if $1 \leqslant s \leqslant t/2$, then $1 + R_t - R_s \leqslant C\log(t/s)$, which proves the last assertion. □

We next derive second and fourth moment bounds for the partition function on each geometric block.

Lemma B.2. *Recall the definition of* Z_i *from* (7.3)*, so in particular* $\mathsf{Z}_i(q_{N_{i-1}}) = Z_{N_{i-1},N_i}(q_{N_{i-1}}, \mathbf{1})$*. Then there exists a constant* $C = C_\rho$ *such that, for any* $1 \leqslant i \leqslant K$*, we have*

$$\begin{aligned}\mathrm{Var}\big(\mathsf{Z}_i(q_{N_{i-1}})\big) &\leqslant \frac{C}{K+1-i}\,, \\ \mathbb{E}\Big[\big|\mathsf{Z}_i(q_{N_{i-1}}) - 1\big|^4\Big] &\leqslant \frac{C}{(K+1-i)^2}\,.\end{aligned} \tag{B.6}$$

Let us stress that Lemma B.2 also treats the initial and final block, which satisfy respectively

$$\mathrm{Var}\big(\mathsf{Z}_1(0)\big) \leqslant \frac{C_\rho}{\log N}\,, \qquad \mathbb{E}\big|\mathsf{Z}_1(0) - 1\big|^4 \leqslant \frac{C_\rho}{(\log N)^2}$$

and

$$\mathrm{Var}\big(\mathsf{Z}_K(q_{N_{K-1}})\big) \leqslant C_\rho, \qquad \mathbb{E}\big|\mathsf{Z}_K(q_{N_{K-1}}) - 1\big|^4 \leqslant C_\rho.$$

Proof. By the first inequality in (4.4),

$$\mathrm{Var}\big(\mathsf{Z}_i(q_{N_{i-1}})\big) = \mathbb{E}\big[\widehat{Z}_{N_{i-1},N_i}(0)^2\big] - 1 \leqslant C\frac{R_{N_i} - R_{N_{i-1}}}{\log(1+N/N_i)}, \qquad 1 \leqslant i \leqslant K.$$

We bound the numerator and denominator separately. By the grid definition (7.1),

$$N\rho^{i-K} \leqslant N_i \leqslant 3N\rho^{i-K}, \qquad 1 \leqslant i \leqslant K.$$

Consequently, for $i \geqslant 2$, we have $N_i/N_{i-1} \leqslant 3\rho$ and hence, using $q_{2r}(0) \leqslant C/r$,

$$R_{N_i} - R_{N_{i-1}} \leqslant C\log\frac{N_i}{N_{i-1}} \leqslant C\log(3\rho).$$

For $i = 1$, the numerator is also bounded by some constant C_ρ, since $N_0 = 0$ and $N_1 \leqslant 2\rho$. Bounding the denominator by $\log(1+N/N_i) \geqslant \log(1+\rho^{K-i}/3) \geqslant c_\rho(K+1-i)$, we conclude that

$$\mathrm{Var}\big(\mathsf{Z}_i(q_{N_{i-1}})\big) \leqslant \frac{C_\rho}{K+1-i}, \qquad 1 \leqslant i \leqslant K.$$

As far as the fourth-moment is concerned, the bound follows from applying [CSZ25, Theorem 1.11], which precisely shows that

$$\mathbb{E}\big[|Z_{N_{i-1},N_i}(q_{N_{i-1}})-1|^4\big] \leqslant \mathfrak{C}_4 \operatorname{Var}\big(Z_{N_{i-1},N_i}(q_{N_{i-1}})\big)^2 \leqslant \frac{C}{(K+1-i)^2}\,,$$

where the last inequality follows from the bound on $\operatorname{Var}(\mathsf{Z}_i(q_{N_{i-1}}))$ obtained above. □

Appendix C. Declaration on the use of AI

The results presented here have been developed through extensive interactions between the authors and AI (ChatGPT 5.5, 5.6 and 6), which contributed to the tightness argument, the proof of convergence to a unique limit, and its non-Gaussianity. We describe below some of the main mathematical contributions and changes of strategy, without attempting to give a complete account of these interactions.

(1) **Tightness and the initial input.** We began by asking whether centring and diffusive rescaling, without an additional amplitude normalisation, suffice for a nontrivial limit. We suggested Gu–Tsai's geometric block factorisation [GT26] as a starting point and also supplied Chen's Gaussian convexity theorem [Che26], which had already been used in [LX26]. From this theorem, AI derived a covariance bound and combined it with negative-moment and second-moment estimates to obtain tightness of the spatial logarithmic field. Further exchanges, prompted by our questions and requests for checks, led to revisions and simplifications.

(2) **Developing the convergence strategy.** Using tightness as an input, we then sought convergence of the discrete field and a construction of the limit directly from the Stochastic Heat Flow (SHF). We suggested revisiting the local-remainder decomposition in [CSZ20], which helped refine a decomposition at two intermediate times. The first time $m = [\delta N]$ defines a regularisation by removing the disorder on the initial interval $(0, m]$. The second separates a local correction from a comparison of two polymer endpoint distributions evolved through the same future environment.

(3) **An unsuccessful switch-off construction.** To pass from the comparison of partition functions to their logarithms, we still needed suitable moment bounds. We suggested introducing auxiliary partition functions with disorder removed on selected geometric time intervals, adapting a construction from [CSZ25], so that conditional Gaussian convexity [Che26] could be applied. AI developed this suggestion into a sparse switch-off construction and explored several refinements. This initially appeared to close the remaining estimate, but a subsequent audit identified a gap in a contraction argument. We abandoned this auxiliary construction.

(4) **Completion and refinement of convergence.** When we asked AI to reconsider the available approaches and look for simpler arguments, it proposed a simpler use of conditional Gaussian convexity. Further exchanges developed this into the logarithmic moment bounds needed to complete the convergence proof, without the additional switch-off construction. The regularisation remained the one introduced above, with disorder removed only on $(0, m]$, and the proof retained the decomposition at two intermediate times and the supporting geometric block estimates. Subsequent exchanges led to simplifications and to the extension to arbitrary regularisation parameters tending to zero. The resulting argument proves Theorem 1.2: the discrete centred log-partition functions converge to a unique limit, constructed as a measurable functional of the SHF by proving that its regularised logarithmic observables are Cauchy.

(5) **Non-Gaussianity.** We initiated the investigation of non-Gaussianity of subsequential limits (before having the proof of convergence to a unique limit) while keeping open the possibility of a Gaussian limit. After several attempts, we suggested revisiting earlier unsuccessful approaches. AI then developed a proof using Gaussian rigidity under bounded perturbations in the terminal condition; a related rigidity result for convex functions of Gaussian vectors is given in [FMP25, Theorem 3.4]. Exchanges led to clarifications and simplifications of the proof.

We supplied structural ideas, assessed the proposed arguments, and guided the choice and revision of approaches. AI developed these suggestions and generated further proof ideas that were not contained in the original prompts. Some proposals were fruitful; others contained errors or introduced unnecessary complications. Checking and revising these arguments was an essential part of the work. We take responsibility for the mathematical claims and the final manuscript.

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