AN INVARIANCE PRINCIPLE FOR RANDOM WALK BRIDGES CONDITIONED TO STAY POSITIVE

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Abstract. We prove an invariance principle for the bridge of a random walk conditioned to stay positive, when the random walk is in the domain of attraction of a stable law, both in the discrete and in the absolutely continuous setting. This includes as a special case the convergence under diffusive rescaling of random walk excursions toward the normalized Brownian excursion, for zero mean, finite variance random walks. The proof exploits a suitable absolute continuity relation together with some local asymptotic estimates for random walks conditioned to stay positive, recently obtained by Vatutin and Wachtel [42] and Doney [21]. We review and extend these relations to the absolutely continuous setting.

1. Introduction

Invariance principles for conditioned random walks have a long history, going back at least to the work of Liggett [35], who proved that the *bridge* of a random walk in the domain of attraction of a stable law, suitably rescaled, converges in distribution toward the bridge of the corresponding stable Lévy process. This is a natural extension of Skorokhod's theorem which proves the same result for non conditioned random walks, cf. [40], itself a generalization to the stable case of Donsker's seminal work [23].

Later on, Iglehart [32], Bolthausen [9] and Doney [19] focused on a different type of conditioning: they proved invariance principles for random walks *conditioned to stay positive* over a finite time interval, obtaining as a limit the analogous conditioning for the corresponding Lévy process, known as *meander*. More recently, such results have been extended to the case when the random walk is conditioned to stay positive *for all time*, cf. Bryn-Jones and Doney [10], Caravenna and Chaumont [13] and Chaumont and Doney [18].

The purpose of this paper is to take a step further, considering the bridge-type conditioning and the constraint to stay positive at the same time. More precisely, given a random walk in the domain of attraction of a stable law, we show that its bridge conditioned to stay positive, suitably rescaled, converges in distribution toward the bridge of the corresponding stable Lévy process conditioned to stay positive. A particular instance of this result, in the special case of attraction to the normal law, has recently been obtained by Sohier [39]. We show in this paper that the result in the general stable case can be proved exploiting

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a suitable absolute continuity relation together with some asymptotic estimates recently obtained in the literature, cf. [42] and [21], that we review and extend.

Besides the great theoretical interest of invariance principles for conditioned processes, a strong motivation for our results comes from statistical physics, with particular reference to (1+1)-dimensional polymer and pinning models interacting with the x-axis, cf. [27, 28, 31]. From a mathematical viewpoint, these models may be viewed as perturbations of the law of a random walk depending on its zero level set. As a consequence, to obtain the scaling limits of such models, one needs invariance principles for random walk excursions, that is random walk bridges conditioned to stay positive that start and end at zero. To the best of our knowledge, such results were previously known only for simple random walks, cf. [33], and were used to obtain the scaling limits of polymer models in [14, 15]. In this paper we deal with bridges that start and end at possibly nonzero points, which makes it possible to deal with polymer models built over non-simple random walks.

The paper is organized as follows.

- In section 2 we state precisely our assumptions and our main results.
- In section 3 we present some preparatory material on fluctuation theory.
- Section 4 is devoted to reviewing some important asymptotic estimates for random walks conditioned to stay positive, in the discrete setting.
- In section 5 we extend the above estimates to the absolutely continuous setting.
- In section 6 we prove the invariance principle.
- Finally, some more technical details are deferred to the appendices.

2. The invariance principle

2.1. **Notation and assumptions.** We set $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Given two positive sequences $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$, we write as usual $b_n \sim c_n$ if $\lim_{n\to\infty} b_n/c_n = 1$, $b_n = o(c_n)$ if $\lim_{n\to\infty} b_n/c_n = 0$ and $b_n = O(c_n)$ if $\limsup_{n\to\infty} b_n/c_n < \infty$.

We recall that a positive sequence $(b_n)_{n\in\mathbb{N}}$ — or a real function $b(x)$ — is said to be *regularly varying with index* $\gamma \in \mathbb{R}$, denoted $(b_n)_{n\in\mathbb{N}} \in R_\gamma$, if $b_n \sim n^\gamma \ell(n)$, where $\ell(\cdot)$ is a slowly varying function, i.e. a positive real function with the property that $\ell(cx)/\ell(x) \to 1$ as $x \to +\infty$ for all fixed $c > 0$, cf. [8] for more details.

Throughout this paper we deal with random walks $(S = \{S_n\}_{n\in\mathbb{N}_0}, \mathbb{P})$ in the domain of attraction of a stable law. Let us write precisely this assumption.

Hypothesis 2.1. *We assume that* $(S = \{S_n\}_{n\in\mathbb{N}_0}, \mathbb{P})$ *is a random walk on* $\mathbb{R}$ *in the domain of attraction of a (strictly) stable law with index* $\alpha \in (0, 2]$ *and positivity parameter* $\varrho \in (0, 1)$. *More precisely, we assume that* $S_0 = 0$, *the real random variables* $\{S_n - S_{n-1}\}_{n\in\mathbb{N}}$ *are independent and identically distributed (i.i.d.) and there exists a sequence* $(a_n)_{n\in\mathbb{N}} \in R_{1/\alpha}$ *such that* $S_n/a_n \Rightarrow X_1$, *where* $(X = \{X_t\}_{t\ge0}, \mathbf{P})$ *denotes a stable Lévy process with index* $\alpha \in (0, 2]$ *and positivity parameter* $\varrho \in (0, 1)$.

Note that, given a random walk $(S, \mathbb{P})$ satisfying this Hypothesis, the limiting stable Lévy process $(X, \mathbf{P})$ is determined only up to a multiplicative constant. In fact, the norming sequence a_n can be multiplied by any positive constant without affecting Hypothesis 2.1.

We recall the general constraint $1 - \frac{1}{\alpha} \le \varrho \le \frac{1}{\alpha}$ (for $\alpha \in (1, 2]$, of course). We also stress that (for $\alpha \in (0, 1]$) we assume that $0 < \varrho < 1$, i.e., we exclude subordinators and cosubordinators. The *Brownian case* corresponds to $\alpha = 2$, $\varrho = \frac{1}{2}$, when the limiting Lévy process X is (a constant times) Brownian motion. This contains the important special instance when $\{S_n - S_{n-1}\}_{n\in\mathbb{N}}$ are i.i.d. zero-mean, finite-variance random variables (the so-called *normal domain of attraction of the normal law*).

Let us denote by $\Omega^{RW} := \mathbb{R}^{\mathbb{N}_0}$ the discrete paths space and by $\Omega := D([0,\infty),\mathbb{R})$ the space of real-valued càdlàg paths on $[0,\infty)$, equipped with the Skorokhod topology, which turns it into a Polish space, and with the corresponding Borel σ-field. We also set $\Omega_N^{RW} := \mathbb{R}^{\{0,\ldots,N\}}$ and $\Omega_t := D([0,t],\mathbb{R})$. For notational simplicity, we assume that $\mathbb{P}$ is a law on Ω^{RW} and $S = \{S_n\}_{n\in\mathbb{N}_0}$ is the coordinate process on this space; we also denote by $\mathbb{P}_x$ the law of the random walk started at $x \in \mathbb{R}$, i.e. the law on Ω_{RW} of $S + x$ under $\mathbb{P}$. Analogously, we assume that $X = \{X_t\}_{t\in[0,\infty)}$ is the coordinate process on Ω, that $\mathbf{P}$ is a law on Ω and we denote by $\mathbf{P}_a$ the law on Ω of $X + a$ under $\mathbf{P}$, for all $a \in \mathbb{R}$. Finally, for every $N \in \mathbb{N}$ we define the rescaling map $\varphi_N : \Omega^{RW} \to \Omega$ by

$$\big(\varphi_N(S)\big)(t) := \frac{S_{\lfloor Nt\rfloor}}{a_N}\,, \tag{2.1}$$

where $(a_N)_{N\in\mathbb{N}}$ is the norming sequence appearing in Hypothesis 2.1. We still denote by φ_N the restriction of this map from Ω_{Nt}^{RW} to Ω_t, for any $t > 0$.

Given $N \in \mathbb{N}$ and $x, y \in [0,\infty)$, by the (law of the) *random walk bridge of length N, conditioned to stay positive, starting at x and ending at y*, we mean either of the following laws on Ω_N^{RW}:

$$\mathbb{P}^{\uparrow,N}_{x,y}(\,\cdot\,) := \mathbb{P}_x(\,\cdot\,|S_1 \ge 0,\ldots,S_{N-1} \ge 0, S_N = y)\,, \tag{2.2}$$

$$\widehat{\mathbb{P}}^{\uparrow,N}_{x,y}(\,\cdot\,) := \mathbb{P}_x(\,\cdot\,|S_1 > 0,\ldots,S_{N-1} > 0, S_N = y)\,. \tag{2.3}$$

In order for the conditioning in the right hand sides of (2.2) and (2.3) to be well-defined, we work in the lattice or in the absolutely continuous setting. More precisely:

Hypothesis 2.2. *We assume that either of the following assumptions hold:*

- ((h,c)-lattice case) *The law of S_1 under $\mathbb{P}$ is supported by the lattice $c+h\mathbb{Z}$, where the span $h > 0$ is chosen to be maximal (i.e., the law of S_1 is not supported by $c' + h'\mathbb{Z}$, for any $h' > h$ and $c' \in \mathbb{R}$). Note that may take $c \in [0,h)$.*
- (absolutely continuous case) *The law of S_1 under $\mathbb{P}$ is absolutely continuous with respect to the Lebsegue measure on $\mathbb{R}$, and there exists $n \in \mathbb{N}$ such that the density $f_n(x) := \mathbb{P}(S_n \in \mathrm{d}x)/\mathrm{d}x$ of S_n is essentially bounded (i.e., $f_n \in L^\infty$).*

We remind that the requirement that $f_n \in L^\infty$ for some $n \in \mathbb{N}$ is the standard necessary and sufficient condition for the uniform convergence of the rescaled density $x \mapsto a_n f_n(a_n x)$ toward the density of X_1, cf. [30, §46].

Plainly, the laws $\mathbb{P}^{\uparrow,N}_{x,y}$ and $\widehat{\mathbb{P}}^{\uparrow,N}_{x,y}$ coincide in the absolutely continuous case, and are very similar in the lattice case: note in fact that $S = (S_1,\ldots,S_N)$ under $\mathbb{P}^{\uparrow,N}_{x_N,y_N}$ has the same law as $S - \varepsilon$ under $\widehat{\mathbb{P}}^{\uparrow,N}_{x_N+\varepsilon,y_N+\varepsilon}$, for $\varepsilon > 0$ small enough. Nevertheless, we consider both laws, for ease of reference.

Through a simple scaling argument, the general (h,c)-lattice case can be easily reduced to the so-called aperiodic setting, in which the span h equals 1. However, the shift parameter c can be nonzero in relevant examples, such as the simple symmetric random walk on $\mathbb{Z}$, for which $\mathbb{P}(S_1 = 1) = \mathbb{P}(S_1 = -1) = \frac{1}{2}$, corresponding to $h = 2$, $c = 1$. See Remark 2.6 below for other examples.

Coming back to relations (2.2) and (2.3), for the laws $\mathbb{P}^{\uparrow,N}_{x,y}$ and $\widehat{\mathbb{P}}^{\uparrow,N}_{x,y}$ to be well-defined, in the lattice case we need that the conditioning event has positive probability: setting

$$\begin{aligned} q^+_N(x,y) &= \mathbb{P}_x(S_1 \geq 0, \dots, S_{N-1} \geq 0, S_N = y)\,, \\ \widehat{q}^+_N(x,y) &= \mathbb{P}_x(S_1 > 0, \dots, S_{N-1} > 0, S_N = y)\,, \end{aligned} \tag{2.4}$$

we need that $q^+_N(x,y) > 0$ and $\widehat{q}^+_N(x,y) > 0$. Analogously, in the absolutely continuous case we require the strict positivity of the density of S_N at y under $\mathbb{P}_x$ and under the positivity constraint: more precisely, we need that $f^+_N(x,y) > 0$, where we set

$$\begin{aligned} f^+_N(x,y) &:= \frac{\mathbb{P}_x(S_1 > 0, \dots, S_{N-1} > 0, S_N \in \mathrm{d}y)}{\mathrm{d}y} \\ &= \int_{\{s_1>0,\dots,s_{N-1}>0\}} \left[f(s_1 - x) \left(\prod_{i=2}^{N-1} f(s_i - s_{i-1}) \right) f(y - s_{N-1}) \right] \mathrm{d}s_1 \cdots \mathrm{d}s_{N-1}\,, \end{aligned} \tag{2.5}$$

and where $f(\cdot) = f_1(\cdot)$ is the density of the random walk step S_1. As a matter of fact, these conditions will always be satisfied in the regimes for x, y that we consider, as it will be clear from the asymptotic estimates that we are going to derive.

Next, for $t \in (0,\infty)$ and $a, b \in [0,\infty)$, we denote by $\mathbf{P}^{\uparrow,t}_{a,b}$ the law on Ω_t corresponding to the *bridge of the Lévy process of length t, conditioned to stay positive, starting at a and ending at b*. Informally, this law is defined in analogy with (2.2) and (2.3), that is

$$\mathbf{P}^{\uparrow,t}_{a,b}(\,\cdot\,) := \mathbf{P}_a(\,\cdot\,| X_s \geq 0 \ \forall s \in [0,t],\ X_t = b)\,,$$

but we stress that some care is required to give to this definition a proper meaning, especially in the case when either $a = 0$ or $b = 0$ (we refer to section 6 for the details). We point out that in the Brownian case $\alpha = 2$, $\varrho = \frac{1}{2}$, when X is a standard Brownian motion, $\mathbf{P}^{\uparrow,1}_{0,0}$ is the law of the so-called *normalized Brownian excursion.*

Remark 2.3. Let $\mathbb{P}^{\uparrow}_x$ and $\widehat{\mathbb{P}}^{\uparrow}_x$ denote respectively the laws on Ω^{RW} of the random walks $(S, \mathbb{P}_x)$ and $(S, \widehat{\mathbb{P}}_x)$ conditioned to stay positive *for all time*, as defined in [6] (cf. also §6.1 below). The laws $\mathbb{P}^{\uparrow,N}_{x,y}$ and $\widehat{\mathbb{P}}^{\uparrow,N}_{x,y}$ may be viewed as bridges of $\mathbb{P}^{\uparrow}_x$ and $\widehat{\mathbb{P}}^{\uparrow}_x$ respectively, i.e.

$$\mathbb{P}^{\uparrow,N}_{x,y}(\,\cdot\,) = \mathbb{P}^{\uparrow}_x(\,\cdot\,| S_N = y)\,, \qquad \widehat{\mathbb{P}}^{\uparrow,N}_{x,y}(\,\cdot\,) = \widehat{\mathbb{P}}^{\uparrow}_x(\,\cdot\,| S_N = y)\,. \tag{2.6}$$

Similarly, if $\mathbf{P}^{\uparrow}_a$ is the law on Ω of the Lévy process $(X, \mathbf{P}_a)$ conditioned to stay positive for all time, as it is defined in [17] (cf. also §6.2 below), then $\mathbf{P}^{\uparrow,t}_{a,b}$ may be viewed as the bridge of $\mathbf{P}^{\uparrow}_a$, i.e.

$$\mathbf{P}^{\uparrow,t}_{a,b}(\,\cdot\,) = \mathbf{P}^{\uparrow}_a(\,\cdot\,| X_t = b)\,. \tag{2.7}$$

In other words, instead of first taking the bridge of a random walk, or a Lévy process, and then conditioning it to stay positive, one can first condition the process to stay positive (for all time) and then consider its bridge, the resulting process being the same.

2.2. **The invariance principle.** Recalling the definition (2.1) of the (restricted) map $\varphi_N : \Omega^{RW}_N \to \Omega_1$, we denote by $\mathbb{P}^{\uparrow,N}_{x,y} \circ \varphi_N^{-1}$ the law on $\Omega_1 = D([0,1],\mathbb{R})$ given by the push-forward of $\mathbb{P}^{\uparrow,N}_{x,y}$ through φ_N, and analogously for $\widehat{\mathbb{P}}^{\uparrow,N}_{x,y}$.

If $\{x_N\}_{N\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ such that $x_N/a_N \to a$ as $N \to \infty$, with $a \geq 0$, it is well known [40] that

$$\mathbb{P}_{x_N} \circ \varphi_N^{-1} \Longrightarrow \mathbf{P}_a\,, \tag{2.8}$$

where "$\Longrightarrow$" denotes weak convergence. Moreover, in [13] we have proved that

$$\mathbb{P}^{\uparrow}_{x_N} \circ \varphi_N^{-1} \Longrightarrow \mathbf{P}^{\uparrow}_a . \tag{2.9}$$

Our first result asserts that such an invariance principle also holds for bridges conditioned to stay positive.

Theorem 2.4. *Assume that Hypothesis 2.1 and 2.2 are satisfied. Let $a, b \in [0,\infty)$ and let $(x_N)_{N\in\mathbb{N}}, (y_N)_{N\in\mathbb{N}}$ be two non-negative sequences such that $x_N/a_N \to a$ and $y_N/a_N \to b$ (in the (h,c)-lattice case, assume that $(y_N - x_N) \in Nc + h\mathbb{Z}$ for all $N \in \mathbb{N}$). Then*

$$\mathbb{P}^{\uparrow,N}_{x_N,y_N} \circ \varphi_N^{-1} \Longrightarrow \mathbf{P}^{\uparrow,1}_{a,b} , \qquad \widehat{\mathbb{P}}^{\uparrow,N}_{x_N,y_N} \circ \varphi_N^{-1} \Longrightarrow \mathbf{P}^{\uparrow,1}_{a,b} \qquad (N \to \infty) . \tag{2.10}$$

Let us note that, by an easy scaling argument, this invariance principle immediately generalizes to bridges of any time length.

In the Brownian case $\alpha = 2$, $\varrho = \frac{1}{2}$, the process $\mathbf{P}^{\uparrow,1}_{a,b}$ has continuous paths. In this situation, it is standard to pass from weak convergence on $D([0,1],\mathbb{R})$ to weak convergence on $C([0,1],\mathbb{R})$, the space of real-valued continuous functions defined on $[0,1]$, endowed with the topology of uniform convergence and with the corresponding Borel σ-field. For this purpose, we introduce the map $\psi_N : \Omega_N^{RW} \to C([0,1],\mathbb{R})$, analogous to φ_N defined in (2.1), but corresponding to *linear interpolation*, i.e.

$$\big(\psi_N(S)\big)(t) := \frac{(1 + \lfloor Nt \rfloor - Nt) S_{\lfloor Nt \rfloor} + (Nt - \lfloor Nt \rfloor) S_{\lfloor Nt \rfloor + 1}}{a_N} , \qquad \forall t \in [0,1] . \tag{2.11}$$

For ease of reference, we state an important special case of Theorem 2.4.

Corollary 2.5 (Brownian case)**.** *Let $(\{X_n\}_{n\in\mathbb{N}}, \mathbb{P})$ be i.i.d. real random variables with zero mean and unit variance and let $S_0 = 0$, $S_n = S_{n-1} + X_n$, $n \in \mathbb{N}$, be the associated random walk, so that Hypothesis 2.1 is satisfied with $a_N = \sqrt{N}$ and X a standard Brownian motion. Assume that Hypothesis 2.2 is satisfied and let $(x_N)_{N\in\mathbb{N}}$, $(y_N)_{N\in\mathbb{N}}$ be non-negative sequences that are $o(\sqrt{N})$ (in the (h,c)-lattice case, assume that $(y_N - x_N) \in Nc + h\mathbb{Z}$ for all $N \in \mathbb{N}$). Then*

$$\mathbb{P}^{\uparrow,N}_{x_N,y_N} \circ \psi_N^{-1} \Longrightarrow \mathbf{P}^{\uparrow,1}_{0,0} , \qquad \widehat{\mathbb{P}}^{\uparrow,N}_{x_N,y_N} \circ \psi_N^{-1} \Longrightarrow \mathbf{P}^{\uparrow,1}_{0,0} \qquad (N \to \infty) .$$

In words: the random walk bridge of length N conditioned to stay positive or non-negative, starting at x_N and ending at y_N, under linear interpolation and diffusive rescaling, converges in distribution on $C([0,1],\mathbb{R})$ toward the normalized Brownian excursion.

Remark 2.6 (Random walk bridge conditioned to stay above a line)**.** Let $(S = \{S_n\}_{n\in\mathbb{N}_0}, \mathbb{P})$ be a random walk satisfying Hypothesis 2.1 and 2.2. For a fixed positive constant $\overline{c} > 0$, let us consider the random walk bridge of length N, conditioned to stay *above the straight line* $\overline{c}\, n$, starting at x and ending at $N\overline{c} + y$:

$$\mathbb{P}_x\big(\,\cdot\, \big| \, S_1 \geq \overline{c},\, S_2 \geq 2\overline{c},\, \ldots,\, S_{N-1} \geq (N-1)\overline{c},\, S_N = N\overline{c} + y \big) . \tag{2.12}$$

An invariance principle for this process can be recovered from our results, in some instances.

An easy example is when Hypothesis 2.1 holds for $\alpha < 1$: in this case, *the "centered" random walk* $(\widetilde{S} = \{\widetilde{S}_n := S_n - n\overline{c}\}_{n\in\mathbb{N}_0}, \mathbb{P})$ *still satisfies Hypothesis 2.1 and 2.2* (because $a_N/N \to +\infty$, see below). Since the conditioning event in (2.12) translates into

$$\{\widetilde{S}_1 \geq 0, \widetilde{S}_2 \geq 0, \ldots, \widetilde{S}_{N-1} \geq 0, \widetilde{S}_N = y\} ,$$

it follows that Theorem 2.4 can be applied to $(\widetilde{S}, \mathbb{P}_x)$. As a consequence, *Theorem 2.4 holds for the law* (2.12) as well, because the rescaled linear centering $n\overline{c}/a_N \leq N\overline{c}/a_N$ vanishes as $N \to \infty$ (recall (2.1)). Intuitively, the fluctuations of the random walk are of super-linear size ($a_N \gg N$), hence the rescaled straight line becomes asymptotically flat: this is why the limiting Lévy bridge is just conditioned to stay positive.

Another interesting example is when the random walk $(S, \mathbb{P})$ has zero mean, finite variance and *locally finite exponential moments* in a right neighborhood of the origin:

$$M(t) := \mathbb{E}[e^{tS_1}] < \infty \qquad \text{for} \quad 0 \le t \le \delta, \qquad \text{with } \delta > 0 .$$

For a given $t \in (0, \delta)$, consider the t-"tilted" random walk $(S = \{S_n\}_{n\in\mathbb{N}_0}, \widetilde{\mathbb{P}}^{(t)})$, whose increments $X_n := S_n - S_{n-1}$ are i.i.d. with marginal laws

$$\frac{\widetilde{\mathbb{P}}^{(t)}(S_1 \in \mathrm{d}x)}{\mathbb{P}(S_1 \in \mathrm{d}x)} := \frac{e^{tx}}{M(t)} .$$

The crucial observation is that *the law* (2.12) *is unchanged if we replace the probability* $\mathbb{P}$ *by* $\widetilde{\mathbb{P}}^{(t)}$*, for any* t: in fact, the Radon-Nikodym density of $(S_1, \ldots, S_N)$ equals

$$\frac{\widetilde{\mathbb{P}}^{(t)}\big((S_1, \ldots, S_N) \in (\mathrm{d}s_1, \ldots, \mathrm{d}s_N)\big)}{\mathbb{P}\big((S_1, \ldots, S_N) \in (\mathrm{d}s_1, \ldots, \mathrm{d}s_N)\big)} = \frac{e^{ts_N}}{M(t)^N} ,$$

therefore it depends only on s_N, which is fixed in (2.12). If we assume that $\mathbb{P}(S_1 > \overline{c}) > 0$, we can choose $\overline{t} > 0$ such that $\overline{c} = (\log M)'(\overline{t})$. In this way

$$\widetilde{\mathbb{E}}^{(\overline{t})}(S_1) = \int_{\mathbb{R}} x \, \frac{e^{\overline{t}x}}{M(\overline{t})} \mathbb{P}(S_1 \in \mathrm{d}x) = \frac{M'(\overline{t})}{M(\overline{t})} = (\log M)'(\overline{t}) = \overline{c} .$$

Then, switching the law to $\widetilde{\mathbb{P}}^{(\overline{t})}$ and applying a linear centering, we get a random walk

$$(\widetilde{S} = \{\widetilde{S}_n := S_n - n\overline{c}\}_{n\in\mathbb{N}_0}, \widetilde{\mathbb{P}}^{(\overline{t})})$$

with zero mean and finite variance $\overline{\sigma}^2 := \widetilde{\mathbb{V}}\mathrm{ar}^{(\overline{t})}(S_1) = (\log M)''(\overline{t}) \in (0, \infty)$, to which Theorem 2.4 can be applied. In particular, we obtain the following invariance principle: if $x = x_N = o(\sqrt{N})$ and $y = y_N = o(\sqrt{N})$, then as $N \to \infty$

$$\left\{ \frac{S_{\lfloor Nt \rfloor} - (N\overline{c})t}{\overline{\sigma}\sqrt{N}} \right\}_{t\in[0,1]} \quad \text{under} \ \ (2.12) \quad \Longrightarrow \quad \mathbf{P}^{\uparrow,1}_{0,0} . \tag{2.13}$$

In words: assuming finite exponential moments, a random walk bridge conditioned to stay above a straight line has diffusive fluctuations (with explicit variance), which converge under rescaling to the normalized Brownian excursion. Intuitively, the fluctuations of the random walk are much smaller than the height of the straight line ($\sqrt{N} \ll N$): this is why the straight line is subtracted in the left hand side of (2.13), to have a non-trivial limit, and the resulting limiting process $\mathbf{P}^{\uparrow,1}_{0,0}$ is just conditioned to stay positive.

As a last observation, note that if the random walk $(S, \mathbb{P})$ is in the standard $(1, 0)$-lattice case (i.e., the law of S_1 is supported by $\mathbb{Z}$ and not by $a + h\mathbb{Z}$, for any $a > 1$ and $h \in \mathbb{R}$), the "centered" random walk $\widetilde{S}$ is in the $(1, c)$-lattice case, with $c = \overline{c} - \lfloor \overline{c} \rfloor \neq 0$ if $\overline{c} \notin \mathbb{Z}$.

The proof of Theorem 2.4 bears on the absolute continuity of $\mathbb{P}^{\uparrow,N}_{x_N,y_N}$ with respect to $\mathbb{P}^{\uparrow}_{x_N}$, cf. (2.6), and exploits the convergence (2.9). In order to apply these arguments, we need a uniform control of the Radon-Nikodym density of $\mathbb{P}^{\uparrow,N}_{x_N,y_N}$ with respect to $\mathbb{P}^{\uparrow}_{x_N}$. This requires precise local estimates of the kernel $f^+_N(x, y)$, in the absolutely continuous case and of the analogous kernels $q^+_N(x, y)$ and $\widehat{q}^+_N(x, y)$ in the lattice case, cf. (2.4) and (2.5).

In the lattice case, such local limit theorems have been proved by Vatutin and Wachtel [42] and Doney [21] and are reviewed in Proposition 4.1. The proof of the local limit theorems for $f_N^+(x,y)$, in the absolutely continuous case, is the second main result of this paper, cf. Theorem 5.1 in section 5. This is obtained from the Stone version of the local limit theorems, also proved in [42, 21], through a careful approximating procedure.

We point out that our approach differs from that of Sohier [39] in the Brownian case, where the weak convergence of the sequence $\mathbb{P}^{\uparrow,N}_{x_N,y_N}$ is established, in a more classical way, proving tightness and convergence of the finite dimensional distributions.

Remark 2.7. For the asymptotic behavior of $f_N^+(x,y)$ in the absolutely continuous case, we need a suitable condition, linked to *direct Riemann integrability*, on a convolution of the random walk step density $f(\cdot)$ (cf. section 5 for details, in particular (5.11)). This is a very mild condition, which is immediately checked if, e.g., there exist $C>0$, $n\in\mathbb{N}$ and $\varepsilon>0$ small enough such that $|f_n(x)|\le C/|x|^{1+\alpha-\varepsilon}$ for every $x\in\mathbb{R}$. As a matter of fact, it turns out that this condition is automatically satisfied with no further assumption beyond Hypotheses 2.1 and 2.2, as it is proved in [12].

3. Preparatory Material

3.1. **An important notation on sequences.** We will frequently deal with sequences $(b_n(z))_{n\in\mathbb{N}}$ indexed by a real parameter z. Given a family of subsets $V_n\subseteq\mathbb{R}$, we write

$$\text{“}b_n(z)=o(1)\ \text{ uniformly for } z\in V_n\text{”} \qquad \text{to mean} \qquad \lim_{n\to\infty}\sup_{z\in V_n}|b_n(z)|=0\,. \tag{3.1}$$

We stress that this is actually equivalent to the seemingly weaker condition

$$\lim_{n\to\infty} b_n(z_n)=0\ \text{ for any fixed sequence } (z_n)_{n\in\mathbb{N}} \text{ such that } z_n\in V_n \text{ for all } n\in\mathbb{N}\,,$$

as one checks by contradiction. We also note that, by a subsequence argument, to prove such a relation it is sufficient to consider sequences $(z_n)_{n\in\mathbb{N}}$ (such that $z_n\in V_n$ for all $n\in\mathbb{N}$) that converge to a (possibly infinite) limit, i.e. such that $z_n\to c\in\mathbb{R}\cup\{\pm\infty\}$.

Given $(b_n(z))_{n\in\mathbb{N}}$, with $z\in\mathbb{R}$, and a fixed positive sequence $(a_n)_{n\in\mathbb{N}}$, it is sometimes customary to write

$$\text{“}b_n(z)=o(1)\ \text{ uniformly for } z=o(a_n)\text{”} \tag{3.2}$$

as a shorthand for

$$b_n(z)=o(1)\ \text{ uniformly for } z\in[0,\varepsilon_n]\text{, for any fixed sequence } \varepsilon_n=o(a_n)\,. \tag{3.3}$$

Again, this is equivalent to the apparently weaker statement

$$b_n(z_n)=o(1)\ \text{ for any fixed sequence } z_n=o(a_n)\,, \tag{3.4}$$

as an easy contradiction argument shows. The formulation (3.2)–(3.3) is usually preferred when stating and applying theorems, while (3.4) is nicer to handle when proving them.

In the sequel, we sometimes write (*const.*), (*const.'*) to denote generic positive constants, whose value may change from place to place.

3.2. **Fluctuation theory for random walks.** For the purpose of this subsection, we only assume that $(S=\{S_n\}_{n\ge 0},\mathbb{P})$ is a random walk on $\mathbb{R}$ starting at zero, that is $S_0=0$ a.s. and $(\{S_n-S_{n-1}\}_{n\ge 1},\mathbb{P})$ are i.i.d. real-valued real random variables. To avoid degeneracies, we assume that the walk is not constant, i.e., $\mathbb{P}(S_1=c)<1$ for all $c\in\mathbb{R}$. We refer to [25] for more details on fluctuation theory.

We denote by $\{\tau_k^\pm\}_{k\ge 0}$ and $\{H_k^\pm\}_{k\ge 0}$ the *weak* ascending $(+)$ and descending $(-)$ ladder epoch and ladder height processes respectively, that is $\tau_0^\pm:=0$, $H_0^\pm:=0$ and for $k\ge 1$

$$\tau_k^\pm:=\inf\left\{n>\tau_{k-1}^\pm:\ \pm S_n\ge \pm S_{\tau_{k-1}^\pm}\right\}, \qquad H_k^\pm:=\pm S_{\tau_k^\pm}\,. \tag{3.5}$$

Note that H_k^- is a non-negative random variable (which, in general, may take the value $+\infty$). In fact, $\{\tau_k^\pm\}_{k\geq 0}$ and $\{H_k^\pm\}_{k\geq 0}$ are *renewal processes*, i.e., random walks with i.i.d. non-negative increments. One can also consider *strict* ladder variables, but we are not going to use them. The interested reader can look at Remark 4.6 below for some connections.

Let us set

$$\begin{aligned}\zeta := \mathbb{P}(H_1^+ = 0) &= \sum_{n=1}^{\infty} \mathbb{P}(S_1 < 0, \ldots, S_{n-1} < 0, S_n = 0) \\ &= \sum_{n=1}^{\infty} \mathbb{P}(S_1 > 0, \ldots, S_{n-1} > 0, S_n = 0) = \mathbb{P}(H_1^- = 0) \in [0,1)\,,\end{aligned} \tag{3.6}$$

where the third inequality holds by time reversal, observing that $\{S_n - S_{n-k}\}_{0\leq k\leq n}$ has the same law as $\{S_k\}_{0\leq k\leq n}$. We denote by $V^\pm(\cdot)$ the weak ascending $(+)$ or descending $(-)$ renewal function associated to $\{H_k^\pm\}_{k\geq 0}$, defined for $x \geq 0$ by

$$V^\pm(x) = \mathbb{E}\big[\#\{k \geq 0 : H_k^\pm \leq x\}\big] = \sum_{k=0}^{\infty} \mathbb{P}(H_k^\pm \leq x) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \mathbb{P}(\tau_k^\pm = n, \pm S_n \leq x)\,. \tag{3.7}$$

Observe that $V^\pm(\cdot)$ is a non-decreasing, right-continuous function and

$$V^\pm(0) = \sum_{k=0}^{\infty} \mathbb{P}(H_k^\pm = 0) = \sum_{k=0}^{\infty} \zeta^k = \frac{1}{1-\zeta}\,. \tag{3.8}$$

We also introduce a *modified* renewal function $\underline{V}^\pm(\cdot)$, defined for $x > 0$ by

$$\underline{V}^\pm(x) = \mathbb{E}\big[\#\{k \geq 0 : H_k^\pm < x\}\big] = \sum_{k=0}^{\infty} \mathbb{P}(H_k^\pm < x) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \mathbb{P}(\tau_k^\pm = n, \pm S_n < x)\,. \tag{3.9}$$

Note that that this function is *left-continuous* on $(0,\infty)$ and it can be recovered from the previously introduced one through a simple limiting procedure: for every $x > 0$

$$\underline{V}^\pm(x) = V^\pm(x-) := \lim_{\varepsilon\downarrow 0} V^\pm(x-\varepsilon)\,. \tag{3.10}$$

Plainly, when the law of S_1 is supported by $\mathbb{Z}$, we have $\underline{V}^\pm(x) = V^\pm(x-1)$. We complete the definition of $\underline{V}^\pm(x)$ by setting, for $x = 0$,

$$\underline{V}^\pm(0) := 1\,. \tag{3.11}$$

The reason for introducing the modified renewal function $\underline{V}^\pm(\cdot)$ is explained by the following classical Lemma, proved e.g. in [34, eq. (23)] (see also [5, part 2.3] and [6, part 2]). We recall that $\mathbb{P}_x$ denotes the law of S started at $x \in \mathbb{R}$, that is $\mathbb{P}_x(S \in \cdot) = \mathbb{P}(S + x \in \cdot)$.

Lemma 3.1. *Assume that the random walk $(\{S_n\}_{n\geq 0}, \mathbb{P})$ does not drift to $-\infty$, that is $\limsup_k S_k = +\infty$, $\mathbb{P}$–a.s.. Then the renewal function $V^-(\cdot)$ is invariant for the semigroup of the random walk killed when it first enters the negative half-line $(-\infty, 0)$, i.e.,*

$$V^-(x) = \mathbb{E}_x\big(V^-(S_N)\, 1_{\{S_1\geq 0,\ldots,S_N\geq 0\}}\big) \qquad \forall x \geq 0, N \in \mathbb{N}\,. \tag{3.12}$$

Analogously, the modified renewal function $\underline{V}^-(x)$ is invariant for the semigroup of the random walk killed when it first enters the non-positive half-line $(-\infty, 0]$:

$$\underline{V}^-(x) = \mathbb{E}_x\big(\underline{V}^-(S_N)\, 1_{\{S_1>0,\ldots,S_N>0\}}\big) \qquad \forall x \geq 0, N \in \mathbb{N}\,. \tag{3.13}$$

We note that by the symmetry $S \to -S$ it follows immediately from (3.12) and (3.13) that if the random walk $(\{S_n\}_{n\geq 0}, \mathbb{P})$ does not drift to $+\infty$ we have

$$V^+(x) = \mathbb{E}_{-x}\big(V^+(-S_N)\, 1_{\{S_1\leq 0,\dots,S_N\leq 0\}}\big) \qquad \forall x\geq 0, N\in\mathbb{N}\,, \tag{3.14}$$
$$\underline{V}^+(x) = \mathbb{E}_{-x}\big(\underline{V}^+(-S_N)\, 1_{\{S_1< 0,\dots,S_N< 0\}}\big) \qquad \forall x\geq 0, N\in\mathbb{N}\,. \tag{3.15}$$

3.3. **Some consequences of our assumptions.** With the notation of Hypothesis 2.1, let $g(\cdot)$ be the density of X_1. We denote by $g^+(\cdot)$ the density of the time-one marginal distribution of the meander [16] of the Lévy process X, which can be defined informally by $g^+(x)\,\mathrm{d}x = \mathbf{P}_0(X_1\in \mathrm{d}x \,|\, \inf_{0\leq s\leq 1} X_s \geq 0)$, see Lemma 4 in [18]. Analogously, $g^-(\cdot)$ is the density of the time-one marginal distribution of the meander of $-X$.

It is well-known [38, 37] that, when Hypothesis 2.1 holds, the random variables τ_1^+ and H_1^+ are in the domain of attraction of stable laws of index ϱ and $\alpha\varrho$.† In particular:

$$\mathbb{P}(\tau_1^+ > n) \in R_{-\varrho}\,, \qquad V^+(x)\in R_{\alpha\varrho}\,. \tag{3.16}$$

An analogous statement holds for the descending ladder variables, changing ϱ with $1-\varrho$:

$$\mathbb{P}(\tau_1^- > n) \in R_{-(1-\varrho)}\,, \qquad V^-(x)\in R_{\alpha(1-\varrho)}\,. \tag{3.17}$$

We point out that, by equation (31) in [42], as $n\to\infty$

$$V^+(a_n) \sim \frac{\mathtt{C}^+}{(1-\zeta)}\, n\,\mathbb{P}(\tau_1^- > n)\,, \qquad V^-(a_n) \sim \frac{\mathtt{C}^-}{(1-\zeta)}\, n\,\mathbb{P}(\tau_1^+ > n)\,, \tag{3.18}$$

where ζ is defined in (3.6). Furthermore, by Theorem 1 of [22], as $\varepsilon\downarrow 0$

$$g^+(\varepsilon) \sim \widetilde{\mathtt{C}}^+\, g(0)\,\varepsilon^{\alpha\varrho}\,, \qquad g^-(\varepsilon) \sim \widetilde{\mathtt{C}}^-\, g(0)\,\varepsilon^{\alpha(1-\varrho)}\,. \tag{3.19}$$

It turns out that the corresponding constants in the preceding relations coincide, i.e.:

$$\mathtt{C}^+ = \widetilde{\mathtt{C}}^+\,, \qquad \mathtt{C}^- = \widetilde{\mathtt{C}}^-\,, \tag{3.20}$$

as we prove in Lemma 4.4 below. We stress that the precise value of these constants is not universal. In fact, if we change the norming sequence, taking $a_n' := c a_n$ with $c>0$, then $S_n/a_n' \Rightarrow X_1' := X_1/c$ (recall Hypothesis 2.1): rewriting (3.18) and (3.19) for a_n' and for the densities associated to X', one sees that the constants $\mathtt{C}^\pm$ and $\widetilde{\mathtt{C}}^\pm$ get divided by $c^{\alpha\varrho}$.

Finally, since $V^\pm(x-\varepsilon)\leq \underline{V}^\pm(x)\leq V^\pm(x)$ for every $\varepsilon>0$, it follows that the functions $\underline{V}^\pm(x)$ and $V^\pm(x)$ have the same asymptotic behavior as $x\uparrow+\infty$. In particular, relations (3.16), (3.17) and (3.18) hold also for $\underline{V}^\pm(\cdot)$.

4. Local limit theorems in the lattice case

In this section we put ourselves in the (h,c)-lattice case, cf. Hypothesis 2.2. We aim at the precise asymptotic behavior of the kernels $q_n^+(x,y)$ and $\widehat{q}_n^+(x,y)$, defined in (2.4).

When both x/a_n and y/a_n stay away from 0 and ∞, this is an easy consequence of Liggett's invariance principle for the bridges [35] and Gnedenko's local limit theorem [30], which states that as $n\to\infty$

$$\mathbb{P}_x(S_n = y) = \frac{h}{a_n}\left\{ g\left(\frac{y-x}{a_n}\right) + o(1)\right\}, \tag{4.1}$$

†Let us mention that more is true: the random vector (τ_1^+, H_1^+) is in the domain of attraction of a bivariate stable law of indexes $(\varrho, \alpha\varrho)$, cf. [29, 20, 19]; see also [18] for a more general result.

uniformly for $x, y \in \mathbb{R}$ with $(y-x) \in nc + h\mathbb{Z}$. In fact, setting for $a, b > 0$

$$C(a,b) := \mathbf{P}_a\left(\inf_{0 \le s \le 1} X_s \ge 0 \,\middle|\, X_1 = b \right),$$

for every fixed $\varepsilon > 0$ we have as $n \to \infty$

$$q_n^+(x,y) = \frac{h}{a_n}\, g\left(\frac{y-x}{a_n}\right) C\left(\frac{x}{a_n}, \frac{y}{a_n}\right)(1+o(1)), \quad \text{unif. for } x, y \in \left(\varepsilon a_n, \frac{1}{\varepsilon} a_n\right), \tag{4.2}$$

always with $(y-x) \in nc + h\mathbb{Z}$. Exactly the same relation holds for $\widehat{q}_n^+(x,y)$.

Note that $C(a,b) > 0$ for all $a, b > 0$, but $C(a,b) \to 0$ if $\min\{a,b\} \to 0$, therefore when either $x = o(a_n)$ or $y = o(a_n)$ relation (4.2) only says that $q_n^+(x,y) = o(1/a_n)$, and analogously $\widehat{q}_n^+(x,y) = o(1/a_n)$. Precise estimates in this regime have been obtained in the last years: for ease of reference, we sum them up explicitly.

Proposition 4.1. *Assume that Hypothesis 2.1 and Hypothesis 2.2 ((h,c)-lattice case) hold. Then the following relations hold as $n \to \infty$, for $x, y \ge 0$ with $(y-x) \in nc + h\mathbb{Z}$:*

$$\left.\begin{aligned} q_n^+(x,y) &= \frac{h\,\mathbb{P}(\tau_1^- > n)}{a_n} V^-(x)\left(g^+\left(\frac{y}{a_n}\right) + o(1)\right) \\ \widehat{q}_n^+(x,y) &= \frac{h\,\mathbb{P}(\tau_1^- > n)}{a_n} \underline{V}^-(x)\left(g^+\left(\frac{y}{a_n}\right) + o(1)\right) \end{aligned}\right\} \text{unif. for } x = o(a_n),\ y \ge 0, \tag{4.3}$$

$$\left.\begin{aligned} q_n^+(x,y) &= \frac{h\,\mathbb{P}(\tau_1^+ > n)}{a_n} V^+(y)\left(g^-\left(\frac{x}{a_n}\right) + o(1)\right) \\ \widehat{q}_n^+(x,y) &= \frac{h\,\mathbb{P}(\tau_1^+ > n)}{a_n} \underline{V}^+(y)\left(g^-\left(\frac{x}{a_n}\right) + o(1)\right) \end{aligned}\right\} \text{unif. for } y = o(a_n),\ x \ge 0, \tag{4.4}$$

$$\left.\begin{aligned} q_n^+(x,y) &= h\,(1-\zeta)\frac{g(0)}{n\,a_n}\, V^-(x)\, V^+(y)\, \big(1+o(1)\big) \\ \widehat{q}_n^+(x,y) &= h\,(1-\zeta)\frac{g(0)}{n\,a_n}\, \underline{V}^-(x)\, \underline{V}^+(y)\, \big(1+o(1)\big) \end{aligned}\right\} \text{uniformly for } x, y = o(a_n), \tag{4.5}$$

where $\zeta \in [0,1)$ is defined in (3.6).

As the above list is somewhat redundant, let us give some directions. The relations in (4.4) can be deduced from those in (4.3) with a symmetry argument, that is, considering the random walk $-S$ instead of S and exchanging x with y and every "+" quantity with the corresponding "−" one. Therefore it is sufficient to focus on (4.3) and (4.5), but there are further simplifications. In fact, for all fixed $n \in \mathbb{N}$ and $x, y \ge 0$, we have $\widehat{q}_n^+(x,y) = q_n^+(x-\varepsilon, y-\varepsilon)$ provided $\varepsilon > 0$ is small enough, thanks to the lattice assumption (recall (2.4)). Taking $\varepsilon = \varepsilon_n \downarrow 0$ and recalling (3.10), it follows that the first equation in (4.5) is a consequence of the second one, provided $x, y > 0$. Analogously, the first equation in (4.3) follows from the second one, when both $x, y > 0$.

Summarizing, for Proposition 4.1 it is sufficient to prove:

- the first relations in (4.3) and (4.5) in the special case $x = 0$ (the case $y = 0$ follows by a symmetry argument);
- the second relations in (4.3) and (4.5) for general x.

The second relations in (4.3) and (4.5) for $x = 0$ were proved in [42, Theorem 5 and Theorem 6], while the first ones can be deduced arguing as in page 100 of [2].[†] (The case $x = y = 0$ of (4.5) has not been considered in [42], but it can be easily deduced, as we show in Appendix A.1 and A.2.) The second relations in (4.3) and (4.5) for general x have been recently proved in [21, Propositions 11 and 24], using a decomposition that allows to express them as a function of the $x = 0$ case. This completes the justification of Proposition 4.1.

We point out that in [42, 21] some of the above relations appear in a slightly different form, being expressed in terms of *strict* rather than weak ladder variables. The interested reader can find more details in Remark 4.6 below.

Remark 4.2. Since the functions $g^{\pm}(z)$ vanish both when $z \downarrow 0$ and when $z \to +\infty$, only when y/a_n is bounded away from 0 and ∞ the two relations in (4.3) give the *precise* asymptotic behavior (i.e., the ratio of the two sides of the equation converges to 1). When $y/a_n \to 0$, that is $y = o(a_n)$, the precise asymptotic behavior is given by (4.5). When $y/a_n \to +\infty$ and $\alpha\varrho < 1$ (which excludes the Brownian case), the precise asymptotic behavior can be derived under additional assumptions, cf. Proposition 13 in [21].

Remark 4.3. In the gaussian case ($\alpha = 2$, $\varrho = \frac{1}{2}$) several explicit expressions are available. For instance, $g^{\pm}(x) = x\, e^{-x^2/2}\, \mathbb{1}_{(0,\infty)}(x)$ and $g(x) = (2\pi)^{-1/2} e^{-x^2/2}$, hence the constants in (3.19) are $\widetilde{\mathtt{C}}^{\pm} = \sqrt{2\pi}$. From equation (2.6) and the last equation on p. 515 in [11], it follows that also for the constants in (3.18) one has $\mathtt{C}^{\pm} = \sqrt{2\pi}$, in agreement with (3.20).

Furthermore, since $\underline{V}^{+}(\cdot)$ and $V^{+}(\cdot) \in R_1$ by (3.16), if y/a_n is bounded away from 0 and ∞ we can write $V^{+}(y) \sim \underline{V}^{+}(y) \sim \frac{y}{a_n} V^{+}(a_n)$. Recalling (3.18) and (3.8), it follows that relations (4.3) and (4.5) can be gathered in the following ones:

$$
\begin{aligned}
q_n^{+}(x,y) &\sim \frac{1-\zeta}{n}\, \mathbb{P}(S_n = y)\, V^{-}(x)\, V^{+}(y)\,, \\
\widehat{q}_n^{+}(x,y) &\sim \frac{1-\zeta}{n}\, \mathbb{P}(S_n = y)\, \underline{V}^{-}(x)\, \underline{V}^{+}(y)\,,
\end{aligned}
\tag{4.6}
$$

uniformly for $x = o(a_n)$ and $y \in [0, M a_n]$, for any fixed $M > 0$ (cf. equation (1.7) in [11]).

It is natural to ask whether relations (4.6) still hold for $\alpha < 2$. Recalling (3.18), (3.19) and (3.20), this is equivalent to asking whether (3.19) can be strengthened to

$$
g^{+}(x) = \widetilde{\mathtt{C}}^{+} x^{\alpha\varrho} g(x) \mathbb{1}_{(0,\infty)}(x)\,, \qquad \forall x \in \mathbb{R}\,. \tag{4.7}
$$

Arguing as in [1], it is not difficult to show that this relation holds when the limiting Lévy process has no positive jumps, i.e. for $\alpha \in (1,2)$ and $\varrho = 1 - 1/\alpha$. We conjecture that relation (4.7) fails whenever $\varrho \neq 1 - 1/\alpha$. In the symmetric Cauchy case $\alpha = 1$, $\varrho = 1/2$ it has been shown that indeed relation (4.7) does not hold (cf. the comments following Proposition 1 in [1]).

We conclude the section proving that the constants $\mathtt{C}^{\pm}$ and $\widetilde{\mathtt{C}}^{\pm}$ indeed coincide.

Lemma 4.4. *Recalling relations* (3.18) *and* (3.19)*, we have* $\mathtt{C}^{+} = \widetilde{\mathtt{C}}^{+}$ *and* $\mathtt{C}^{-} = \widetilde{\mathtt{C}}^{-}$.

For the proof we need the following very general result.

Lemma 4.5. *Let* $\{f_n(\cdot)\}_{n\in\mathbb{N}}$ *be an arbitrary sequence of real functions, all defined on the same subset* $I \subseteq \mathbb{R}$. *Assume that for every* $z \in I$ *and for every sequence* $\{z_n\}_{n\in\mathbb{N}}$ *of* I*, such*

[†]Previously, these relations were proved in the gaussian case ($\alpha = 2$, $\varrho = \frac{1}{2}$), cf. [10, Proposition 1] and [11, Theorem 4] for (4.3) and [2, equation (9)] for (4.5). See also [24] for related results.

that $z_n \to z$, the limit $f(z) := \lim_{n\to\infty} f_n(z_n)$ exists and does not depend on the sequence $\{z_n\}_{n\in\mathbb{N}}$, but only on the limit point z. Then the function $f : I \to \mathbb{R}$ is continuous.

Proof. We proceed by contradiction. If f is not continuous, there exist $\varepsilon > 0$, $\bar{z} \in I$ and a sequence $\bar{z}^{(k)} \to \bar{z}$ such that $|f(\bar{z}) - f(\bar{z}^{(k)})| > 2\varepsilon$ for every $k \in \mathbb{N}$. For every fixed $k \in \mathbb{N}$ we have $\lim_{n\to\infty} f_n(\bar{z}^{(k)}) = f(\bar{z}^{(k)})$ by assumption, hence there exists $\widetilde{n}(k) \in \mathbb{N}$ such that $|f(\bar{z}^{(k)}) - f_{\widetilde{n}(k)}(\bar{z}^{(k)})| < \varepsilon$. By the triangle inequality, we then have

$$|f_{\widetilde{n}(k)}(\bar{z}^{(k)}) - f(\bar{z})| \;\geq\; |f(\bar{z}) - f(\bar{z}^{(k)})| \;-\; |f(\bar{z}^{(k)}) - f_{\widetilde{n}(k)}(\bar{z}^{(k)})| \;>\; \varepsilon\,, \quad \forall k \in \mathbb{N}\,. \tag{4.8}$$

Observe that $\widetilde{n}(k)$ can be taken as large as we wish, hence we may assume that $k \mapsto \widetilde{n}(k)$ is increasing. We also set $\widetilde{n}(0) := 0$ for convenience. Let us finally define the sequence $\{z_n\}_{n\in\mathbb{N}}$ by $z_n := \bar{z}^{(k)}$, where $k \in \mathbb{N}$ is the only index such that $\widetilde{n}(k-1) < n \leq \widetilde{n}(k)$. By construction $\bar{z}^{(k)} \to \bar{z}$, hence also $z_n \to \bar{z}$ and it follows by assumption that $f_n(z_n) \to f(\bar{z})$. However, this is impossible because the subsequence $\{f_{\widetilde{n}(k)}(z_{\widetilde{n}(k)})\}_{k\in\mathbb{N}}$ does not converge to $f(\bar{z})$, as $|f_{\widetilde{n}(k)}(z_{\widetilde{n}(k)}) - f(\bar{z})| = |f_{\widetilde{n}(k)}(\bar{z}^{(k)}) - f(\bar{z})| > \varepsilon$ for every $k \in \mathbb{N}$ by (4.8). ☐

Proof of Lemma 4.4. Let us set for $n \in \mathbb{N}$ and $z \in [0, \infty)$

$$f_n(z) \;:=\; \frac{n\, a_n}{V^+(\lfloor a_n z \rfloor)}\, q_n^+(0, \lfloor a_n z \rfloor)\,.$$

Observe that, if $z > 0$ and $z_n \to z$, by (3.16) and (3.18) we have, as $n \to \infty$,

$$V^+(\lfloor a_n z_n \rfloor) \;\sim\; z^{\alpha\varrho}\, V^+(a_n) \;\sim\; z^{\alpha\varrho}\, \frac{\mathtt{C}^+}{(1-\zeta)}\, n\, \mathbb{P}(\tau_1^- > n)\,.$$

It follows that the sequence of real functions $\{f_n(\cdot)\}_{n\in\mathbb{N}}$, all defined on $I = [0, \infty)$, satisfies the assumptions of Lemma 4.5: in fact, by (3.8) and the first relations in (4.3) and (4.5), for every $z \in [0, \infty)$ and every sequence $z_n \to z$ we have that

$$\exists f(z) \;:=\; \lim_{n\to\infty} f_n(z_n) \;=\; \begin{cases} h\, \dfrac{g^+(z)}{\mathtt{C}^+\, z^{\alpha\varrho}} & \text{if } z > 0 \\ h\, g(0) & \text{if } z = 0 \end{cases}\,.$$

By Lemma 4.5, the function $f(\cdot)$ is continuous, hence $\lim_{z\downarrow 0} f(z) = f(0)$. Recalling (3.19), it follows that $\mathtt{C}^+ = \widetilde{\mathtt{C}}^+$. With almost identical arguments one shows that $\mathtt{C}^- = \widetilde{\mathtt{C}}^-$. ☐

Remark 4.6 (Strict ladder variables)**.** For notational simplicity, so far we have only worked with *weak* ladder variables, and we will continue to do so. However, since *strict* ladder variables appear often in the literature, it might be useful to point out some connections.

The strict ladder variables $\{\widehat{\tau}_k^\pm\}_{k\geq 0}$ and $\{\widehat{H}_k^\pm\}_{k\geq 0}$ are defined by $\widehat{\tau}_0^\pm := 0$, $\widehat{H}_0^\pm := 0$ and, for $k \geq 1$, by the same relation (3.5), in which the weak inequality $\geq$ is replaced by the strict one $>$. The laws of $H_1^\pm$ and $\widehat{H}_1^\pm$ are closely related: recalling the definition (3.6) of ζ,

$$\mathbb{P}(H_1^\pm \in \mathrm{d}x) = \zeta \delta_0(\mathrm{d}x) + (1-\zeta)\mathbb{P}(\widehat{H}_1^\pm \in \mathrm{d}x)\,.$$

The renewal function $\widehat{V}^\pm(x)$ associated to $\{\widehat{H}_k^\pm\}_{k\geq 0}$ is defined like in (3.7), setting for $x \geq 0$

$$\widehat{V}^\pm(x) = \mathbb{E}\big[\#\{k \geq 0 : \widehat{H}_k^\pm \leq x\}\big] = \sum_{k=0}^{\infty} \mathbb{P}(\widehat{H}_k^\pm \leq x) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \mathbb{P}(\widehat{\tau}_k^\pm = n, \pm S_n \leq x); \tag{4.9}$$

in particular $\widehat{V}^\pm(0) = 1$. Likewise, the modified renewal function $\underline{\widehat{V}}^\pm(x)$ is defined for $x > 0$ like in (3.9), replacing $H_k^\pm$ by $\widehat{H}_k^\pm$, and for $x = 0$ by setting $\underline{\widehat{V}}^\pm(0) := 1 - \zeta = \mathbb{P}(H_1^\pm > 0)$.

These "strict" renewal functions turn out to be multiples of the "weak" ones:

$$\widehat{V}^{\pm}(x) = (1-\zeta)V^{\pm}(x)\,, \qquad \widehat{\underline{V}}^{\pm}(x) = (1-\zeta)\underline{V}^{\pm}(x)\,, \qquad \forall x \geq 0\,, \tag{4.10}$$

(cf. equation (1.13) in [25, § XII.1]). In particular, relation (3.10) holds for the strict renewal functions $\widehat{\underline{V}}^{\pm}(\cdot), \widehat{V}^{\pm}(\cdot)$ as well. Concerning the ladder epochs, we have

$$\mathbb{P}(\widehat{\tau}_1^- > n) \sim \frac{1}{(1-\zeta)}\mathbb{P}(\tau_1^- > n) \qquad \text{as} \quad n \to \infty\,, \tag{4.11}$$

as we prove in Appendix A.3.

Thanks to (4.10) and (4.11), every relation depending on $V^{\pm}(\cdot)$, $\underline{V}^{\pm}(\cdot)$, $\mathbb{P}(\tau_1^- > n)$ can be equivalently rephrased in terms of $\widehat{V}^{\pm}(\cdot)$, $\widehat{\underline{V}}^{\pm}(\cdot)$, $\mathbb{P}(\widehat{\tau}_1^- > n)$. This applies, for instance, to all the asymptotic relations in Proposition 4.1. In particular, for both relations in (4.5), the factor $(1-\zeta)$ can be "absorbed" into one of the renewal functions $V^{\pm}(\cdot)$ and $\underline{V}^{\pm}(\cdot)$, by (4.10) (this is the way these relations appear in [42, 21]).

5. Local limit theorems in the absolutely continuous case

In this section we focus on the absolutely continuous case, cf. Hypothesis 2.2. Our goal is to derive local asymptotic relations for the kernel $f_N^+(x,y)$, recall (2.5), that are closely analogous to the relations stated in Proposition 4.1 for the lattice case. Since the law of S_1 has no atom, the modified renewal functions $\underline{V}^{\pm}(x)$ coincide with the ordinary ones $V^{\pm}(x)$, hence everything will be expressed as a function of V^+ and V^-. More precisely, we are going to prove the following important result:

Theorem 5.1. *Assume that Hypothesis 2.1 and Hypothesis 2.2 (absolutely continuous case) hold. Then the following relations hold as $n \to \infty$, for $x, y \in [0,\infty)$:*

$$f_n^+(x,y) = \frac{\mathbb{P}(\tau_1^- > n)}{a_n} V^-(x)\left(g^+\left(\frac{y}{a_n}\right) + o(1)\right), \quad \textit{unif. for } x = o(a_n),\ y \in [0,\infty) \tag{5.1}$$

$$f_n^+(x,y) = \frac{g(0)}{n a_n} V^-(x) V^+(y)\,(1+o(1))\,, \quad \textit{unif. for } x = o(a_n),\ y = o(a_n)\,. \tag{5.2}$$

It is convenient to introduce the measure

$$F_n^+(x, \mathrm{d}y) := \mathbb{P}_x(S_1 > 0, \ldots, S_n > 0, S_n \in \mathrm{d}y)\,,$$

so that, cf. (2.5),

$$f_n^+(x,y) = \frac{F_n^+(x,\mathrm{d}y)}{\mathrm{d}y}\,.$$

Our starting point is the "Stone version" of the lattice estimates in Proposition 4.1, proved by Vatutin and Wachtel [42] and Doney [21], that read as

(1) for any fixed $\Delta > 0$, uniformly for $x = o(a_n)$, $y \in [0,\infty)$ one has, as $n \to \infty$,

$$F_n^+(x, [y, y+\Delta)) = \frac{\mathbb{P}(\tau_1^- > n)}{a_n} V^-(x)\left(g^+\left(\frac{y}{a_n}\right)\Delta + o(1)\right), \tag{5.3}$$

(2) for any fixed $\Delta > 0$, uniformly for $x = o(a_n)$, $y = o(a_n)$ one has, as $n \to \infty$,

$$F_n^+(x, [y, y+\Delta)) = \frac{g(0)}{n a_n} V^-(x)\left(\int_{[y,y+\Delta)} V^+(z)\,\mathrm{d}z\right)(1+o(1))\,. \tag{5.4}$$

The idea of the proof of Theorem 5.1 is to derive the asymptotic relations for $f_n^+(x,y)$ from the "integrated" relations (5.3), (5.4), by letting $\Delta \downarrow 0$. The delicate point is that interchanging the limits $\Delta \downarrow 0$ and $n \to \infty$ requires a careful justification.

5.1. **Strategy of the proof.** We choose $\bar{k} \in \mathbb{N}$ sufficiently large but *fixed*, as we specify below (cf. §5.2), and we write for all $n \geq \bar{k}$

$$f_n^+(x,y) \;=\; \int_{[0,\infty)} F_{n-\bar{k}}^+(x,\mathrm{d}z)\, f_{\bar{k}}^+(z,y)\,. \tag{5.5}$$

Next we approximate this integral by a Riemann sum over small intervals. More precisely, we set for $z, y \geq 0$ and $\Delta > 0$

$$\underline{f}_\Delta^+(z,y) := \inf_{u \in [z,z+\Delta)} f_{\bar{k}}^+(u,y)\,, \qquad \overline{f}_\Delta^+(z,y) := \sup_{u \in [z,z+\Delta)} f_{\bar{k}}^+(u,y)\,, \tag{5.6}$$

so that for every $\Delta > 0$, $n \in \mathbb{N}$ and $x, y \geq 0$ we can write

$$s_{n,\Delta}^+(x,y) \;\leq\; f_n^+(x,y) \;\leq\; S_{n,\Delta}^+(x,y)\,, \tag{5.7}$$

where

$$s_{n,\Delta}^+(x,y) \;:=\; \sum_{z \in \Delta\mathbb{N}_0} F_{n-\bar{k}}^+(x,[z,z+\Delta))\, \underline{f}_\Delta^+(z,y)\,, \tag{5.8}$$

$$S_{n,\Delta}^+(x,y) \;:=\; \sum_{z \in \Delta\mathbb{N}_0} F_{n-\bar{k}}^+(x,[z,z+\Delta))\, \overline{f}_\Delta^+(z,y)\,. \tag{5.9}$$

The idea is to replace $F_{n-\bar{k}}^+(x,[z,z+\Delta))$ by its asymptotic behavior, given in (5.3) and (5.4), and to show that $\underline{f}_\Delta^+(z,y) \simeq \overline{f}_\Delta^+(z,y)$ if Δ is small. This is of course to be made precise. The delicate point is that we need uniformity in z.

5.2. **The choice of $\bar{k}$.** For the choice of $\bar{k}$ appearing in (5.5) we impose two conditions.

The first condition on $\bar{k}$ is that $f_{\bar{k}-1}$ is a bounded function, that is $\|f_{\bar{k}-1}\|_\infty < \infty$, which we can do by Hypothesis 2.2. This is enough to ensure that $f_{\bar{k}}^+(z,y)$ is uniformly continuous in z, uniformly in y. By this we mean that for every $\varepsilon > 0$ there exists $\Delta > 0$ such that for all $z, z' \geq 0$ with $|z - z'| \leq \Delta$ and for every $y \geq 0$

$$|f_{\bar{k}}^+(z',y) - f_{\bar{k}}^+(z,y)| \leq \varepsilon\,.$$

The proof is simple: since $|f_n^+(a,b)| \leq |f_n(b-a)| \leq \|f_n\|_\infty$ for all $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$,

$$\begin{aligned}
|f_{\bar{k}}^+(z',y) - f_{\bar{k}}^+(z,y)| &= \left| \int_{w \in [0,\infty)} (f(w-z') - f(w-z)) f_{\bar{k}-1}^+(w,y)\, \mathrm{d}w \right| \\
&\leq \|f_{\bar{k}-1}\|_\infty \left| \int_{w \in [0,\infty)} (f(w-z') - f(w-z))\, \mathrm{d}w \right| \\
&\leq \|f_{\bar{k}-1}\|_\infty \left| \int_{w \in \mathbb{R}} (f(w+(z'-z)) - f(w))\, \mathrm{d}w \right| = \|f_{\bar{k}-1}\|_\infty \|\Theta_{(z'-z)} f - f\|_{L^1}\,,
\end{aligned}$$

where $(\Theta_h f)(x) := f(x-h)$ denotes the translation operator. Since this is continuous in L^1, the claim follows.

Let us now set, for $\bar{k} \in \mathbb{N}$, $\alpha' > 0$, $\Delta > 0$ and $z \in \mathbb{R}$,

$$\widetilde{C}_\Delta^{\alpha'}(z) := \sup_{u \in [z,z+\Delta)} (1+|u|)^{\alpha'}\, f_{\bar{k}}(u)\,. \tag{5.10}$$

The second condition on $\bar{k}$ is that (it is large enough so that) for some $\alpha' \in (\varrho\alpha, \alpha)$ and for some (hence any) $\Delta > 0$ the following upper Riemann sum is finite:

$$\sum_{w \in \Delta\mathbb{Z}} \Delta \cdot \widetilde{C}_\Delta^{\alpha'}(w) \;<\; \infty\,. \tag{5.11}$$

In other words, we require that the function $(1+|u|)^{\alpha'} f_{\bar{k}}(u)$ is *directly Riemann integrable*, cf. [25, §XI.1]. We point out that this condition is always satisfied if $\bar{k}$ is large enough, with no further assumptions beyond Hypothese 2.1 and 2.2, as it is proved in [12]. Of course, an immediate sufficient condition, very common in concrete applications, is that there exists $\alpha'' \in (\varrho\alpha, \alpha)$ such that $f_{\bar{k}}(x) \le (const.)/|x|^{1+\alpha''}$.

A direct consequence of (5.11) is that, for any fixed $\Delta > 0$, the contribution to the sum of the terms with $|w| > M$ is small, provided M is large. The interesting point is that M *can be chosen independently of (bounded)* Δ: more precisely,

$$\forall \Delta_0 > 0,\ \forall \eta > 0 \quad \exists M > 0 : \quad \sum_{\substack{w \in \Delta\mathbb{Z} \\ |w| > M}} \Delta \cdot \widetilde{C}^{\alpha'}_{\Delta}(w) \,<\, \eta\,, \quad \forall \Delta \in (0, \Delta_0)\,. \tag{5.12}$$

This relation can be obtained combining (5.11) for $\Delta = \Delta_0$ with the following result, which describes the *monotonicity properties of Riemann sums.*

Lemma 5.2. *Let μ be a Borel measure on $\mathbb{R}$. Given $\Delta > 0$ and a positive function $h : \mathbb{R} \to [0, +\infty)$, define the upper Riemann sum*

$$\mathcal{S}_{\mu,\Delta}(h) := \sum_{w \in \Delta\mathbb{Z}} \left(\sup_{z \in [w, w+\Delta)} h(z) \right) \cdot \mu\big([w, w+\Delta)\big)\,. \tag{5.13}$$

Then, setting $(\Theta_z h)(x) := h(x - z)$, we have

$$\mathcal{S}_{\mu,\Delta}(h) \,\le\, \mathcal{S}_{\mu,\Delta_0}(h) + \mathcal{S}_{\mu,\Delta_0}(\Theta_{-\Delta_0} h) + \mathcal{S}_{\mu,\Delta_0}(\Theta_{\Delta_0} h)\,, \qquad \forall \Delta \in (0, \Delta_0)\,. \tag{5.14}$$

In particular, when μ is the Lebesgue measure, the right hand side of this relation equals $3\,\mathcal{S}_{\mu,\Delta_0}(h)$, because the three terms coincide.

Proof. Let us denote by $\overline{h}_\Delta : \mathbb{R} \to [0, +\infty)$ the piecewise constant function that, in every interval $[w, w+\Delta)$, with $w \in \Delta\mathbb{Z}$, takes the value $\sup_{z \in [w, w+\Delta)} h(z)$. Then

$$\mathcal{S}_{\mu,\Delta}(h) = \int_{\mathbb{R}} \overline{h}_\Delta(x)\, \mu(\mathrm{d}x)\,.$$

Plainly, if $0 < \Delta < \Delta_0$, each interval of the kind $[w, w+\Delta)$ can intersect at most two intervals of the kind $[w', w'+\Delta_0)$. It follows that

$$\overline{h}_\Delta(x) \,\le\, \overline{h}_{\Delta_0}(x) \,+\, \overline{(\Theta_{\Delta_0} h)}_{\Delta_0}(x) \,+\, \overline{(\Theta_{-\Delta_0} h)}_{\Delta_0}(x)\,, \qquad \forall x \in \mathbb{R}\,.$$

Integrating this relation over μ yields the conclusion. □

5.3. **Some preliminary results.** A useful observation is that the function V^+ is increasing and sub-additive (as every renewal function), hence

$$\frac{V^+(y+\delta)}{V^+(y)} \,\le\, 1 + \frac{V^+(\delta)}{V^+(y)} \,\le\, 1 + \frac{V^+(\delta)}{V^+(0)} \,=\, 1 + V^+(\delta)\,, \qquad \forall \delta, y \ge 0\,. \tag{5.15}$$

Let us first derive an upper bound from (5.3) and (5.4).

Lemma 5.3. *Fix any sequence $x_n = o(a_n)$ and $\Delta > 0$. There exists a constant $C \in (0, \infty)$ such that*

$$F^+_n(x_n, [y, y+\Delta)) \,\le\, \frac{C\,\Delta}{n a_n}\, V^-(x_n)\, V^+(y)\,, \qquad \forall n \in \mathbb{N},\ \forall y \ge 0\,. \tag{5.16}$$

Proof. We proceed by contradiciton. If (5.16) doesn't hold, there exist $\Delta > 0$ and sequences $C_n \to +\infty$, $y_n \ge 0$ such that along a subsequence $n = n_k \to +\infty$ we have

$$F_n^+(x_n, [y_n, y_n + \Delta)) \,>\, \frac{C_n\,\Delta}{n a_n}\, V^-(x_n)\, V^+(y_n)\,. \tag{5.17}$$

For ease of notation, we implicitly assume that $n = n_k$ until the end of the proof. Extracting a suitable subsequence, we may assume that $y_n/a_n \to c \in [0, +\infty]$ and we show that in each case $c = 0$, $c \in (0, \infty)$ and $c = +\infty$ we obtain a contradiction.

- If $c = 0$ then $y_n = o(a_n)$ and (5.17) contradicts (5.4), because, by (5.15),
$$\int_{[y_n, y_n+\Delta)} V^+(z)\,\mathrm{d}z \,\le\, \Delta\, V^+(y_n + \Delta) \,\le\, (const.)\, V^+(y_n)\,.$$
- If $0 < c < \infty$ then $y_n \sim c\, a_n$ and $V^+(y_n) \sim c^{\alpha\varrho} V^+(a_n)$, because $V^+ \in R_{\alpha\varrho}$. By (3.18) we know that $V^+(a_n) \sim (const.)\, n\mathbb{P}(\tau_1^- > n)$, hence from (5.17) we get
$$F_n^+(x_n, [y_n, y_n + \Delta)) \,>\, (const.')\frac{C_n\,\Delta}{a_n}\, V^-(x_n)\, \mathbb{P}(\tau_1^- > n)\,. \tag{5.18}$$
This is in contradiction with (5.3), because g^+ is bounded.
- The case $c = \infty$ is analogous and even simpler. In fact in this case $V^+(y_n) \ge V^+(a_n)$ for large n (recall that V^+ is increasing) hence we still have (5.18), which is again in contradiction with (5.3) (which holds also for $y_n \gg a_n$).

This completes the proof. □

Remark 5.4. It is worth stressing that, in general, estimate (5.16) *it is not uniform in* Δ (i.e., the constant C depends on Δ). In fact, being a direct consequence of (5.3) and (5.4), relation (5.16) holds for general non-lattice random walks (satisfying Hypothesis 2.1).

In the absolutely continuous case, under Hypothesis 2.2, it will follow *a posteriori*, by our Theorem 5.1, that relation (5.16) does hold uniformly in Δ. However, since this is not granted *a priori*, throughout the proof we need to work with a *fixed* Δ.

We now prove a crucial approximation result: we show that the sum in (5.9) can be truncated to values of z at a finite distance from y, losing a negligible contribution.

Lemma 5.5. *Fix any sequence* $x_n = o(a_n)$ *and* $\Delta_0 > 0$. *For every* $\eta > 0$ *there exists* $M \in (0, \infty)$ *such that for all* $n > \overline{k}$, $\Delta \in (0, \Delta_0)$ *and* $y \ge 0$

$$\sum_{\substack{z \in \Delta\mathbb{N}_0 \\ |z-y| > M}} F_{n-\bar{k}}^+(x_n, [z, z+\Delta))\, \overline{f}_\Delta^+(z, y) \,\le\, \eta\left(\frac{g(0)}{n a_n} V^-(x_n) V^+(y)\right). \tag{5.19}$$

Proof. The left hand side of (5.19) is an upper Riemann sum $\mathcal{S}_{\mu,\Delta}(h)$ like in (5.13), with $\mu([a,b)) = F_{n-\bar{k}}^+(x_n, [a,b))$ and $h(z) = f_{\bar{k}}^+(z,y) 1_{\{|z-y|>M\}}$ (recall (5.6)). Therefore, applying relation (5.14), it suffices to prove (5.19) *for a fixed* $\Delta = \Delta_0 > 0$ and with $\overline{f}_{\Delta_0}^+(z,y)$ replaced by $\overline{f}_{\Delta_0}^+(z + a\Delta_0, y)$, with $a \in \{-1, 0, +1\}$. For simplicity, in the sequel we only focus on the case $a = 0$ (the case $a = \pm 1$ is almost identical).

We are left with proving (5.19) for fixed $\Delta > 0$. For any $M > 0$, by (5.16) we can write

$$\sum_{\substack{z \in \Delta\mathbb{N}_0 \\ |z-y| > M}} F_{n-\bar{k}}^+(x_n, [z, z+\Delta))\, \overline{f}_\Delta^+(z, y) \,\le\, \frac{C}{n a_n}\, V^-(x_n) \sum_{\substack{z \in \Delta\mathbb{N}_0 \\ |z-y| > M}} \Delta \cdot V^+(z)\, \overline{f}_\Delta^+(z, y)\,.$$

Since V^+ is sub-additive and increasing, for $z > y$ we have $V^+(z) \le V^+(y) + V^+(z-y)$, while for $z < y$ we have $V^+(z) \le V^+(y)$. Therefore we can bound

$$\begin{aligned} &\sum_{\substack{z\in\Delta\mathbb{N}_0\\|z-y|>M}} \Delta\cdot V^+(z)\,\overline{f}^+_\Delta(z,y)\\ &\qquad \le V^+(y)\sum_{\substack{z\in\Delta\mathbb{N}_0\\z<y-M}} \Delta\cdot\overline{f}^+_\Delta(z,y) + \sum_{\substack{z\in\Delta\mathbb{N}_0\\z>y+M}} \big(V^+(y)+V^+(z-y)\big)\Delta\cdot\overline{f}^+_\Delta(z,y)\\ &\qquad = V^+(y)\sum_{\substack{z\in\Delta\mathbb{N}_0\\|z-y|>M}} \Delta\cdot\overline{f}^+_\Delta(z,y) + \sum_{\substack{z\in\Delta\mathbb{N}_0\\z>y+M}} V^+(z-y)\,\Delta\cdot\overline{f}^+_\Delta(z,y)\,. \end{aligned} \tag{5.20}$$

Let us bound the second sum. Recalling (5.6) and observe that $f^+_{\bar k}(u,y) \le f_{\bar k}(y-u)$ for all $u, y \ge 0$; moreover, for any fixed $\alpha' \in (\varrho\alpha, \alpha)$ one has $V^+(w) \le |w|^{\alpha'}$ for large w, because $V^+ \in R_{\alpha\varrho}$. Recalling (5.10), it follows that $V^+(z-y)\,\overline{f}^+_\Delta(z,y) \le (const.)\,\widetilde{C}^{\alpha'}_\Delta(y-z)$. Coming back to (5.20), we use the bound $\overline{f}^+_\Delta(z,y) \le \widetilde{C}^{\alpha'}_\Delta(y-z)$ in the first sum of the last line. Since $1 = V^+(0) \le V^+(y)$, it follows that for some constant $c > 0$

$$\sum_{\substack{z\in\Delta\mathbb{N}_0\\|z-y|>M}} \Delta\cdot V^+(z)\,\overline{f}^+_\Delta(z,y) \;\le\; c\,V^+(y)\sum_{\substack{w\in\Delta\mathbb{Z}\\|w|>M}} \Delta\cdot\widetilde{C}^{\alpha'}_\Delta(w)\,.$$

We now apply (5.12) with η replaced by $\eta g(0)/(cC)$, getting

$$\sum_{\substack{z\in\Delta\mathbb{N}_0\\|z-y|>M}} F^+_{n-\bar k}(x_n,[z,z+\Delta))\,\overline{f}^+_\Delta(z,y) \;\le\; \eta\,\frac{g(0)}{na_n}\,V^-(x_n)V^+(y+\Delta)\,,$$

which is precisely what we want to prove (recall (5.15)). □

5.4. **Proof of (5.2).** We can finally prove (5.2). From the discussion of section 3.1, it suffices to show that, if we *fix* any two sequences $x_n = o(a_n)$ and $y_n = o(a_n)$, for every $\varepsilon > 0$ we have, for large n,

$$(1-\varepsilon)V^+(y_n) \;\le\; \frac{f^+_n(x_n,y_n)}{\frac{g(0)}{na_n}V^-(x_n)} \;\le\; (1+\varepsilon)V^+(y_n)\,. \tag{5.21}$$

Let $\varepsilon \in (0,1)$ be fixed. We start choosing $\Delta_0 := 1$ and we let M_0 denote the constant M in Lemma 5.5 corresponding to $\eta = \varepsilon/4$. The reason for this will be clear later.

Observe that, by time-reversal, relation (3.14) for $N = \bar k$ can be rewritten as

$$\int_{[0,\infty)} V^+(u)\,f^+_{\bar k}(u,y)\,\mathrm{d}u \;=\; V^+(y)\,, \qquad \forall y \ge 0\,. \tag{5.22}$$

For notational convenience, let us set $V^+(u) := 0$ for $u < 0$, so that the domain of integration can be extended to $\mathbb{R}$. We claim that there exists $\bar M > 0$ such that

$$\int_{|u-y|>\bar M-1} V^+(u)\,f^+_{\bar k}(u,y)\,\mathrm{d}u \;\le\; \frac{\varepsilon}{4}V^+(y)\,, \qquad \forall y \ge 0\,. \tag{5.23}$$

In fact, observe that $f^+_n(x,y) \le f_n(y-x)$ and that for every $y \ge 0$ and $s \in \mathbb{R}$

$$V^+(y-s) \;\le\; V^+(y) + V^+(|s|) \;\le\; 2\,V^+(y)V^+(|s|)\,,$$

by sub-additivity of $V^+(\cdot)$ and the fact that $1 = V^+(0) \le V^+(x)$ for every $x \ge 0$, hence

$$\begin{aligned}\int_{|u-y|>M-1} V^+(u)\, f^+_{\bar k}(u,y)\,\mathrm{d}u \;&\le\; \int_{|u-y|>M-1} V^+(u)\, f_{\bar k}(y-u)\,\mathrm{d}u \\ &= \mathbb{E}(V^+(y-S_{\bar k})1_{\{|S_{\bar k}|>M-1\}}) \;\le\; 2\,V^+(y)\,\mathbb{E}(V^+(|S_{\bar k}|)\,1_{\{|S_{\bar k}|>M-1\}})\,.\end{aligned}$$

It is well-known that Hypothesis 2.1 entails that $\mathbb{E}(|X_1|^{\alpha'}) < \infty$, for all $\alpha' < \alpha$, and consequently $\mathbb{E}(|S_{\bar k}|^{\alpha'}) < \infty$. Since $V^+(\cdot) \in R_{\alpha\varrho}$ by (3.16) and since $\varrho < 1$, it follows that $\mathbb{E}(V^+(|S_{\bar k}|)) < \infty$. We can then choose $\bar M > 0$ large enough, with $\bar M > M_0$ (that was fixed above), so that $\mathbb{E}(V^+(|S_{\bar k}|)\,1_{\{|S_{\bar k}|>\bar M-1\}}) \le \varepsilon/8$. Relation (5.23) is proved.

It follows immediately from (5.22), (5.23) that for every $0 < \Delta \le \Delta_0$ and $y \ge 0$ one has

$$\left| \sum_{\substack{z\in\Delta\mathbb{N}_0\\ |z-y|\le\bar M}} \left(\int_{[z,z+\Delta)} V^+(u)\, f^+_{\bar k}(u,y)\,\mathrm{d}u \right) \;-\; V^+(y) \right| \;\le\; \frac{\varepsilon}{4}\, V^+(y)\,, \tag{5.24}$$

just because the intervals $[z, z+\Delta)$, as z varies in $\Delta\mathbb{N}_0$ with $|z-y| \le \bar M$, are disjoint and their union contains $[y-(\bar M-1), y+(\bar M-1))$.

Next observe that, for any $\Delta \le \Delta_0 = 1$ and $y \ge 0$, by (5.15)

$$\begin{aligned}\sum_{\substack{z\in\Delta\mathbb{N}_0\\ |z-y|\le\bar M}} \int_{[z,z+\Delta)} V^+(u)\,\mathrm{d}u \;&\le\; \int_{y-\bar M}^{y+\bar M+\Delta} V^+(u)\,\mathrm{d}u \;\le\; (2\bar M+\Delta)\, V^+(y+\bar M+\Delta) \\ &\le\; (2\bar M+1)\,(1+V^+(\bar M+1))\,V^+(y) \;=:\; \bar C\, V^+(y)\,,\end{aligned}$$

where we stress that $\bar C > 0$ is a constant depending only on ε (through $\bar M$). Recall the definition (5.6). Since we have chosen $\bar k$ large enough so that $f^+_{\bar k}(u,y)$ is uniformly continuous in u, uniformly in y, we can choose $0 < \bar\Delta < \Delta_0$ small enough so that

$$\forall z,y \ge 0,\ \forall u \in [z, z+\bar\Delta): \qquad \overline{f}^+_{\bar\Delta}(z,y) - \frac{\varepsilon}{4\bar C} \;\le\; f^+_{\bar k}(u,y) \;\le\; \underline{f}^+_{\bar\Delta}(z,y) + \frac{\varepsilon}{4\bar C}\,.$$

Inserting these estimates in (5.24), it follows that, for every $y \ge 0$,

$$\sum_{\substack{z\in\bar\Delta\mathbb{N}_0\\ |z-y|\le\bar M}} \left(\int_{[z,z+\bar\Delta)} V^+(u)\,\mathrm{d}u \right) \overline{f}^+_{\bar\Delta}(z,y) \;\le\; \left(1+\frac{\varepsilon}{2}\right) V^+(y) \tag{5.25}$$

$$\sum_{\substack{z\in\bar\Delta\mathbb{N}_0\\ |z-y|\le\bar M}} \left(\int_{[z,z+\bar\Delta)} V^+(u)\,\mathrm{d}u \right) \underline{f}^+_{\bar\Delta}(z,y) \;\ge\; \left(1-\frac{\varepsilon}{2}\right) V^+(y)\,. \tag{5.26}$$

Fix two sequences $x_n, y_n = o(a_n)$ and consider the previous relations with $y = y_n$. Note that the sum over z ranges over a finite number of points, all at finite distance from y_n, hence each z in the sum is $o(a_n)$. Then it follows from (5.4) that there exists $n_0 = n_0(\varepsilon) < \infty$ such that for all $n \ge n_0$ and for all $z \in \bar\Delta\mathbb{N}_0$ with $|z - y_n| \le \bar M$

$$\frac{1+\frac{\varepsilon}{2}}{1+\frac{3\varepsilon}{4}}\, \frac{F^+_{n-\bar k}(x_n, [z, z+\bar\Delta))}{\frac{g(0)}{na_n} V^-(x_n)} \;\le\; \int_{[z,z+\bar\Delta)} V^+(u)\,\mathrm{d}u \;\le\; \frac{1-\frac{\varepsilon}{2}}{1-\varepsilon}\, \frac{F^+_{n-\bar k}(x_n, [z, z+\bar\Delta))}{\frac{g(0)}{na_n} V^-(x_n)}\,.$$

Relations (5.25), (5.26) then yield

$$\sum_{\substack{z\in\bar\Delta\mathbb{N}_0\\ |z-y_n|\le\bar M}} \frac{F^+_{n-\bar k}(x_n,[z,z+\bar\Delta))}{\frac{g(0)}{na_n}V^-(x_n)}\,\overline{f}^+_{\bar\Delta}(z,y_n) \,\le\, \left(1+\frac{3\varepsilon}{4}\right)V^+(y_n) \tag{5.27}$$

$$\sum_{\substack{z\in\bar\Delta\mathbb{N}_0\\ |z-y_n|\le\bar M}} \frac{F^+_{n-\bar k}(x_n,[z,z+\bar\Delta))}{\frac{g(0)}{na_n}V^-(x_n)}\,\underline{f}^+_{\bar\Delta}(z,y_n) \,\ge\, (1-\varepsilon)V^+(y_n)\,. \tag{5.28}$$

Recall that our choice of $\bar M \ge M_0$ was such that Lemma 5.5 holds for $\eta = \varepsilon/4$. Therefore we can drop the restriction $|z-y_n|\le \bar M$ in (5.27), provided we replace $\frac{3\varepsilon}{4}$ by ε in the right hand side. Plainly, the restriction $|z-y_n|\le\bar M$ can be dropped from the sum in (5.28) with no further modification. Looking back at (5.7)–(5.9), it follows that (5.21) holds true for $n\ge n_0$. This completes the proof of relation (5.2).

5.5. **Proof of** (5.1)**.** The proof of (5.1) is close in spirit to that of (5.2) just given. It suffices to show that, if we *fix* any two sequences $x_n = o(a_n)$ and $y_n\ge 0$, for every $\varepsilon>0$ we have, for large n,

$$g^+\!\left(\frac{y_n}{a_n}\right) - \varepsilon \,\le\, \frac{f^+_n(x_n,y_n)}{\frac{1}{a_n}\mathbb{P}(\tau_1^- > n)V^-(x_n)} \,\le\, g^+\!\left(\frac{y_n}{a_n}\right) + \varepsilon\,. \tag{5.29}$$

As we remarked in §3.1, it suffices to consider sequences that have a (possibly infinite) limit, so we assume that $y_n/a_n \to \kappa\in[0,+\infty]$. The case $\kappa=0$, i.e. $y_n=o(a_n)$, is a consequence of relation (5.2), which is a stronger statement, so there is nothing to prove. We then treat separately the cases $\kappa\in(0,\infty)$ and $\kappa=\infty$, starting from the former.

Let $\varepsilon\in(0,1)$ be fixed. We choose $\Delta_0 := 1$ and we let M_0 denote the constant M in Lemma 5.5 corresponding to $\eta = \varepsilon/(8g(0)\mathtt{C}^+\kappa^{\alpha\varrho})$ (the reason for this choice will be clear later), where we recall that $\mathtt{C}^+$ is the constant appearing in (3.18).

Recall that $y_n\sim\kappa a_n$ with $\kappa\in(0,\infty)$ and note that $g^+(\kappa)>0$. We claim that we can choose $n_0\in\mathbb{N}_0$ and $\bar M>0$ such that

$$1-\frac{\varepsilon}{4\,g^+(\kappa)} \,\le\, \int_{|u-y_n|\le\bar M-1} f^+_{\bar k}(u,y_n)\,\mathrm{d}u \,\le\, 1\,, \qquad \forall n\ge n_0\,. \tag{5.30}$$

In fact, if we denote by $S^* := -S$ the reflected walk, if $y_n\ge M-1$ we can write

$$\begin{aligned}&\int_{|u-y_n|\le M-1} f^+_{\bar k}(u,y_n)\,\mathrm{d}u\\ &\qquad = \mathbb{P}_{y_n}(S^*_1\ge 0,\ldots,S^*_{\bar k-1}\ge 0, S^*_{\bar k}\in[y_n-(M-1),y_n+(M-1)])\\ &\qquad = \mathbb{P}_0(S^*_1\ge -y_n,\ldots,S^*_{\bar k-1}\ge -y_n, S^*_{\bar k}\in[-(M-1),+(M-1)])\,,\end{aligned}$$

from which the upper bound in (5.30) follows trivially. For the lower bound, note that

$$\begin{aligned}&1-\int_{|u-y_n|\le M-1} f^+_{\bar k}(u,y_n)\,\mathrm{d}u\\ &\qquad \le \mathbb{P}(\{S^*_1\ge -y_n,\ldots,S^*_{\bar k-1}\ge -y_n\}^c) + \mathbb{P}(S^*_{\bar k}\notin[-(M-1),+(M-1)])\,.\end{aligned}$$

Since $\bar k$ is fixed and $y_n\to+\infty$, the first term in the right hand side vanishes as $n\to\infty$, hence we can choose n_0 such that for $n\ge n_0$ it is less than $\varepsilon/(8\,g^+(\kappa))$. Analogously, we choose $\bar M$ large enough, with $\bar M > M_0$ (that was fixed above), such that for $M\ge\bar M$ the second term in the right hand side is less than $\varepsilon/(8\,g^+(\kappa))$. Equation (5.30) is proved.

It follows immediately from (5.30) that for every $0 < \Delta \le \Delta_0$ and $n \ge n_0$ one has

$$1 - \frac{\varepsilon}{4\, g^+(\kappa)} \;\le\; \sum_{\substack{z \in \Delta\mathbb{N}_0 \\ |z-y_n| \le \bar{M}}} \bigg(\int_{[z,z+\Delta)} f^+_{\bar{k}}(u, y_n)\, \mathrm{d}u \bigg) \;\le\; 1\,, \tag{5.31}$$

just because the intervals $[z, z+\Delta)$, as z varies in $\Delta\mathbb{N}_0$ with $|z - y_n| \le \bar{M}$, are disjoint and their union contains $[y_n - (\bar{M} - 1), y_n + (\bar{M} - 1))$.

Recall the definition (5.6). Since we have chosen $\bar{k}$ large enough so that $f^+_{\bar{k}}(u, y)$ is uniformly continuous in u, uniformly in y, we can choose $0 < \bar{\Delta} < \Delta_0$ small enough so that for all $n \ge n_0$, $z \ge 0$ and for every $u \in [z, z + \bar{\Delta})$

$$\overline{f}^+_{\bar{\Delta}}(z, y_n) - \frac{\varepsilon}{4\,(2\bar{M}+1)\, g^+(\kappa)} \;\le\; f^+_{\bar{k}}(u, y_n) \;\le\; \underline{f}^+_{\bar{\Delta}}(z, y_n) + \frac{\varepsilon}{4\,(2\bar{M}+1)\, g^+(\kappa)}\,.$$

Plugging these estimates into (5.31) and observing that there are at most $(2\bar{M} + 1)/\bar{\Delta}$ values of $z \in \bar{\Delta}\mathbb{N}_0$ such that $|z - y_n| \le \bar{M}$, we obtain

$$\sum_{\substack{z \in \bar{\Delta}\mathbb{N}_0 \\ |z-y_n| \le \bar{M}}} \Delta\, \overline{f}^+_{\bar{\Delta}}(z, y_n) \;\le\; 1 + \frac{\varepsilon}{4\, g^+(\kappa)}\,, \qquad \sum_{\substack{z \in \bar{\Delta}\mathbb{N}_0 \\ |z-y_n| \le \bar{M}}} \Delta\, \underline{f}^+_{\bar{\Delta}}(z, y_n) \;\ge\; 1 - \frac{\varepsilon}{2\, g^+(\kappa)}\,. \tag{5.32}$$

Observe that the sum in both the preceding relations ranges over a finite number of z, all at finite distance from y_n, hence each z in the sum is such that $z/a_n \to \kappa \in (0, \infty)$. Therefore $g^+(z/a_n) \to g(\kappa)$ as $n \to \infty$, uniformly over z in the sum range. It follows that there exists $n_1 \ge n_0$ such that for all $n \ge n_1$

$$\sum_{\substack{z \in \bar{\Delta}\mathbb{N}_0 \\ |z-y_n| \le \bar{M}}} g^+\Big(\frac{z}{a_n}\Big)\, \Delta\, \overline{f}^+_{\bar{\Delta}}(z, y_n) \;\le\; g^+(\kappa) + \frac{\varepsilon}{2}\,, \tag{5.33}$$

$$\sum_{\substack{z \in \bar{\Delta}\mathbb{N}_0 \\ |z-y_n| \le \bar{M}}} g^+\Big(\frac{z}{a_n}\Big)\, \Delta\, \underline{f}^+_{\bar{\Delta}}(z, y_n) \;\ge\; g^+(\kappa) - \frac{3\varepsilon}{4}\,. \tag{5.34}$$

Next observe that, as $n \to \infty$, we have $g^+(y_n/a_n) \to g^+(\kappa)$, and by (5.3)

$$\left| g^+\Big(\frac{z}{a_n}\Big)\, \Delta \;-\; \frac{F^+_{n-\bar{k}}(x_n, [z, z+\bar{\Delta}))}{\frac{1}{a_n}\mathbb{P}(\tau_1^- > n) V^-(x_n)} \right| \longrightarrow 0\,,$$

uniformly over z in the sum range of (5.33) and (5.34). It follows that there exists $n_2 \ge n_1$ such that for all $n \ge n_2$

$$\sum_{\substack{z \in \bar{\Delta}\mathbb{N}_0 \\ |z-y_n| \le \bar{M}}} \frac{F^+_{n-\bar{k}}(x_n, [z, z+\bar{\Delta}))}{\frac{1}{a_n}\mathbb{P}(\tau_1^- > n) V^-(x_n)}\, \overline{f}^+_{\bar{\Delta}}(z, y_n) \;\le\; g^+\Big(\frac{y_n}{a_n}\Big) + \frac{3\varepsilon}{4}\,, \tag{5.35}$$

$$\sum_{\substack{z \in \bar{\Delta}\mathbb{N}_0 \\ |z-y_n| \le \bar{M}}} \frac{F^+_{n-\bar{k}}(x_n, [z, z+\bar{\Delta}))}{\frac{1}{a_n}\mathbb{P}(\tau_1^- > n) V^-(x_n)}\, \underline{f}^+_{\bar{\Delta}}(z, y_n) \;\ge\; g^+\Big(\frac{y_n}{a_n}\Big) - \varepsilon\,. \tag{5.36}$$

Dropping the restriction $|z - y_n| \le \bar{M}$ in the sum in (5.36) and recalling (5.7)–(5.9), it follows that the lower bound in (5.29) holds true for $n \ge n_2$.

In order to drop the restriction $|z - y_n| \le \bar{M}$ in the sum in (5.35) as well, we need to control the contribution of the terms with $|z - y_n| > \bar{M}$. Recall that $\bar{M}$ was chosen greater than M_0, in such a way that Lemma 5.5 holds with $\eta = \varepsilon/(8g(0)\mathtt{C}^+\kappa^{\alpha\varrho})$. Since $y_n \sim \kappa a_n$ and $V^+ \in R_{\alpha\varrho}$ by (3.16), it follows that $V^+(y_n) \sim \kappa^{\alpha\varrho} V^+(a_n) \sim \mathtt{C}^+\kappa^{\alpha\varrho} n\mathbb{P}(\tau_1^- > n)$, having applied (3.18). Therefore there exists $n_3 \geq n_2$ such that for $n \geq n_3$ one has $V^+(y_n) \leq 2\mathtt{C}^+\kappa^{\alpha\varrho} n\mathbb{P}(\tau_1^- > n)$, hence by Lemma 5.5

$$\sum_{\substack{z\in\bar\Delta\mathbb{N}_0\\ |z-y_n|>\bar M}} \frac{F^+_{n-\bar k}(x_n,[z,z+\bar\Delta))}{\frac{1}{a_n}\mathbb{P}(\tau_1^- > n)V^-(x_n)}\, \overline{f}^+_{\bar\Delta}(z,y_n) \,\leq\, \frac{\varepsilon}{4}\,.$$

This means that we can drop the restriction $|z-y_n| \leq \bar M$ in (5.35), provided we replace $\frac{3\varepsilon}{4}$ by ε in the right hand side. Recalling (5.7)–(5.9), we have proved that the upper bound in (5.29) holds true for $n \geq n_3$.

Finally, it remains to prove (5.29) in the case when $\kappa = \lim_{n\to\infty} y_n/a_n = +\infty$. Since $g^+(x) \to 0$ as $x \to +\infty$, it suffices to show that for every $\varepsilon > 0$, for n large,

$$\frac{f_n^+(x_n,y_n)}{\frac{1}{a_n}\mathbb{P}(\tau_1^- > n)V^-(x_n)} \,\leq\, \varepsilon\,, \tag{5.37}$$

where we recall that $x_n = o(a_n)$ is a fixed sequence. We fix an arbitrary Δ (say $\Delta = 1$) and note that, by the upper bound in (5.7), we can write

$$\frac{f_n^+(x_n,y_n)}{\frac{1}{a_n}\mathbb{P}(\tau_1^- > n)V^-(x_n)} \,\leq\, \sum_{z\in\Delta\mathbb{N}_0} \frac{F^+_{n-\bar k}(x_n,[z,z+1))}{\frac{1}{a_n}\mathbb{P}(\tau_1^- > n)V^-(x_n)}\, \overline{f}^+_{\Delta}(z,y_n)\,. \tag{5.38}$$

Since the function $g^+(\cdot)$ is bounded, by (5.3)

$$\hat c \,:=\, \sup_{z\in[0,\infty),\, n>\bar k} \frac{F^+_{n-\bar k}(x_n,[z,z+\Delta))}{\frac{1}{a_n}\mathbb{P}(\tau_1^- > n)V^-(x_n)\Delta} \,<\, \infty\,.$$

Observing that $f_n^+(x,y) \leq f_n(y-x)$ and recalling (5.6), we can write

$$\begin{aligned}\sum_{\substack{z\in\Delta\mathbb{N}_0\\ |z-y_n|>M}} \frac{F^+_{n-\bar k}(x_n,[z,z+1))}{\frac{1}{a_n}\mathbb{P}(\tau_1^- > n)V^-(x_n)}\, \overline{f}^+_{\Delta}(z,y_n) \,&\leq\, \hat c \sum_{\substack{z\in\Delta\mathbb{Z}\\ |z-y_n|>M}} \Delta \sup_{u\in[y_n-z,y_n-z+\Delta)} f_{\bar k}(u)\\ &= \hat c \sum_{\substack{w\in\Delta\mathbb{Z}+y_n\\ |w|>M}} \Delta \sup_{u\in[w,w+\Delta)} f_{\bar k}(u) \,\leq\, 2\hat c \sum_{\substack{w\in\Delta\mathbb{Z}\\ |w|>M}} \Delta \sup_{u\in[w,w+\Delta)} f_{\bar k}(u)\,,\end{aligned}$$

where the factor 2 in the last inequality is due to the lattice shift, from $\Delta\mathbb{Z} + y_n$ to $\Delta\mathbb{Z}$. Recalling (5.10) and (5.11), it follows that the last sum is convergent, hence we can choose M large enough so that it is less than $\varepsilon/2$. Let us now focus on the contribution to (5.38) of the terms with $|z-y_n| \leq M$. Note that there are only a finite number of such terms. Since each z with $|z-y_n| \leq M$ is such that $z/a_n \to +\infty$, it follows by (5.3) that

$$\lim_{n\to\infty}\left(\sup_{z\in\Delta\mathbb{N}_0,\, |z-y_n|\leq M} \frac{F^+_{n-\bar k}(x_n,[z,z+1))}{\frac{1}{a_n}\mathbb{P}(\tau_1^- > n)V^-(x_n)}\right) = 0\,.$$

By construction the function $f_{\bar k}$ is bounded, hence there exists n_4 such that for $n \geq n_4$

$$\sum_{\substack{z\in\Delta\mathbb{N}_0\\ |z-y_n|\leq M}} \frac{F^+_{n-\bar k}(x_n,[z,z+1))}{\frac{1}{a_n}\mathbb{P}(\tau_1^- > n)V^-(x_n)}\, \overline{f}^+_{\Delta}(z,y_n) \,\leq\, \frac{\varepsilon}{2}\,.$$

Recalling (5.38), it follows that (5.37) holds true for $n \geq n_4$, completing the proof.

6. Proof of Theorem 2.4

This section is devoted to the proof of the invariance principle in Theorem 2.4. We recall that, by Hypothesis 2.1, $(S = \{S_n\}_{n\geq 0}, \mathbb{P})$ is a random walk in the domain of attraction of a Lévy process $(X = \{X_t\}_{t\geq 0}, \mathbf{P})$ with index $\alpha \in (0,2]$ and positivity parameter $\varrho \in (0,1)$. We denote by $(a_n)_{n\in\mathbb{N}} \in R_{1/\alpha}$ the norming sequence, so that $S_n/a_n \Rightarrow X_1$.

6.1. **Random walks conditioned to stay positive.** We remind that for convenience we assume that $\mathbb{P}$ is a law on the space $\Omega^{RW} := \mathbb{R}^{\mathbb{N}_0}$, $S = \{S_n\}_{n\in\mathbb{N}}$ is the coordinate process on this space and $\mathbb{P}_x$ the law on Ω_{RW} of $S + x$ under $\mathbb{P}$, for all $x \in \mathbb{R}$. We also set $\Omega^{RW}_N := \mathbb{R}^{\{0,\dots,N\}}$ for $N \in \mathbb{N}_0$.

Let us recall the definitions of the bridges of length N of the random walk $\mathbb{P}$ from x to y conditioned to stay non-negative or strictly positive, cf. (2.2), (2.3): these are the laws $\mathbb{P}^{\uparrow,N}_{x,y}$ and $\widehat{\mathbb{P}}^{\uparrow,N}_{x,y}$ on Ω^{RW}_N, defined for $x, y \geq 0$ and $N \in \mathbb{N}$ by

$$\mathbb{P}^{\uparrow,N}_{x,y}(\,\cdot\,) := \mathbb{P}_x(\,\cdot\,|S_1 \geq 0, \dots, S_{N-1} \geq 0, S_N = y)\,, \tag{6.1}$$

$$\widehat{\mathbb{P}}^{\uparrow,N}_{x,y}(\,\cdot\,) := \mathbb{P}_x(\,\cdot\,|S_1 > 0, \dots, S_{N-1} > 0, S_N = y)\,. \tag{6.2}$$

Other basic laws on Ω_{RW} are $\mathbb{P}^{\uparrow}_x$ and $\widehat{\mathbb{P}}^{\uparrow}_x$, the laws of the random walk $\mathbb{P}$ started at x and conditioned to stay non-negative or strictly positive for all time (cf. [6, 13]): these are defined for $x \geq 0$ by setting, for all $N \in \mathbb{N}$ and $B \in \sigma(S_0, \dots, S_N)$,

$$\mathbb{P}^{\uparrow}_x(B) := \frac{1}{V^-(x)} \mathbb{E}_x(1_B\, V^-(S_N)\, 1_{\{S_1 \geq 0, \dots, S_N \geq 0\}})\,, \tag{6.3}$$

$$\widehat{\mathbb{P}}^{\uparrow}_x(B) := \frac{1}{\underline{V}^-(x)} \mathbb{E}_x(1_B\, \underline{V}^-(S_N)\, 1_{\{S_1 > 0, \dots, S_N > 0\}})\,, \tag{6.4}$$

where the renewal functions $V^-(\cdot)$ and $\underline{V}^-(\cdot)$ have been introduced in §3.2. Lemma 3.1 ensures that the laws $\mathbb{P}^{\uparrow}_x$ and $\widehat{\mathbb{P}}^{\uparrow}_x$ are well-defined. As we already observed, the laws $\mathbb{P}^{\uparrow,N}_{x,y}$ and $\widehat{\mathbb{P}}^{\uparrow,N}_{x,y}$ may be viewed as bridges of $\mathbb{P}^{\uparrow}_x$ and $\widehat{\mathbb{P}}^{\uparrow}_x$, respectively:

$$\mathbb{P}^{\uparrow,N}_{x,y}(\,\cdot\,) = \mathbb{P}^{\uparrow}_x(\,\cdot\,|S_N = y)\,, \qquad \widehat{\mathbb{P}}^{\uparrow,N}_{x,y}(\,\cdot\,) = \widehat{\mathbb{P}}^{\uparrow}_x(\,\cdot\,|S_N = y)\,. \tag{6.5}$$

The following lemma is a direct consequence of the time reversal property of random walks. Let $\widetilde{\mathbb{P}}^{\uparrow,N}_{x,y}$ be the law of the bridge of the reflected walk $S^* := -S$ conditioned to stay non-negative, as it is defined in (6.1) for S.

Lemma 6.1. *For all $x, y \geq 0$, and for all $N \geq 1$, under the law $\mathbb{P}^{\uparrow,N}_{x,y}$, the process $(S_N - S_{N-M},\, 0 \leq M \leq N)$ has law $\widetilde{\mathbb{P}}^{\uparrow,N}_{y,x}$.*

6.2. **Lévy processes conditioned to stay positive.** We recall that $\Omega := D([0,\infty), \mathbb{R})$ is the space of real-valued càdlàg paths which are defined on $[0,\infty)$, $X = \{X_t\}_{t\geq 0}$ is the corresponding coordinate process and $\mathbf{P}$ is the law on Ω under which X is the stable Lévy process appearing in Hypothesis 2.1. We denote by $\mathbf{P}_a$ the law on Ω of $X + a$ under $\mathbf{P}$, for all $a \in \mathbb{R}$. We also denote by $\Omega_t := D([0,t], \mathbb{R})$ for $t \geq 0$ the space of paths of length t.

In analogy to the discrete case, we can define the law of the Lévy process started at $a > 0$ and conditioned to stay positive for all time [16, 13] to be the law $\mathbf{P}^{\uparrow}_a$ on Ω such that, for all $t > 0$ and $B \in \sigma(X_s, 0 \leq s \leq t)$,

$$\mathbf{P}^{\uparrow}_a(B) := \frac{1}{U^-(a)} \mathbf{E}_a(1_B\, U^-(X_t)\, 1_{\{X_s \geq 0,\, \forall 0 \leq s \leq t\}})\,, \tag{6.6}$$

where $U^-(\cdot)$ is the renewal function associated to the descending ladder height process of $(X, \mathbf{P})$. Since X is stable we have $U^-(x) = x^{\alpha(1-\varrho)}$. Note that for Lévy processes there is no distinction between staying non-negative and strictly positive. Although (6.6) does not make sense when $a = 0$, because $U^-(0) = 0$, the law $\mathbf{P}_0^{\uparrow}$ can still be defined, see [16], and we have

$$\mathbf{P}_a^{\uparrow} \Rightarrow \mathbf{P}_0^{\uparrow}, \quad \text{as } a \downarrow 0, \tag{6.7}$$

where $\Rightarrow$ denotes weak convergence on Ω.

Then we will define the law $\mathbf{P}_{a,b}^{\uparrow,t}$, on Ω_t of the bridge of the Lévy process $(X, \mathbf{P})$, with length $t > 0$, between $a \geq 0$ and $b \geq 0$, conditioned to stay positive. The following definitions and results are mainly excerpt from [41] to which we refer for details. Set $\widetilde{X} := -X$ and denote by $\widetilde{U}^-(x) = x^{\alpha\varrho}$ the renewal function associated to the descending ladder heights process of $(\widetilde{X}, \mathbf{P})$. Define the measure

$$\lambda^{\uparrow}(\mathrm{d}z) = U^-(z)\widetilde{U}^-(z)\,\mathrm{d}z = z^{\alpha}\,\mathrm{d}z\,, \tag{6.8}$$

on $[0, \infty)$ and let $g_t^{\uparrow}(a, b)$ be the unique version of the semigroup density of $\mathbf{P}^{\uparrow}$ with respect to the measure $\lambda^{\uparrow}(\mathrm{d}z)$, i.e.

$$g_t^{\uparrow}(a, b)\, \lambda^{\uparrow}(\mathrm{d}b) := \mathbf{P}_a^{\uparrow}(X_t \in \mathrm{d}b)\,,$$

which satisfies the Chapman-Kolmogorov equation:

$$g_{s+t}^{\uparrow}(a, b) = \int_0^{\infty} g_s^{\uparrow}(a, z) g_t^{\uparrow}(z, b)\, \lambda^{\uparrow}(\mathrm{d}z)\,, \quad \text{for all } s, t > 0 \text{ and } a \geq 0,\, b \geq 0. \tag{6.9}$$

By (6.6), for $a, b > 0$, this density may be written as

$$g_t^{\uparrow}(a, b) := \frac{1}{U^-(a)\widetilde{U}^-(b)} g_t^{+}(a, b)\,, \tag{6.10}$$

where $g_t^{+}(a, b)$ is the semigroup of the Lévy process $(X, \mathbf{P})$ killed at its first passage time below 0, i.e. for $a, b > 0$,

$$g_t^{+}(a, b)\,\mathrm{d}b := \mathbf{P}_a(X_t \in \mathrm{d}b,\, X_s \geq 0,\, \forall 0 \leq s \leq t)\,.$$

By Lemma 3 in [41], for each $t > 0$, the densities $g_t^{\uparrow}(a, b)$ are strictly positive and continuous on $[0, \infty) \times [0, \infty)$ (including $a = 0$ and $b = 0$). Then for all $a \geq 0$ and $b \geq 0$, the bridge of the Lévy process conditioned to stay positive is formally defined as follows: for $\varepsilon > 0$ and $B \in \sigma(X_s, 0 \leq s \leq t - \varepsilon)$,

$$\mathbf{P}_{a,b}^{\uparrow,t}(B) := \frac{1}{g_t^{\uparrow}(a, b)} \mathbf{E}_a^{\uparrow}(1_B\, g_{\varepsilon}^{\uparrow}(X_{t-\varepsilon}, b))\,, \tag{6.11}$$

and we may check, thanks to the Chapman-Kolmogorov equation (6.9), that this relation indeed defines a regular version of the conditional law $\mathbf{P}_a^{\uparrow}(\,\cdot \mid X_t = b)$ on Ω_t. Moreover, from [41], the measures $\mathbf{P}_{a,b}^{\uparrow,t}$ are weakly continuous in a and b on Ω_t.

Let us denote by $\widetilde{\mathbf{P}}_{a,b}^{\uparrow,t}$ the law of the bridge of $(\widetilde{X}, \mathbf{P}) = (-X, \mathbf{P})$ conditioned to stay positive. Then we derive, from the duality property which is proved in Lemma 1 of [41] and from Corollary 1 in [26], the following time reversal property, which is the continuous time counterpart of Lemma 6.1.

Lemma 6.2. *With the convention* $0- = 0$*, for all* $a, b \geq 0$*, and for all* $t > 0$*, under the law* $\mathbf{P}_{a,b}^{\uparrow,t}$*, the process* $(X_t - X_{(t-s)-},\, 0 \leq s \leq t)$ *has law* $\widetilde{\mathbf{P}}_{b,a}^{\uparrow,t}$*.*

Now let us focus on the special case where $a = b = 0$. In [16], Lemma 2, the law $\mathbf{P}^{\uparrow,t}_{0,0}$ is interpreted as the weak limit

$$\mathbf{P}^{\uparrow,t}_{0,0}(\cdot) = \lim_{\varepsilon\downarrow 0} \mathbf{P}^{\uparrow}_{0}(\,\cdot\,|\, 0 \le X_t \le \varepsilon)\,,$$

and, when the Lévy process $(X, \mathbf{P})$ has no negative jumps, this law is identified to the law of the normalized excursion of the reflected process at its past infimum. In particular, when X is the standard Brownian motion, it corresponds to the normalized Brownian excursion.

Then for $0 < \varepsilon < t$ and $x \ge 0$ we set

$$f_{\varepsilon,t}(x) := \frac{g^{\uparrow}_{\varepsilon}(x,0)}{g^{\uparrow}_{t}(0,0)}\,, \tag{6.12}$$

so that, by (6.11), $f_{\varepsilon,t}(X_{t-\varepsilon})$ is the Radon-Nikodym density of $\mathbf{P}^{\uparrow,t}_{0,0}$ with respect to $\mathbf{P}^{\uparrow}_{0}$ on the sigma field $\sigma(X_s, 0 \le s \le t-\varepsilon)$, i.e. for $B \in \sigma(X_s, 0 \le s \le t-\varepsilon)$,

$$\mathbf{P}^{\uparrow,t}_{0,0}(B) := \mathbf{E}^{\uparrow}_{0}(1_B\, f_{\varepsilon,t}(X_{t-\varepsilon},0))\,. \tag{6.13}$$

We recall that $g^{-}(\cdot)$ is the density of the law of the terminal value of the meander with length 1 of $-X$. Recall also from (3.19) and (3.20) that $g^{-}(x) \sim \mathtt{C}^{-} g(0) x^{\alpha(1-\varrho)}$, as $x \downarrow 0$. Then we have the following result, proved in Appendix B.

Lemma 6.3. *The function $f_{\varepsilon,t}(\cdot)$ is continuous on $[0,\infty)$ and is given by:*

$$f_{\varepsilon,t}(x) \;=\; \frac{(t/\varepsilon)^{1+1/\alpha}}{\mathtt{C}^{-}\, g(0)}\, \frac{g^{-}(\varepsilon^{-1/\alpha}x)}{(\varepsilon^{-1/\alpha}x)^{\alpha(1-\varrho)}}\,, \quad \textit{for } x > 0\,, \tag{6.14}$$

$$f_{\varepsilon,t}(0) \;=\; \left(\frac{t}{\varepsilon}\right)^{1+\frac{1}{\alpha}}\,. \tag{6.15}$$

In the sequel we will simply denote $f_{\varepsilon}(x) := f_{\varepsilon,1}(x)$.

Remark 6.4. In the Brownian case $\alpha = 2$, $\varrho = \frac{1}{2}$ everything is explicit. The density of the Brownian meander is $g^{-}(x) = x\, e^{-x^2/2}\, 1_{(0,\infty)}(x)$, while the density of $\mathbf{P}^{\uparrow}_{0}$ (the Bessel(3) process) at time t is

$$\frac{\mathbf{P}^{\uparrow}_{0}(X_t \in \mathrm{d}x)}{\mathrm{d}x} = \sqrt{\frac{2}{\pi t}}\,\frac{x^2}{t}\, e^{-x^2/(2t)} = \sqrt{\frac{2}{\pi}}\,\frac{x}{t}\, g^{-}\!\left(\frac{x}{\sqrt{t}}\right),$$

cf. [36, §3 in Chapter VI]. Since $\lambda^{\uparrow}(\mathrm{d}z) = z^2\,\mathrm{d}z$ (recall (6.8)), we obtain

$$g^{\uparrow}_{t}(0,x) \;:=\; \frac{\mathbf{P}^{\uparrow}_{0}(X_t \in \mathrm{d}x)}{\lambda^{\uparrow}(\mathrm{d}x)} \;=\; \sqrt{\frac{2}{\pi}}\,\frac{1}{t^{3/2}}\, e^{-x^2/(2t)}\,.$$

By symmetry $g^{\uparrow}_{t}(x,0) = g^{\uparrow}_{t}(0,x)$, and recalling (6.12) we find

$$f_{\varepsilon,t}(x) \;=\; \frac{g^{\uparrow}_{\varepsilon}(x,0)}{g^{\uparrow}_{t}(0,0)} \;=\; \frac{t^{3/2}}{\varepsilon^{3/2}}\, e^{-x^2/(2\varepsilon)}\,.$$

which coincides precisely with the expression in (6.14), because $g(0) = 1/\sqrt{2\pi}$ and $\mathtt{C}^{\pm} = \sqrt{2\pi}$, as it was shown in Remark 4.3.

Also note that the density of $\mathbf{P}^{\uparrow,1}_{0,0}$ (the Brownian excursion of length 1) at time $1-\varepsilon$ is

$$\frac{\mathbf{P}^{\uparrow,1}_{0,0}(X_{1-\varepsilon} \in \mathrm{d}x)}{\mathrm{d}x} = \frac{2}{\sqrt{2\pi}}\,\frac{1}{1-\varepsilon} g^{-}\!\left(\frac{x}{\sqrt{1-\varepsilon}}\right) \frac{1}{\varepsilon} g^{-}\!\left(\frac{x}{\sqrt{\varepsilon}}\right),$$

cf. [36, §3 in Chapter IX]. Recalling that $f_\varepsilon(x) = f_{\varepsilon,1}(x)$ is the Radon-Nikodym derivative of the law $\mathbf{P}^{\uparrow,1}_{0,0}$ with respect to $\mathbf{P}^{\uparrow}_0$ at time $1-\varepsilon$, we can write

$$f_\varepsilon(x) \;=\; \frac{\mathbf{P}^{\uparrow,1}_{0,0}(X_{1-\varepsilon} \in \mathrm{d}x)}{\mathbf{P}^{\uparrow}_0(X_{1-\varepsilon} \in \mathrm{d}x)} = \frac{1}{\varepsilon\, x}\, g^-\!\left(\frac{x}{\sqrt{\varepsilon}}\right) = \frac{1}{\varepsilon^{3/2}}\, e^{-x^2/(2\varepsilon)}\,,$$

finding again the expression (6.14) (in the special case $t = 1$).

6.3. **Proof of Theorem 2.4.** For simplicity, we restrict our attention to the $(1,0)$-lattice case of Hypothesis 2.2, i.e., we assume that the law of S_1 is supported by $\mathbb{Z}$ and is aperiodic. The general (h,c)-lattice case just requires some heavier notation, as the lattice $(nc + h\mathbb{Z})$ supporting the law of S_n varies with n. Also the proof in the absolutely continuous case is almost identical, except that the local limit theorems of Proposition 4.1 must be replaced by the corresponding ones of Theorem 5.1.

Let us fix $a, b \in [0,\infty)$ and $(x_N)_{N\in\mathbb{N}}, (y_N)_{N\in\mathbb{N}}$ positive sequences (with $y_N - x_N \in \mathbb{Z}$) such that $x_N/a_N \to a$ and $y_N/a_N \to b$ as $N \to \infty$. We recall that $\varphi_N : \Omega^{RW} \to \Omega$ is the map defined by

$$(\varphi_N(S))(t) := \frac{S_{\lfloor Nt\rfloor}}{a_N}\,,$$

for $N \in \mathbb{N}$, and by extension we still denote by φ_N the analogous map defined from Ω^{RW}_N to Ω_1, or more generally from Ω^{RW}_{NT} to Ω_T, for any fixed $T > 0$.

Our goal is to prove (2.10). It is enough to focus on the second relation in (2.10), i.e. conditioning the random walk to stay strictly positive, because the non-negative case can be easily deduced from it: note in fact that $S = (S_1, \ldots, S_N)$ under $\mathbb{P}^{\uparrow,N}_{x_N,y_N}$ has the same law as $S - \varepsilon$ under $\widehat{\mathbb{P}}^{\uparrow,N}_{x_N+\varepsilon,y_N+\varepsilon}$, provided $\varepsilon = \varepsilon_N > 0$ is taken sufficiently small.

When both $a > 0$ and $b > 0$, there is nothing to prove, as (2.10) follows directly by Liggett's invariance principle for the bridges [35]. In fact, the latter states that as $N \to \infty$

$$\mathbb{P}^N_{x_N,y_N} \circ \varphi_N^{-1} \;\Longrightarrow\; \mathbf{P}^1_{a,b}\,, \tag{6.16}$$

where $\mathbb{P}^N_{x,y}(\cdot) := \mathbb{P}_x(S \in \cdot\,|S_N = y)$ and $\mathbf{P}^1_{a,b}(\cdot) := \mathbf{P}_a(X \in \cdot\,|X_1 = b)$ are the bridges of the random walk and Lévy process respectively. Note that we can write

$$\widehat{\mathbb{P}}^{\uparrow,N}_{x_N,y_N}(\cdot) \;=\; \mathbb{P}^N_{x_N,y_N}(\cdot\,|S_1 > 0, \ldots, S_N > 0)\,, \qquad \mathbf{P}^{\uparrow,1}_{a,b}(\cdot) \;=\; \mathbf{P}^1_{a,b}\Big(\,\cdot\;\Big|\inf_{0\le t\le 1} X_t \ge 0\Big)\,.$$

Since for $a, b > 0$ the conditioning to stay positive has a non-vanishing probability under $\mathbf{P}^1_{a,b}$, relation (2.10) follows from (6.16).

We now focus on the case $a = b = 0$. The cases $a = 0$, $b > 0$ and $a > 0$, $b = 0$ are similar and simpler, so we skip them for brevity. By (6.5), the law $\widehat{\mathbb{P}}^{\uparrow,N}_{x,y}$ is absolutely continuous with respect to $\widehat{\mathbb{P}}^{\uparrow}_x$: more precisely, recalling relations (2.4) and (6.4), for all $0 < M < N$ and for all $B \in \sigma(S_0, \ldots, S_M)$, we can write

$$\widehat{\mathbb{P}}^{\uparrow,N}_{x,y}(B) = \frac{1}{\widehat{\mathbb{P}}^{\uparrow}_x(S_N = y)}\mathbb{E}^{\uparrow}_x(\mathbf{1}_B\, \widehat{\mathbb{P}}^{\uparrow}_{S_M}(S_{N-M} = y)) = \frac{\underline{V}^-(x)}{\widehat{q}^+_N(x,y)}\mathbb{E}^{\uparrow}_x\left(\mathbf{1}_B\, \frac{\widehat{q}^+_{N-M}(S_M, y)}{\underline{V}^-(S_M)}\right), \tag{6.17}$$

where we recall for clarity that $\widehat{q}^+_n(x,y) := \mathbb{P}_x(S_1 > 0, \ldots, S_{n-1} > 0, S_n = y)$. If we introduce for convenience the laws on Ω and Ω_1 respectively, given by

$$\mathcal{P}^{\uparrow,(N)}_x := \widehat{\mathbb{P}}^{\uparrow}_{a_N x} \circ \varphi_N^{-1}\,, \qquad \mathcal{P}^{\uparrow,(N)}_{x,y} := \widehat{\mathbb{P}}^{\uparrow,N}_{a_N x,\, a_N y} \circ \varphi_N^{-1}\,, \tag{6.18}$$

then from (6.17) with $M = \lfloor (1-\varepsilon)N \rfloor$, ($N \geq 2$, $0 < \varepsilon < 1/2$), we infer that $\mathcal{P}^{\uparrow,(N)}_{x/a_N,\, y/a_N}$ is absolutely continuous with respect to $\mathcal{P}^{\uparrow,(N)}_{x/a_N}$ (restricted to Ω_1) on the σ-field $\sigma(X_s, 0 \leq s \leq N^{-1}\lfloor (1-\varepsilon)N \rfloor)$, for all $\varepsilon > 0$. More precisely, for $B \in \sigma(X_s, 0 \leq s \leq N^{-1}\lfloor (1-\varepsilon)N \rfloor)$ we can write

$$\mathcal{P}^{\uparrow,(N)}_{x_N/a_N,\, y_N/a_N}(B) = \mathcal{E}^{\uparrow,(N)}_{x_N/a_N}\big(1_B\, f^{(N)}_\varepsilon(X_{N^{-1}\lfloor(1-\varepsilon)N\rfloor})\big)\,, \tag{6.19}$$

where $\mathcal{E}^{\uparrow,(N)}_x$ denotes the expectation under $\mathcal{P}^{\uparrow,(N)}_x$ and with $\varepsilon(N) = N - \lfloor (1-\varepsilon)N \rfloor$,

$$f^{(N)}_\varepsilon(z) := \frac{\underline{V}^-(x_N)}{\widehat{q}^{\,+}_N(x_N, y_N)}\, \frac{\widehat{q}^{\,+}_{\varepsilon(N)}(\lfloor z a_N \rfloor, y_N)}{\underline{V}^-(z a_N)}\,. \tag{6.20}$$

Recall the definition of $f_\varepsilon := f_{\varepsilon,1}$ in (6.14). We state the following basic lemma.

Lemma 6.5. *The following uniform convergence holds:*

$$\lim_{N\to\infty} \sup_{z\in\mathbb{R}} |f^{(N)}_\varepsilon(z) - f_\varepsilon(z)| = 0\,. \tag{6.21}$$

The proof is given below. To complete the proof of Theorem 2.4, we first prove that, for all $\varepsilon > 0$ and for every bounded and continuous functional F on Ω_1 which is measurable with respect to $\sigma(X_s, 0 \leq s \leq 1-\varepsilon)$, one has

$$\lim_{N\to\infty} \mathcal{E}^{\uparrow,(N)}_{x_N/a_N, y_N/a_N}(F) \;=\; \mathbf{E}^{\uparrow,1}_{0,0}(F)\,. \tag{6.22}$$

Let $\varepsilon' < \varepsilon$ and N sufficiently large so that $1-\varepsilon < N^{-1}\lfloor (1-\varepsilon')N \rfloor$. Then from (6.13),

$$\mathcal{E}^{\uparrow,(N)}_{x_N/a_N, y_N/a_N}(F) \;=\; \mathcal{E}^{\uparrow,(N)}_{x_N/a_N}(f^{(N)}_{\varepsilon'}(X_{N^{-1}\lfloor(1-\varepsilon')N\rfloor}) \cdot F)\,.$$

On the other hand, from (6.19) and the Markov property,

$$\begin{aligned} \mathbf{E}^{\uparrow,1}_{0,0}(F) &= \mathbf{E}^{\uparrow}_0(f_\varepsilon(X_{1-\varepsilon}) \cdot F) \\ &= \mathbf{E}^{\uparrow}_0(f_{\varepsilon'}(X_{1-\varepsilon'}) \cdot F)\,. \end{aligned}$$

Looking at (2.9), we see that (6.22) is equivalent to showing that

$$\lim_{N\to\infty} \mathcal{E}^{\uparrow,(N)}_{x_N/a_N}(f^{(N)}_{\varepsilon'}(X_{N^{-1}\lfloor(1-\varepsilon')N\rfloor}) \cdot F) = \mathbf{E}^{\uparrow}_0(f_{\varepsilon'}(X_{1-\varepsilon'}) \cdot F)\,. \tag{6.23}$$

By (6.21), for every $\eta > 0$ there exists $N_0 < \infty$ such that $|f^{(N)}_{\varepsilon'}(z) - f_{\varepsilon'}(z)| \leq \eta$, for all $N \geq N_0$ and $z \in [0,\infty)$. It follows that for $N \geq N_0$,

$$\begin{aligned} &|\mathcal{E}^{\uparrow,(N)}_{x_N/a_N}(f^{(N)}_{\varepsilon'}(X_{N^{-1}\lfloor(1-\varepsilon')N\rfloor}) \cdot F) - \mathbf{E}^{\uparrow}_0(f_{\varepsilon'}(X_{1-\varepsilon'}) \cdot F)| \\ &\qquad \leq \eta + |\mathcal{E}^{\uparrow,(N)}_{x_N/a_N}(f_{\varepsilon'}(X_{N^{-1}\lfloor(1-\varepsilon')N\rfloor}) \cdot F) - \mathbf{E}^{\uparrow}_0(f_{\varepsilon'}(X_{1-\varepsilon'}) \cdot F)|\,. \end{aligned}$$

By (2.9) and Skorokhod's representation theorem, there exist processes $Y^{(N)}$, Y on a probability space $(\Omega', \mathcal{F}, \mathrm{P})$, such that $(Y^{(N)}, \mathrm{P}) \overset{d}{=} (X, \mathcal{P}^{\uparrow,(N)}_{x_N/a_N})$, $(Y, \mathrm{P}) \overset{d}{=} (X, \mathbf{P}^{\uparrow}_0)$, and such that $Y^{(N)}$ converges P–a.s. toward Y. Since $f_{\varepsilon'}(\cdot)$ is continuous and Y is P–a.s. continuous at time $1-\varepsilon'$, the sequence $f_{\varepsilon'}(Y^{(N)}_{N^{-1}\lfloor(1-\varepsilon')N\rfloor})$ converges P–a.s. toward $f_{\varepsilon'}(Y_{1-\varepsilon'})$, so that by dominated convergence (recall that F is bounded), for all $N \geq N_1$,

$$\begin{aligned} &|\,\mathrm{E}(f_{\varepsilon'}(Y^{(N)}_{N^{-1}\lfloor(1-\varepsilon')N\rfloor}) \cdot F) - \mathrm{E}(f_{\varepsilon'}(Y_{1-\varepsilon'}) \cdot F)| = \\ &|\mathcal{E}^{\uparrow,(N)}_{x_N/a_N}(f_{\varepsilon'}(X_{N^{-1}\lfloor(1-\varepsilon')N\rfloor}) \cdot F) - \mathbf{E}^{\uparrow}_0(f_{\varepsilon'}(X_{1-\varepsilon'}) \cdot F)| \leq \eta\,. \end{aligned}$$

Hence for $N \geq \max(N_0, N_1)$, we have

$$|\mathcal{E}^{\uparrow,(N)}_{x_N/a_N}(f^{(N)}_{\varepsilon'}(X_{N^{-1}\lfloor(1-\varepsilon')N\rfloor}) \cdot F) - \mathbf{E}^{\uparrow}_0(f_{\varepsilon'}(X_{1-\varepsilon'}) \cdot F)| \leq 2\eta\,.$$

Since $\eta > 0$ was arbitrary, (6.23) is proved.

Relation (6.22) shows that the sequence of probability distributions $(\mathcal{P}^{\uparrow,(N)}_{x_N/a_N,y_N/a_N})$ restricted to $\Omega_{1-\varepsilon}$ converges weakly on this space, as $N \to +\infty$ toward $(\mathbf{P}^{\uparrow,1}_{0,0})$, for all $\varepsilon \in (0,1)$. In particular, the sequence of probability distributions $(\mathcal{P}^{\uparrow,(N)}_{x_N/a_N,y_N/a_N})$ converges on Ω_1 in the sense of finite dimensional distributions toward $(\mathbf{P}^{\uparrow,1}_{0,0})$, as $N \to +\infty$. Then in order to prove the weak convergence of this sequence, it remains to check that it is tight on Ω_1. From Theorem 15.3 of Billingsley [7], it suffices to show that for all $\eta > 0$,

$$\lim_{\delta\to 0}\limsup_{N\to+\infty}\mathcal{P}^{\uparrow,(N)}_{x_N/a_N,y_N/a_N}\left(\sup_{s,t\in[1-\delta,1]}|X_t - X_s| > \eta\right) = 0\,. \tag{6.24}$$

But, this follows from the time reversal properties, i.e. Lemma 6.1 and Lemma 6.2. Indeed, by Lemma 6.1, under the law $\mathcal{P}^{\uparrow,(N)}_{x_N/a_N,y_N/a_N}$ the process $(X_t - X_{(t-s)-},\ 0 \leq s \leq t)$ has law $\widetilde{\mathcal{P}}^{\uparrow,(N)}_{y_N/a_N,x_N/a_N}$, with an obvious notation. Moreover, from what have been proved above applied to $-S$ and $-X$, we obtain that the sequence of probability distributions $(\widetilde{\mathcal{P}}^{\uparrow,(N)}_{y_N/a_N,x_N/a_N})$ restricted to $\Omega_{1-\varepsilon}$ converges weakly on this space, as $N \to +\infty$ toward $(\widetilde{\mathbf{P}}^{\uparrow,1}_{0,0})$, for all $\varepsilon \in (0,1)$. Hence from Theorem 15.3 of Billingsley [7], we have

$$\lim_{\delta\to 0}\limsup_{N\to+\infty}\widetilde{\mathcal{P}}^{\uparrow,(N)}_{y_N/a_N,x_N/a_N}\left(\sup_{s,t\in[0,\delta]}|X_t - X_s| > \eta\right) = 0\,, \tag{6.25}$$

which is precisely (6.24), thanks to our time reversal argument.

Proof of Lemma 6.5. From the argument recalled in section 3.1, relation (6.21) is equivalent to the convergence

$$\lim_{N\to\infty}|f^{(N)}_{\varepsilon}(z_N) - f_{\varepsilon}(z_N)| = 0\,, \tag{6.26}$$

for every $\overline{z} \in [0,+\infty]$ *and for every sequence* $(z_N)_{N\in\mathbb{N}}$ *in* $[0,\infty)$ *such that* $z_N \to \overline{z}$.

We start with the case $\overline{z} \in (0,\infty)$. Since by assumption both $x_N, y_N = o(a_N)$, by the second relation in (4.5) we have as $N \to \infty$

$$\widehat{q}^{+}_N(x_N, y_N) \sim (1-\zeta)\,\underline{V}^-(x_N)\,\underline{V}^+(y_N)\,\frac{g(0)}{N\,a_N}\,. \tag{6.27}$$

By the second relation in (4.4) we get

$$\widehat{q}^{+}_{\varepsilon(N)}(\lfloor z_N a_N\rfloor, y_N) \sim \underline{V}^+(y_N)\,\frac{\mathbb{P}(\tau^+_1 > \varepsilon(N))}{a_{\varepsilon(N)}}\,g^-\left(z_N\frac{a_N}{a_{\varepsilon(N)}}\right), \tag{6.28}$$

therefore we can write, as $N \to \infty$

$$f^{(N)}_{\varepsilon}(z_N) \sim \frac{1}{g(0)}\,\frac{a_N}{a_{\varepsilon(N)}}\,\frac{N\,\mathbb{P}(\tau^+_1 > \varepsilon(N))}{(1-\zeta)\,\underline{V}^-(z_N a_N)}\,g^-\left(z_N\frac{a_N}{a_{\varepsilon(N)}}\right), \tag{6.29}$$

where we recall that $\varepsilon(N) := N - \lfloor(1-\varepsilon)N\rfloor$. We know from §3.3, cf. in particular (3.16) and (3.17), that

$$a_N \in R_{1/\alpha}\,, \qquad \underline{V}^-(\cdot) \in R_{\alpha(1-\varrho)}\,, \qquad \mathbb{P}(\tau^+_1 > N) \in R_{-\varrho}\,,$$

therefore as $N \to \infty$, since $z_N \to \overline{z} \in (0,\infty)$,

$$a_{\varepsilon(N)} \sim \varepsilon^{1/\alpha} a_N\,, \quad \underline{V}^-(z_N a_N) \sim \overline{z}^{\alpha(1-\varrho)}\, \underline{V}^-(a_N)\,, \quad \mathbb{P}(\tau_1^+ > \varepsilon(N)) \sim \varepsilon^{-(1-\varrho)}\, \mathbb{P}(\tau_1^+ > N)\,.$$

By (3.18),

$$N\,\mathbb{P}(\tau_1^+ > N) \sim (1-\zeta)\,\frac{\underline{V}^-(a_N)}{\mathtt{C}^-}\,. \tag{6.30}$$

Since $z_N \to \overline{z} \in (0,\infty)$ by assumption, from the preceding relations we get

$$\lim_{N\to\infty} f_\varepsilon^{(N)}(z_N) = \frac{1}{\mathtt{C}^-\, g(0)}\,\varepsilon^{-1}\,\varepsilon^{-1/\alpha}\,\frac{g^-(\varepsilon^{-1/\alpha}\,\overline{z})}{(\varepsilon^{-1/\alpha}\overline{z})^{\alpha(1-\varrho)}} = f_\varepsilon(\overline{z}) = \lim_{N\to\infty} f_\varepsilon(z_N)\,, \tag{6.31}$$

where the second equality follows by (3.20) and the definition (6.14) of $f_\varepsilon = f_{\varepsilon,1}$. We have shown that (6.26) holds true when $\overline{z} \in (0,\infty)$.

Next we consider the case $\overline{z} = 0$, so that $z_N a_N = o(a_N)$. By the second relation in (4.5), we have

$$\widehat{q}^{\,+}_{\varepsilon(N)}(\lfloor z_N a_N \rfloor, y_N) \sim (1-\zeta)\,\underline{V}^-(z_N a_N)\,\underline{V}^+(y_N)\,\frac{g(0)}{\varepsilon(N)\,a_{\varepsilon(N)}}\,, \tag{6.32}$$

therefore by (6.27) and (6.20) we obtain as $N \to \infty$

$$f_\varepsilon^{(N)}(z_N) \sim \frac{a_N}{\varepsilon a_{\varepsilon(N)}} \sim \frac{1}{\varepsilon^{1+1/\alpha}} = f_\varepsilon(0) = \lim_{N\to\infty} f_\varepsilon(z_N)\,,$$

by (6.15) and the continuity of $f_\varepsilon(\cdot)$. This shows that (6.26) holds true also when $\overline{z} = 0$.

Finally, we have to consider the case $\overline{z} = +\infty$. Since $g^-(z) \to 0$ as $z \to +\infty$, by the second relation in (4.4) we can write

$$\widehat{q}^{\,+}_{\varepsilon(N)}(\lfloor z_N a_N \rfloor, y_N) = \underline{V}^+(y_N)\,\frac{\mathbb{P}(\tau_1^+ > \varepsilon(N))}{a_{\varepsilon(N)}}\,o(1)\,. \tag{6.33}$$

Recalling (6.27) and (6.20), we obtain as $N \to \infty$

$$f_\varepsilon^{(N)}(z_N) = \frac{1}{g(0)}\,\frac{a_N}{a_{\varepsilon(N)}}\,\frac{N\,\mathbb{P}(\tau_1^+ > \varepsilon(N))}{(1-\zeta)\,\underline{V}^-(z_N a_N)}\,o(1)\,.$$

Observe that $a_N/a_{\varepsilon(N)}$ is bounded and $\underline{V}^-(z_N a_N)/\underline{V}^-(a_N) \to +\infty$, because $z_N \to +\infty$. Recalling (6.30), it follows that $f_\varepsilon^{(N)}(z_N) \to 0$, i.e. (6.26) holds also when $\overline{z} = +\infty$. ☐

Appendix A. Complements on the local limit theorems in the lattice case

A.1. **Proof of the second relation in** (4.5) **in the case** $x = y = 0$**.** Recall that we are in the (h,c)-lattice case (cf. Hypothesis 2.1), so that the law of S_n is supported by $nc + h\mathbb{Z}$. Summing over the location of S_{n-1}, we can write

$$\begin{aligned} \widehat{q}^{\,+}_n(0,0) &= \sum_{y>0} \widehat{q}_{n-1}(0,y)\cdot\mathbb{P}(S_1 = -y) \\ &= \sum_{0<y\le\delta_n a_n} \widehat{q}_{n-1}(0,y)\cdot\mathbb{P}(S_1=-y) + \sum_{y>\delta_n a_n} \widehat{q}_{n-1}(0,y)\cdot\mathbb{P}(S_1=-y)\,, \end{aligned} \tag{A.1}$$

where we agree that the sums are restricted to $y \in (nc + h\mathbb{Z})$, and the sequence $(\delta_n)_{n\in\mathbb{N}}$ will be fixed in a moment. Applying (4.3), we can estimate the second sum by

$$\sum_{y>\delta_n a_n} \widehat{q}_{n-1}(0,y)\cdot\mathbb{P}(S_1=-y) \le \frac{(const.)}{a_n}\,\mathbb{P}(\tau_1^- > n)\,\mathbb{P}(S_1 < -\delta_n a_n)\,. \tag{A.2}$$

Observe that $n\,\mathbb{P}(S_1 < -a_n) \to (const.') \in [0,\infty)$ as $n \to \infty$, cf. [25, eq. (8.4), (8.5), (8.12)], hence $\mathbb{P}(S_1 < -a_n) \le (const.'')/n$. Since $\mathbb{P}(\tau_1^- > n) \in R_{-(1-\varrho)}$ by (3.17), it follows that, if $\delta_n \downarrow 0$ slowly enough, then $\delta_n a_n \uparrow +\infty$ fast enough so that the right hand side of (A.2) is $o(1/(na_n))$. Coming back to the first sum in (A.1), applying (4.5) for $y > 0$ yields

$$\sum_{0<y\le\delta_n a_n} \widehat{q}_{n-1}(0,y)\cdot\mathbb{P}(S_1=-y) = \big(1+o(1)\big)\,h\,(1-\zeta)\,\frac{g(0)}{n\,a_n}\sum_{0<y\le\delta_n a_n}\underline{V}^+(y)\,\mathbb{P}(S_1=-y)$$

$$= \big(1+o(1)\big)\,h\,(1-\zeta)\,\frac{g(0)}{n\,a_n}\,\mathbb{E}_0\big(\underline{V}^+(-S_1)\,1_{\{0>S_1\ge-\delta_n a_n\}}\big)\,.$$

Applying (3.15), since $\delta_n a_n \uparrow +\infty$, we see that the last expectation converges to $\underline{V}^+(0)$ as $n \to \infty$. The second relation in (4.5) for $x = y = 0$ is proved.

A.2. **Proof of the first relation in** (4.5) **in the case** $x = y = 0$**.** We set

$$K(n) := \widehat{q}_n^+(0,0) = \mathbb{P}(\tau_1^- = n, H_1^- = 0)$$

and note that $\sum_{n\in\mathbb{N}} K(n) = \mathbb{P}(H_1^- = 0) = \zeta \in (0,1)$, so that $K(\cdot)$ may be viewed as a defective probability on $\mathbb{N}$. Summing over the location of the times $t \le n$ at which $S_t = 0$, we obtain for all $n \in \mathbb{N}$

$$q_n^+(0,0) = \sum_{m=1}^{\infty}\sum_{0=:t_0<t_1<\ldots<t_m=n} K(t_1)K(t_2-t_1)\cdots K(t_m-t_{m-1}) = \sum_{m=0}^{\infty} K^{*m}(n)\,, \tag{A.3}$$

where $K^{*m}(\cdot)$ denotes the k-fold convolution of $K(\cdot)$ with itself. This shows that $q_n^+(0,0)$ may be viewed as the renewal mass function associated to the renewal process with inter-arrival *defective* probability $K(\cdot)$. Since $K(n) \in R_{-1-1/\alpha}$ by the second relation in (4.5), the asymptotic behavior of $q_n^+(0,0)$ as $n \to \infty$ is a classical result in heavy-tailed renewal theory (cf. [27, Theorem A.4] or [3, Proposition 12] for a more general result):

$$q_n^+(0,0) \sim \frac{1}{(1-\sum_{n\in\mathbb{N}} K(n))^2}\,K(n) = \frac{1}{(1-\zeta)^2}\,h\,(1-\zeta)\,\underline{V}^-(0)\,\underline{V}^+(0)\,\frac{g(0)}{n\,a_n}\,.$$

Since $V^{\pm}(0) = (1-\zeta)^{-1}$ and $\underline{V}^{\pm}(0) = 1$, cf. (3.8) and (3.11), the first relation in (4.5) is proved for $x = y = 0$.

A.3. **Proof of relation** (4.11)**.** Let us rewrite (4.11) for convenience: as $n \to \infty$

$$\mathbb{P}(\widehat{\tau}_1^- > n) \sim (1-\zeta)^{-1}\mathbb{P}(\tau_1^- > n)\,. \tag{A.4}$$

Summing on the position of the last epoch before n at which the random walk hits zero, we can write

$$\mathbb{P}(\widehat{\tau}_1^- > n) = \sum_{\ell=0}^{n}\mathbb{P}(S_i \ge 0\ \forall 0\le i\le\ell\,,\ S_\ell = 0)\,\mathbb{P}(\tau_1^- > n-\ell) = \sum_{\ell=0}^{n} q_\ell^+(0,0)\,\mathbb{P}(\tau_1^- > n-\ell)\,.$$

We have $\sum_{\ell=0}^{\infty} q_\ell^+(0,0) = \sum_{k=0}^{\infty}\zeta^k = (1-\zeta)^{-1}$, by (A.3) and the preceding lines. Since $\mathbb{P}(\tau_1^- > n-\ell) \sim \mathbb{P}(\tau_1^- > n)$ as $n \to \infty$ uniformly for $\ell \le \log n$ (we recall that $\mathbb{P}(\tau_1^- > n)$ is regularly varying), it follows that

$$\sum_{\ell=0}^{\lfloor\log n\rfloor} q_\ell^+(0,0)\,\mathbb{P}(\tau_1^- > n-\ell) \sim (1-\zeta)^{-1}\mathbb{P}(\tau_1^- > n)\,,$$

in agreement with (A.4). It remains to show that the terms with $\ell \geq \log n$ give a negligible contribution. Since $\mathbb{P}(\tau_1^- > n-\ell) \leq (const.)\mathbb{P}(\tau_1^- > n)$ for $\log n \leq \ell \leq n/2$, we can write

$$\sum_{\ell=\lfloor \log n \rfloor}^{\lfloor n/2 \rfloor} q_\ell^+(0,0)\, \mathbb{P}(\tau_1^- > n-\ell) \leq (const.)\bigg(\sum_{\ell \geq \log n} q_\ell^+(0,0)\bigg)\mathbb{P}(\tau_1^- > n) = o\big(\mathbb{P}(\tau_1^- > n)\big)\,.$$

Finally, for $\ell \geq n/2$ we have $q_\ell^+(0,0) \leq (const.)/(na_n)$ by (4.5). Furthermore, we also have $\sum_{\ell=\lfloor n/2 \rfloor}^{n} \mathbb{P}(\tau_1^- > n-\ell) \sim \varrho^{-1} n \mathbb{P}(\tau_1^- > n/2)$, because $\mathbb{P}(\tau_1^- > m)$ is regularly varying with index $-(1-\varrho) > -1$, cf. (3.17). It follows that

$$\sum_{\ell=\lfloor n/2 \rfloor}^{n} q_\ell^+(0,0)\, \mathbb{P}(\tau_1^- > n-\ell) \leq (const.)\frac{1}{na_n}\, n\, \mathbb{P}(\tau_1^- > n) = o\big(\mathbb{P}(\tau_1^- > n)\big)\,,$$

and the proof of (A.4) is complete.

Appendix B. Proof of Lemma 6.3

The continuity of the function $x \mapsto f_{\varepsilon,t}(x)$ on $[0,\infty)$ is a consequence of its definition:

$$f_{\varepsilon,t}(x) = \frac{g_\varepsilon^\uparrow(x,0)}{g_t^\uparrow(0,0)}\,, \tag{B.1}$$

thanks to the continuity of $x \mapsto g_\varepsilon^\uparrow(x,0)$, which is proved in Lemma 3 of [41].

We recall that $\lambda^\uparrow(\mathrm{d}z) = z^\alpha\, \mathrm{d}z$ and

$$g_t^\uparrow(x,y) \;:=\; \frac{\mathbf{P}_x^\uparrow(X_t \in \mathrm{d}y)}{\lambda^\uparrow(\mathrm{d}y)}\,.$$

Since $\{c^{-1/\alpha} X_{ct}\}_{t\geq 0}$ has the same law as $\{X_t\}_{t\geq 0}$, it follows that

$$g_t^\uparrow(x,y) \;=\; \frac{1}{t^{1+1/\alpha}}\, g_1^\uparrow\bigg(\frac{x}{t^{1/\alpha}}, \frac{y}{t^{1/\alpha}}\bigg)\,, \qquad \forall t>0,\ \forall x,y \in [0,\infty)\,. \tag{B.2}$$

In particular

$$g_t^\uparrow(0,0) = t^{-1-1/\alpha} g_1^\uparrow(0,0) \tag{B.3}$$

and equation (6.15) follows:

$$f_{\varepsilon,t}(0) \;=\; \frac{g_\varepsilon^\uparrow(0,0)}{g_t^\uparrow(0,0)} \;=\; \left(\frac{t}{\varepsilon}\right)^{1+1/\alpha}\,. \tag{B.4}$$

It only remains to prove (6.14). Let us consider the reflected Lévy process $\widetilde{X} := -X$. By part 1 of Theorem 1 in [16], the law of the terminal value of the meander of this process, that is $g^-(x)\,\mathrm{d}x$, is absolutely continuous with respect to the law of $\widetilde{X}_1$ under $\mathbf{P}^\uparrow$, that is $\widetilde{g}_1^\uparrow(0,x)\,\lambda^\uparrow(\mathrm{d}x)$ (with obvious notations), with Radon-Nikodym density proportional to $1/\widetilde{U}^-(x) = x^{-\alpha\varrho}$. Therefore there is a constant $C>0$ such that for all $x \geq 0$

$$g^-(x) \;=\; C\, \widetilde{g}_1^\uparrow(0,x)\, x^\alpha\, \frac{1}{x^{\alpha\varrho}}\,, \qquad \forall x \geq 0\,.$$

Now observe that $g_t^\uparrow(a,b) = g_t^\uparrow(b,a)$ for all $a,b \geq 0$, as it is proved in Lemma 1 of [41]. (More directly, it is enough to check this relation for $a,b>0$, by continuity, and this is evident from (6.10).) Therefore we can write

$$g_1^\uparrow(x,0) \;=\; \frac{1}{C}\,\frac{g^-(x)}{x^{\alpha(1-\varrho)}}\,,$$

and by the scaling relation (B.2) it follows that

$$g_\varepsilon^\uparrow(x,0) = \frac{\varepsilon^{-\varrho-1/\alpha}}{C} \frac{g^-(\varepsilon^{-1/\alpha}x)}{x^{\alpha(1-\varrho)}} .$$

Looking back at (B.1) and (B.3) we then obtain

$$f_{\varepsilon,t}(x) \;=\; \frac{\varepsilon^{-\varrho-1/\alpha}}{C\, g_1^\uparrow(0,0)\, t^{1-1/\alpha}} \frac{g^-(\varepsilon^{-1/\alpha}x)}{x^{\alpha(1-\varrho)}} \;=\; \frac{(t/\varepsilon)^{1+1/\alpha}}{C\, g_1^\uparrow(0,0)} \frac{g^-(\varepsilon^{-1/\alpha}x)}{(\varepsilon^{-1/\alpha}x)^{\alpha(1-\varrho)}} . \tag{B.5}$$

Recalling (3.19) and (3.20), it follows from this expression that

$$\lim_{x\downarrow 0} f_{\varepsilon,t}(x) \;=\; \frac{\mathtt{C}^- g(0)}{C\, g_1^\uparrow(0,0)}\, (t/\varepsilon)^{1+1/\alpha} ,$$

and looking back at (B.4) we see that $C\, g_1^\uparrow(0,0) = \mathtt{C}^- g(0)$. But then (B.5) coincides with (6.14), and the proof is completed.

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