

TIGHTNESS CONDITIONS FOR POLYMER MEASURES

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ABSTRACT. We give sufficient conditions for tightness in the space $C([0, 1])$ for sequences of probability measures which enjoy a suitable decoupling between zero level set and excursions. Applications of our results are given in the context of (homogeneous, periodic and disordered) random walk models for polymers and interfaces.

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1. INTRODUCTION

In this note we want to prove tightness under diffusive rescaling for a sequence of processes (\mathbf{P}_N) with the following property: conditionally on the zero level set, the excursions of the process between consecutive zeros are independent and each excursion is distributed according to the same fixed law (corresponding to the given excursion length).

Let us be more precise. We first need three main ingredients:

- the *zero level set law* p_N is, for each $N \in \mathbb{N}$, a probability measure on the subsets of $\{1, \dots, N\}$;
- the *bulk excursion law* P_t is, for each $t \in \mathbb{N}$, a probability measure on \mathbb{R}^t such that

$$P_t(y \in \mathbb{R}^t : y_1 > 0, \dots, y_{t-1} > 0, y_t = 0) = 1;$$

- the *final excursion law* P_t^f is, for each $t \in \mathbb{N}$, a probability measure on \mathbb{R}^t such that

$$P_t^f(y \in \mathbb{R}^t : y_1 > 0, \dots, y_t > 0) = 1.$$

We also set for convenience

$$\mathcal{T}_N := \bigcup_{k=1}^N \{(T_1, \dots, T_k) : T_i \in \mathbb{N}, 0 =: T_0 < T_1 < \dots < T_k \leq N\}.$$

Then, for each $N \in \mathbb{N}$, we introduce the measure \mathbf{P}_N on \mathbb{R}^N defined for B_1, \dots, B_N Borel sets in \mathbb{R} by the relation:

$$\begin{aligned} \mathbf{P}_N(B_1 \times \dots \times B_N) &= \sum_{(T_i) \in \mathcal{T}_N} p_N(\{T_1, T_2, \dots, T_k\}) \cdot \\ &\cdot \left\{ \prod_{i=1}^k P_{T_i - T_{i-1}}(B_{T_{i-1}+1} \times \dots \times B_{T_i - T_{i-1}}) \right\} \cdot P_{N-T_k}^f(B_{T_k+1} \times \dots \times B_N). \end{aligned} \tag{1.1}$$

This means that under \mathbf{P}_N :

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- the zero level set, defined for $y \in \mathbb{R}^N$ by $\mathcal{A} = \mathcal{A}(y) := \{\ell = 1, \dots, N : y_\ell = 0\}$, is distributed according to the law p_N ;
- conditionally on $\mathcal{A} = \{T_1, \dots, T_k\}$, with $(T_i) \in \mathcal{T}_N$, the family of excursions $\{e_i := (y_j, j = T_{i-1} + 1, \dots, T_i)\}_{i=1, \dots, k+1}$ is independent;
- conditionally on $\mathcal{A} = \{T_1, \dots, T_k\}$, for all $i = 1, \dots, k$, e_i has law $P_{T_i - T_{i-1}}$; if $T_k < N$ then e_{k+1} has law $P_{N - T_k}^f$.

We are interested in the diffusive rescaling of \mathbf{P}_N , that we call \mathbf{Q}_N . More precisely, let us define the map $X^N : \mathbb{R}^N \mapsto C([0, 1])$:

$$X_t^N(y) = \frac{y_{\lfloor Nt \rfloor}}{N^{1/2}} + (Nt - \lfloor Nt \rfloor) \frac{y_{\lfloor Nt \rfloor + 1} - y_{\lfloor Nt \rfloor}}{N^{1/2}}, \quad t \in [0, 1],$$

where $\lfloor r \rfloor$ denotes the integer part of $r \in \mathbb{R}_+$ and $y_0 := 0$. Notice that $X_t^N(y)$ is nothing but the linear interpolation of $\{y_{\lfloor Nt \rfloor} / \sqrt{N}\}_{t \in \frac{\mathbb{N}}{N} \cap [0, 1]}$. Then we set

$$\mathbf{Q}_N := \mathbf{P}_N \circ (X^N)^{-1}.$$

Analogously, we denote by Q_N and Q_N^f the diffusive rescaling of P_N and P_N^f respectively.

Our aim is to give sufficient conditions for tightness of (\mathbf{Q}_N) in $C([0, 1])$: in the next section we state and prove our main result. Applications to random walk models for polymers and interface are discussed in Section 3.

2. MAIN RESULT

Theorem 2.1. *Suppose that:*

- the sequences (Q_N) and (Q_N^f) are tight in $C([0, 1])$;
- the following relation holds true:

$$\lim_{a \rightarrow \infty} C(a) = 0 \quad \text{where } C(a) := \sup_n E_n \left(\left(\max_{0 \leq i \leq n} \frac{y_i^2}{n} \right) \mathbf{1}_{\{\max_{0 \leq i \leq n} y_i^2/n > a\}} \right). \quad (2.1)$$

Then the sequence (\mathbf{Q}_N) is tight in $C([0, 1])$.

We stress that we make no hypothesis on the law (p_N) . Before proving the theorem, we introduce for $\delta > 0$ the *continuity modulus* $\Gamma(\delta)$, i.e. the real-valued functional defined for $x \in C([0, 1])$ by

$$\Gamma(\delta) = \Gamma(\delta)[x] := \sup_{\substack{s, t \in [0, 1] \\ |t-s| \leq \delta}} |x_t - x_s|.$$

We are going to check the standard necessary and sufficient condition for tightness on $C([0, 1])$ (Theorems of Prohorov and Ascoli-Arzelà): for every $\gamma > 0$

$$\lim_{\delta \rightarrow 0} \sup_{N \in \mathbb{N}} \mathbf{Q}_N(\Gamma(\delta) > \gamma) = 0. \quad (2.2)$$

It is actually convenient to work with a modified continuity modulus $\tilde{\Gamma}(\delta)$: for $x \in C([0, 1])$ and $s, t \in [0, 1]$ we set $s \sim_x t$ iff $x_u \neq 0$ for every $u \in (s, t)$, i.e. iff s and t belong to the same excursion of x , and we define

$$\tilde{\Gamma}(\delta) = \tilde{\Gamma}(\delta)[x] := \sup_{\substack{s, t \in [0, 1], s \sim_x t \\ |t-s| \leq \delta}} |x_t - x_s|.$$

Clearly $\tilde{\Gamma}(\delta) \leq \Gamma(\delta) \leq 2\tilde{\Gamma}(\delta)$, therefore it suffices to prove that for every $\gamma > 0$

$$\lim_{\delta \rightarrow 0} \sup_{N \in \mathbb{N}} \mathbf{Q}_N \left(\tilde{\Gamma}(\delta) > \gamma \right) = 0. \quad (2.3)$$

Proof of Theorem 2.1. The path we follow is rather general. The crucial property that we exploit is the independence of the excursions conditionally on the zero level set \mathcal{A} . Setting $\mathcal{N}_N := \bigcup_{k=0}^{\infty} \{(t_1, \dots, t_{k+1}) : t_i \in \mathbb{N}, \sum_{i=1}^{k+1} t_i = N\}$, by (1.1) we can write

$$\begin{aligned} \mathbf{Q}_N \left(\tilde{\Gamma}(\delta) \leq \gamma \right) &= \sum_{(t_i) \in \mathcal{N}_N} \left\{ \prod_{\ell=1}^k Q_{t_\ell} \left(\Gamma \left(\frac{N}{t_\ell} \delta \right) \leq \gamma \sqrt{\frac{N}{t_\ell}} \right) \right\} \cdot Q_{t_{k+1}}^f \left(\Gamma \left(\frac{N}{t_{k+1}} \delta \right) \leq \gamma \sqrt{\frac{N}{t_{k+1}}} \right) \\ &\quad \cdot p_N(\{t_1, t_1 + t_2, \dots, t_1 + \dots + t_{k+1}\}). \end{aligned} \quad (2.4)$$

Next we perform a very drastic bound: we set

$$\begin{aligned} f_\gamma(\delta) &:= \inf_{N \in \mathbb{N}, (t_i) \in \mathcal{N}_N} \prod_{\ell=1}^k Q_{t_\ell} \left(\Gamma \left(\frac{N}{t_\ell} \delta \right) \leq \gamma \sqrt{\frac{N}{t_\ell}} \right), \\ g_\gamma(\delta) &:= \inf_{N \in \mathbb{N}, 1 \leq t \leq N} Q_t^f \left(\Gamma \left(\frac{N}{t} \delta \right) \leq \gamma \sqrt{\frac{N}{t}} \right). \end{aligned} \quad (2.5)$$

By (2.4) we have $\mathbf{Q}_N(\tilde{\Gamma}(\delta) \leq \gamma) \geq f_\gamma(\delta) \cdot g_\gamma(\delta)$. Therefore if we show that $f_\gamma(\delta)g_\gamma(\delta) \rightarrow 1$ as $\delta \rightarrow 0$, for any fixed $\gamma > 0$, equation (2.3) follows and the proof is completed.

We start by proving that $\liminf_{\delta \searrow 0} f_\gamma(\delta) = 1$ for all $\gamma > 0$. We introduce an auxiliary (small) parameter $\eta > 0$ and we define the set

$$\mathcal{S}^\eta = \mathcal{S}^\eta(N, (t_i)) := \{\ell \in \{1, \dots, k\} : t_\ell > \eta N\}, \quad N \in \mathbb{N}, (t_i) \in \mathcal{N}_N.$$

Notice that $\Gamma(\cdot)$ is non-decreasing and that we have trivially $\Gamma(\delta')[x] \leq 2 \max_{t \in [0,1]} |x_t|$. Splitting the product in the r.h.s. of (2.5) and using these observations, we obtain

$$\begin{aligned} \prod_{\ell=1}^k Q_{t_\ell} \left(\Gamma \left(\frac{N}{t_\ell} \delta \right) \leq \gamma \sqrt{\frac{N}{t_\ell}} \right) &\geq \left\{ \prod_{\ell \in \mathcal{S}^\eta} Q_{t_\ell} \left(\Gamma \left(\frac{\delta}{\eta} \right) \leq \gamma \right) \right\} \\ &\quad \cdot \left\{ \prod_{\ell \in \{1, \dots, k\} \setminus \mathcal{S}^\eta} P_{t_\ell} \left(\max_{0 \leq i \leq t_\ell} |y_i| \leq \frac{\gamma}{2} \sqrt{N} \right) \right\}. \end{aligned} \quad (2.6)$$

Suppose now that we can prove the following:

$$\forall \gamma > 0 : \lim_{\eta \searrow 0} \inf_{N, (t_i) \in \mathcal{N}_N} \prod_{\ell \in \{1, \dots, k\} \setminus \mathcal{S}^\eta} P_{t_\ell} \left(\max_{0 \leq i \leq t_\ell} |y_i| \leq \frac{\gamma}{2} \sqrt{N} \right) = 1. \quad (2.7)$$

In other words, for any fixed $\gamma > 0$, the parameter η can be chosen in order to make the second term in the r.h.s. of (2.6) as close to 1 as we wish, *uniformly in N and (t_i)*. If (2.7) is proven, then we can fix $\eta > 0$ and it is easy to see that one can choose δ in order to make also the first term in the r.h.s. of (2.6) as close to 1 as we wish, *uniformly in N and (t_i)*: this is just because the number of factors in the product (i.e. the cardinality of the set \mathcal{S}^η) is bounded by construction by $1/\eta < \infty$ and because by hypothesis the sequences (Q_N) and (Q_N^f) are tight in $C([0, 1])$ (we recall that $t_\ell \geq \eta N$ for $\ell \in \mathcal{S}^\eta$). This shows that indeed $f_\gamma(\delta) \rightarrow 1$ as $\delta \rightarrow 0$, for any fixed $\gamma > 0$, completing the proof.

Therefore we are left with proving (2.7): we have to show that for any fixed γ and $\alpha > 0$ we can choose the parameter η such that

$$\inf_{N \in \mathbb{N}, (t_i) \in \mathcal{N}_N} \left(\prod_{\ell \in \{1, \dots, k\} \setminus \mathcal{S}^\eta} P_{t_\ell} \left(\max_{0 \leq i \leq t_\ell} |y_i| \leq \frac{\gamma}{2} \sqrt{N} \right) \right) \geq 1 - \alpha. \quad (2.8)$$

By (a somewhat enhanced) Chebychev inequality we have

$$P_n \left(\max_{0 \leq i \leq n} \frac{|y_i|}{\sqrt{n}} > a \right) \leq \frac{c_n(a)}{a^2}, \quad c_n(a) := \mathbf{E}_n \left(\left(\max_{0 \leq i \leq n} \frac{y_i^2}{n} \right) \mathbf{1}_{\{\max_{0 \leq i \leq n} y_i^2/n > a\}} \right).$$

Since $c_n(\cdot)$ is non-increasing and $t_\ell \leq \eta N$ for $\ell \notin \mathcal{S}^\eta$, we obtain

$$\prod_{\ell \in \{1, \dots, k\} \setminus \mathcal{S}^\eta} P_{t_\ell} \left(\max_{0 \leq i \leq t_\ell} |y_i| \leq \frac{\gamma}{2} \sqrt{N} \right) \geq \prod_{\ell \in \{1, \dots, k\} \setminus \mathcal{S}^\eta} \left(1 - c_{t_\ell} \left(\frac{\gamma}{2\sqrt{\eta}} \right) \frac{4}{\gamma^2} \frac{t_\ell}{N} \right). \quad (2.9)$$

Notice now that $C(a) = \sup_{n \in \mathbb{N}} c_n(a)$ by the definition (2.1) of C . Choosing η sufficiently small, so that $c_{t_\ell} \left(\frac{\gamma}{2\sqrt{\eta}} \right) \frac{4}{\gamma^2} \frac{t_\ell}{N} \leq C \left(\frac{\gamma}{2\sqrt{\eta}} \right) \frac{4}{\gamma^2} \eta \leq \frac{1}{2}$, and observing that $1 - x \geq \exp(-2x)$ for $0 \leq x \leq \frac{1}{2}$, we can finally bound (2.9) by

$$\begin{aligned} \prod_{\ell \in \{1, \dots, k\} \setminus \mathcal{S}^\eta} P_{t_\ell} \left(\max_{0 \leq i \leq t_\ell} |y_i| \leq \frac{\gamma}{2} \sqrt{N} \right) &\geq \prod_{\ell \in \{1, \dots, k\} \setminus \mathcal{S}^\eta} \exp \left(-2C \left(\frac{\gamma}{2\sqrt{\eta}} \right) \frac{4}{\gamma^2} \frac{t_\ell}{N} \right) \\ &= \exp \left(-2C \left(\frac{\gamma}{2\sqrt{\eta}} \right) \frac{4}{\gamma^2} \sum_{\ell \in \{1, \dots, k\} \setminus \mathcal{S}^\eta} \frac{t_\ell}{N} \right) \geq \exp \left(-\frac{8}{\gamma^2} C \left(\frac{\gamma}{2\sqrt{\eta}} \right) \right), \end{aligned} \quad (2.10)$$

and equation (2.8) follows from the hypothesis $\lim_{a \rightarrow \infty} C(a) = 0$.

It remains to prove that $\liminf_{\delta \searrow 0} g_\gamma(\delta) = 1$, for all $\gamma > 0$. Let $\theta \in (0, 1)$. Then

$$g_\gamma^{(1)}(\delta) := \inf_{N \in \mathbb{N}, \theta N \leq t \leq N} Q_t^f \left(\Gamma \left(\frac{N}{t} \delta \right) \leq \gamma \sqrt{\frac{N}{t}} \right) \geq \inf_t Q_t^f \left(\Gamma \left(\frac{\delta}{\theta} \right) \leq \gamma \right)$$

so that, by tightness of (Q_N^f) in $C([0, 1])$, we have $\liminf_{\delta \searrow 0} g_\gamma^{(1)}(\delta) = 1$ for all $\gamma > 0$ and $\theta \in (0, 1)$. Now:

$$g_\gamma^{(2)}(\delta) := \inf_{N \in \mathbb{N}, 1 \leq t < \theta N} Q_t^f \left(\Gamma \left(\frac{N}{t} \delta \right) \leq \gamma \sqrt{\frac{N}{t}} \right) \geq \inf_t Q_t^f \left(\sup_{s \in [0, 1]} |x_s| \leq \frac{\gamma}{2\sqrt{\theta}} \right),$$

and again by tightness of (Q_N^f) in $C([0, 1])$, we have that for any $\alpha \in (0, 1)$ we can find $\theta \in (0, 1)$ such that $\liminf_{\delta \searrow 0} g_\gamma^{(2)}(\delta) \geq 1 - \alpha$. It follows that $\liminf_{\delta \searrow 0} g_\gamma(\delta) \geq 1 - \alpha$ for all $\alpha \in (0, 1)$, and the proof is completed. \square

3. APPLICATION TO POLYMER MEASURES

One direct application of Theorem 2.1, and the main motivation of this note, is in the context of $(1+1)$ -dimensional random walk models for polymer chains and interfaces. We look in particular at the *copolymer near a selective interface model*, both in the disordered [3, 1] and in the periodic [4, 5] setting, but also at the *interface wetting models* considered in [7, 6] and at *pinning models* based on random walks, described e.g. in [9] (to which we refer for a detailed overview on all these models).

Notice that, for the purpose of proving tightness in $C[0, 1]$, one can safely focus on the absolute value of the process. With this observation in mind, we have the basic fact that

all the above mentioned models satisfy equation (1.1), for suitable choices of the laws p_N , P_N and P_N^f . More precisely, in all these cases we have that for every Borel set $B \in \mathbb{R}^t$

$$\begin{aligned} P_t(B) &= \mathbf{P}((S_1, \dots, S_t) \in B \mid S_1 > 0, \dots, S_{t-1} > 0, S_t = 0) \\ P_t^f(B) &= \mathbf{P}((S_1, \dots, S_t) \in B \mid S_1 > 0, \dots, S_t > 0), \end{aligned} \quad (3.1)$$

where $(\{S_n\}_{n \geq 0}, \mathbf{P})$ is a real random walk. (In particular, in the non-homogeneous cases, all the dependence on the environment is contained in the law p_N .) Then by Theorem 2.1 the tightness under diffusive rescaling for all the above models is reduced to showing that the sequences (P_N) and (P_N^f) are tight and that equation (2.1) holds.

We are going to check these conditions in the special instance when $(\{S_n\}_{n \geq 0}, \mathbf{P})$ is a non-trivial symmetric random walk with $S_1 \in \{-1, 0, +1\}$, thus proving tightness for the (disordered and periodic) copolymer near a selective interface model. Notice that the law of the walk is identified by $p := \mathbf{P}(S_1 = +1) \in (0, 1/2]$. To lighten notations, in what follows we actually assume that $p \in (0, 1/2)$. The tightness for the sequences (Q_N) and (Q_N^f) in this case is a classical result, cf. [10] and [2]. Therefore it remains to prove that equation (2.1) holds true, which follows as a simple consequence of the following lemma.

Lemma 3.1. *There exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ and $a > 0$ we have:*

$$f_n(a) := \mathbf{P} \left(\max_{1 \leq i \leq n} \frac{S_i^2}{n} \geq a \mid S_1 > 0, \dots, S_{n-1} > 0, S_n = 0 \right) \leq C \frac{1}{1 + a^2}.$$

Proof. Set $\tau := \inf\{n > 0 : S_n^2 \geq na\}$ and $T := \inf\{n > 0 : S_n \leq 0\}$. Then for $n \geq 1$:

$$f_n(a) = \mathbf{P}(\tau \leq n \mid T = n) = \frac{\mathbf{P}(\tau \leq n, T = n)}{\mathbf{P}(T = n)}$$

By the reflection principle, the denominator is equal to:

$$\mathbf{P}(T = n) = p^2 [\mathbf{P}(S_{n-2} = 0) - \mathbf{P}(S_{n-2} = 2)].$$

By symmetry and by the strong Markov property, we can estimate the numerator:

$$\begin{aligned} \mathbf{P}(\tau \leq n, T = n) &\leq 2 \mathbf{P}(\tau \leq n/2, T = n) \\ &= 2 \sum_{j=1}^{n/2} \mathbf{P}(\tau = j \leq n/2 < T) \mathbf{P}(S_{n-j} = a_n, T > n - j) \end{aligned}$$

where $a_n := \lfloor \sqrt{an} \rfloor$ is the integer part of \sqrt{an} . Again by the reflection principle:

$$\mathbf{P}(S_{n-j} = a_n, T > n - j) = p [\mathbf{P}(S_{n-j-1} = a_n - 1) - \mathbf{P}(S_{n-j-1} = a_n + 1)].$$

Th. 16 in Ch. VII of [11] says that when $k \rightarrow \infty$, uniformly in $b \in \mathbb{N}/\sqrt{k}$:

$$\left(1 + \left|\frac{b}{\sigma}\right|^4\right) \left(\sigma\sqrt{k} \mathbf{P}(S_k = b\sqrt{k}) - \frac{1}{\sqrt{2\pi}} \exp(-b^2/(2\sigma)) - \sum_{\nu=1}^2 \frac{q_\nu(b/\sigma)}{k^{\nu/2}}\right) = o(k^{-1}),$$

where $\sigma = \sqrt{2p}$. By using the last two formulas we obtain that there exist positive constants c_1 and c_2 such that for all $j \leq n/2$

$$\mathbf{P}(T > n - j, S_{n-j} = a_n) \leq \frac{c_1}{n} \left(\exp(-c_2 a) + \frac{1}{1 + a^2} \right), \quad \forall a > 0. \quad (3.2)$$

Now:

$$\sum_{j=1}^{n/2} \mathbf{P}(\tau = j \leq n/2 < T) \leq \mathbf{P}(\tau < T) = \frac{1}{1 + a_n} \leq \frac{1}{\sqrt{an}},$$

where the equality can be proved with a martingale argument. Similarly, we have that

$$\exists \lim_{n \rightarrow \infty} n^{3/2} \mathbf{P}(T = n) =: c_K \in (0, \infty). \quad (3.3)$$

(see also [8, Ch. XII.7]). It is now easy to conclude. \square

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