A POLYMER IN A MULTI-INTERFACE MEDIUM

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Abstract. We consider a model for a polymer chain interacting with a sequence of equispaced flat interfaces through a pinning potential. The intensity $\delta \in \mathbb{R}$ of the pinning interaction is constant, while the interface spacing $T = T_N$ is allowed to vary with the size N of the polymer. Our main result is the explicit determination of the scaling behavior of the model in the large N limit, as a function of $(T_N)_N$ and for fixed $\delta > 0$. In particular, we show that a transition occurs at $T_N = O(\log N)$. Our approach is based on renewal theory.

1. Introduction and main results

1.1. **The model.** In this paper we study a $(1+1)$–dimensional model for a polymer chain dipped in a medium constituted by infinitely many horizontal interfaces. The possible configurations of the polymer are modeled by the trajectories $\{(i, S_i)\}_{i\geq 0}$ of the simple symmetric random walk on $\mathbb{Z}$, with law denoted by $\mathbf{P}$, i.e., $S_0 = 0$ and $(S_i - S_{i-1})_{i\geq 1}$ is an i.i.d. sequence of Bernoulli trials satisfying $\mathbf{P}(S_1 = \pm 1) = 1/2$. We assume that the interfaces are equispaced, i.e., at the same distance $T \in 2\mathbb{N}$ from each other (note that T is even by assumption, for notational convenience, due to the periodicity of the simple random walk).

The interaction between the polymer and the medium is described by the following Hamiltonian:

$$H^T_{N,\delta}(S) \;:=\; \delta \sum_{i=1}^{N} \mathbf{1}_{\{S_i \in T\mathbb{Z}\}} \;=\; \delta \sum_{k\in\mathbb{Z}} \sum_{i=1}^{N} \mathbf{1}_{\{S_i = k\,T\}}, \tag{1.1}$$

where $N \in \mathbb{N}$ is the size of the polymer and $\delta \in \mathbb{R}$ is the intensity of the energetic reward (if $\delta > 0$) or penalty (if $\delta < 0$) that the polymer receives when touching the interfaces. More precisely, the model is defined by the following probability law $\mathbf{P}^T_{N,\delta}$ on $\mathbb{R}^{\mathbb{N}\cup\{0\}}$:

$$\frac{\mathrm{d}\mathbf{P}^T_{N,\delta}}{\mathrm{d}\mathbf{P}}(S) \;:=\; \frac{\exp\big(H^T_{N,\delta}(S)\big)}{Z^T_{N,\delta}}, \tag{1.2}$$

where $Z^T_{N,\delta} = \mathbf{E}\big(\exp(H^T_{N,\delta}(S))\big)$ is the normalizing constant, called the *partition function*.

It should be clear that the effect of the Hamiltonian $H^T_{N,\delta}$ is to favor or penalize, according to the sign of δ, the trajectories $\{(n, S_n)\}_n$ that have a lot of intersections with the interfaces, located at heights $T\mathbb{Z}$ (we refer to Figure 1 for a graphical description). Although in this work we give a number of results that do not depend on the sign of δ, we stress from now that our main concern is with the case $\delta > 0$.

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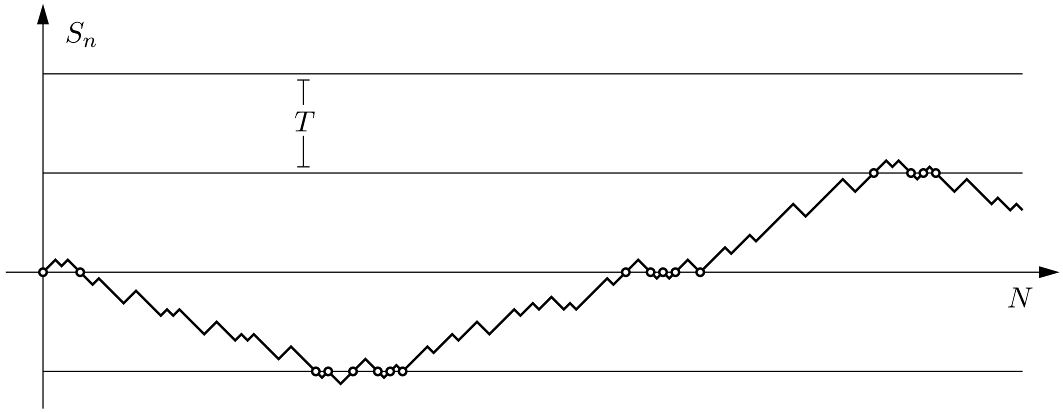

FIGURE 1. A typical path of the polymer measure $\mathbf{P}^T_{N,\delta}$ with $N = 158$ and $T = 16$. The circles represent the points where the polymer touches the interfaces, which are favored (resp. disfavored) when $\delta > 0$ (resp. $\delta < 0$).

If we let $T \to \infty$ in (1.2) *for fixed* N (in fact it suffices to take $T > N$), we obtain a well defined limiting model $\mathbf{P}^\infty_{N,\delta}$:

$$\frac{\mathrm{d}\mathbf{P}^\infty_{N,\delta}}{\mathrm{d}\mathbf{P}}(S) := \frac{\exp\big(H^\infty_{N,\delta}(S)\big)}{Z^\infty_{N,\delta}} \qquad \text{where} \qquad H^\infty_{N,\delta}(S) := \delta \sum_{i=1}^N \mathbf{1}_{\{S_i=0\}} \,. \tag{1.3}$$

$\mathbf{P}^\infty_{N,\delta}$ is known in the literature as a *homogeneous pinning model* and it describes a polymer chain interacting with a single flat interface, namely the x-axis. This model, together with several variants (like the *wetting model*, where $\{S_n\}_n$ is also constrained to stay non-negative), has been studied in depth, first in the physical literature, cf. [11] and references therein, and more recently in the mathematical one [13, 9, 4, 12]. In particular, it is well-known that a *phase transition* between a delocalized regime and a localized one occurs as δ varies and this transition can be characterized in terms of the path properties of $\mathbf{P}^\infty_{N,\delta}$.

The aim of this paper is to answer the same kind of questions for the model $\mathbf{P}^{T_N}_{N,\delta}$, as a function of δ and of the interface spacing $T = T_N$, which is allowed to vary with N. We denote the full sequence by $\boldsymbol{T} := (T_N)_{N\in\mathbb{N}}$ (taking values in $2\mathbb{N}$) and, with no essential loss of generality (one could focus on subsequences), we assume that $\boldsymbol{T}$ has a limit as $N \to \infty$:

$$\exists \lim_{N\to\infty} T_N =: T_\infty \in 2\mathbb{N} \cup \{+\infty\} \,. \tag{1.4}$$

(Of course, if $T_\infty < \infty$ the sequence $(T_N)_N$ must take eventually the constant value T_∞.) For notational convenience, we also assume that $T_N \le N$: again, this is no real loss of generality, since for $T_N > N$ the law $\mathbf{P}^{T_N}_{N,\delta}$ reduces to the just mentioned $\mathbf{P}^\infty_{N,\delta}$.

Before stating precisely the results we obtain in this paper, let us describe briefly the motivations behind our model and its context. Several models for a polymer interacting with a single linear interface have been investigated in the past 20 years, both in the physical and in the mathematical literature (see [11] and [12] for two excellent surveys). The two most popular classes among them are probably the so-called *copolymer at a selective interface* separating two selective solvents and the *pinning of a polymer at an interface*, of which the homogeneous pinning model $\mathbf{P}^\infty_{N,\delta}$ is the simplest and most basic example. Although some questions still remain open, notably when *disorder* is present, important progress has been made and there is now a fairly good comprehension of the mechanism leading to phase transitions for these models.

More recently, some generalizations have been introduced, to account for interactions taking place on more general structures than a single linear interface. In the *copolymer class*, we mention [6, 7] and [17], where the medium is constituted by an emulsion, and especially [8], where the single linear interface is replaced by infinitely many equi-spaced flat interfaces, separating alternate layers of each selective solvent. Our model $\mathbf{P}_{N,\delta}^{T}$ provides a closely analogous generalization in the *pinning class*, with the important difference that the model considered in [8] is disordered. In a sense, what we consider is the simplest case of a pinning model interacting with infinitely many interfaces. In analogy with the single interface case [12], we believe that understanding in detail this basic example is the first step toward a comprehension of the more sophisticated disordered case.

Let us describe briefly the results obtained in [8]. The authors focus on the case when the interface spacing T_N diverges as $N \to \infty$ and they show that the free energy of the model is the same as in the case of one single linear interface. Then, under stronger assumption on $(T_N)_N$, namely $T_N / \log\log N \to \infty$ and $T_N / \log N \to 0$, they show that the polymer visits infinitely many different interfaces and the asymptotic behavior of the time needed to hop from an interface to a neighboring one is shown to behave like e^{cT_N}.

In this paper we consider analogous questions for our model $\mathbf{P}_{N,\delta}^{T}$. In our non-disordered setting, we obtain stronger results: in particular, we are able to describe precisely the path behavior of the polymer in the large N limit, for an arbitrary sequence $\boldsymbol{T} = (T_N)_N$ and for $\delta > 0$ (i.e., we focus on the case of attractive interfaces). In fact there is a subtle interplay between the pinning reward δ and the speed T_N at which the interfaces depart, which is responsible for the scaling behavior of the polymer. It turns out that there are three different regimes, determined by comparing T_N with $\frac{\log N}{c_\delta}$, where $c_\delta > 0$ is computed explicitly. We refer to Theorem 2 and to the following discussion for a detailed explanation of our results. Let us just mention that, as T_N increases from $O(1)$ to the critical speed $\frac{\log N}{c_\delta}$, the scaling constants of S_N decrease smoothly from the diffusive behavior $\sqrt{N}$ to $\log N$, while if $T_N \gg \frac{\log N}{c_\delta}$ then $S_N = O(1)$. This means, on the one hand, that by accelerating the growth of the interface spacing the scaling of S_N decreases, and, on the other hand, that scaling behaviors for S_N intermediate between $O(1)$ and $\log N$ (such as, e.g., $\log\log N$) are not possible in our model. We also stress that our model is *sub-diffusive* as soon as $T_N \to \infty$. Sub-diffusive behaviors appear in a variety of models dealing with random walks subject to some form of penalization: from the (very rich) literature we mention for instance [15] and [1] on the mathematical side and [16] on the physical side.

Our approach is mainly based on renewal theory. The use of this kind of techniques in the field of polymer models has proved to be extremely successful, starting from [9] and [4], and has been generalized more recently to cover Markovian settings, cf. [3] and [2]. The key point is to get sharp estimates on suitable renewal functions.

The same approach can be applied to deal with the depinning case $\delta < 0$, i.e., when touching an interface entails a penalty rather than a reward. However, in this case the limiting model $\mathbf{P}_{N,\delta}^{\infty}$ is *delocalized* and this fact generates additional non-trivial difficulties. For this reason, the analysis of the $\delta < 0$ case is given in a separate paper [5], where we show that there are remarkable differences with respect to the $\delta > 0$ case that we consider here. In particular, the critical speed of T_N above which the polymer gives up visiting infinitely many different interfaces is no longer of order $\log N$, but rather of order $N^{1/3}$ (see Theorem 1 in [5] for a precise statement).

1.2. **The free energy.** The standard way of studying the effect of the interaction (1.1) for large N is to look at the *free energy* of the model, defined as the limit

$$\phi(\delta, \boldsymbol{T}) \;:=\; \lim_{N\to\infty} \phi_N(\delta, \boldsymbol{T})\,, \qquad \text{where} \qquad \phi_N(\delta, \boldsymbol{T}) \;:=\; \frac{1}{N} \log Z^{T_N}_{N,\delta}\,. \tag{1.5}$$

The existence of such a limit, for any choice of $\delta \in \mathbb{R}$ and $\boldsymbol{T}$ satisfying (1.4), is proven in Section 2. To understand why one should look at ϕ, we introduce the random variable

$$L_{N,T} \;:=\; \sum_{i=1}^{N} \mathbf{1}_{\{S_i \in T\mathbb{Z}\}} \;=\; \sum_{k\in\mathbb{Z}} \sum_{i=1}^{N} \mathbf{1}_{\{S_i = kT\}}\,, \tag{1.6}$$

and we observe that an easy computation yields

$$\frac{\partial}{\partial\delta}\phi_N(\delta, \boldsymbol{T}) \;=\; \mathbf{E}^{T_N}_{N,\delta}\bigg(\frac{L_{N,T_N}}{N}\bigg)\,, \qquad\qquad \frac{\partial^2}{\partial\delta^2}\phi_N(\delta, \boldsymbol{T}) \;=\; N\ \mathrm{var}_{\mathbf{P}^{T_N}_{N,\delta}}\bigg(\frac{L_{N,T_N}}{N}\bigg) \;\geq\; 0\,.$$

In particular, $\phi_N(\delta, \boldsymbol{T})$ is a convex function of δ, for every $N \in \mathbb{N}$. Hence $\phi(\delta, \boldsymbol{T})$ is convex too and by elementary convex analysis it follows that as soon as $\phi(\delta, \boldsymbol{T})$ is differentiable

$$\frac{\partial}{\partial\delta}\phi(\delta, \boldsymbol{T}) \;=\; \lim_{N\to\infty} \mathbf{E}^{T_N}_{N,\delta}\bigg(\frac{L_{N,T_N}}{N}\bigg)\,. \tag{1.7}$$

Thus, the first derivative of $\phi(\delta, \boldsymbol{T})$ gives the asymptotic proportion of time spent by the polymer on the interfaces, which explains the interest of looking at $\phi(\delta, \boldsymbol{T})$. In fact a basic problem is the determination of the set of values of δ (if any) where $\phi(\delta, \boldsymbol{T})$ is not analytic, which correspond physically to the occurrence of a *phase transition* in the system.

This issue is addressed by our first result, which provides an explicit formula for $\phi(\delta, \boldsymbol{T})$. Let us introduce for $T \in 2\mathbb{N} \cup \{+\infty\}$ the random variable τ_1^T defined by

$$\tau_1^T \;:=\; \inf\big\{n > 0 :\ S_n \in \{-T, 0, +T\}\big\}\,, \tag{1.8}$$

and denote by $Q_T(\lambda)$ its Laplace transform under the simple random walk law $\mathbf{P}$:

$$Q_T(\lambda) \;:=\; \mathbf{E}\big(e^{-\lambda \tau_1^T}\big) \;=\; \sum_{n=1}^{\infty} e^{-\lambda n}\, \mathbf{P}\big(\tau_1^T = n\big)\,. \tag{1.9}$$

When $T = +\infty$, the variable τ_1^∞ is nothing but the first return time of the simple random walk to zero, and it is well-known that $Q_\infty(\lambda) = +\infty$ for $\lambda < 0$ while $Q_\infty(\lambda) = 1 - \sqrt{1 - e^{-2\lambda}}$ for $\lambda \geq 0$, cf. [10]. We point out that $Q_T(\lambda)$ can be given a closed explicit expression also for finite T, see Appendix A and in particular equation (A.4). Here it is important to stress that for $T < \infty$ the function $Q_T(\lambda)$ is *analytic and decreasing* on $(\lambda_0^T, +\infty)$, where $\lambda_0^T < 0$ (see eq. (A.6)), and $Q_T(\lambda) \to +\infty$ as $\lambda \downarrow \lambda_0^T$ while $Q_T(\lambda) \to 0$ as $\lambda \to \infty$. In particular, when $T < \infty$ the inverse function $\big(Q_T\big)^{-1}(\cdot)$ is (analytic and) defined on the whole $(0, \infty)$, while $\big(Q_\infty\big)^{-1}(\cdot)$ is (analytic and) defined only on $(0, 1]$.

Theorem 1. *Denoting by $T_\infty = \lim_{N\to\infty} T_N$, the free energy $\phi(\delta, \boldsymbol{T}) = \phi(\delta, T_\infty)$ depends only on δ and T_∞ and is given by*

$$\phi(\delta, T_\infty) \;=\; \begin{cases} \big(Q_{T_\infty}\big)^{-1}(e^{-\delta}) & \text{if } T_\infty < +\infty \\ \big(Q_\infty\big)^{-1}(e^{-\delta} \wedge 1) & \text{if } T_\infty = +\infty\,. \end{cases} \tag{1.10}$$

It follows that for $T_\infty < +\infty$ the function $\delta \mapsto \phi(\delta, \boldsymbol{T})$ is analytic on the whole real line, while for $T_\infty = +\infty$ it is not analytic only at $\delta = 0$.

So there are no phase transitions in our model, except in the $T_\infty = +\infty$ case, where $\phi(\delta,\infty)$ is not analytic at $\delta = 0$. This fact is well-known, because $\phi(\delta,\infty)$ is nothing but the free energy of the classical homogeneous pinning model $\mathbf{P}^\infty_{N,\delta}$, cf. [12]. In fact the explicit formula for $Q_\infty(\cdot)$ mentioned above yields

$$\phi(\delta,\infty) \;=\; \left(\frac{\delta}{2} - \log\sqrt{2-e^{-\delta}}\right)\mathbf{1}_{\{\delta\geq 0\}}\,. \tag{1.11}$$

Also in the case when $T_\infty < \infty$, some general properties of $\phi(\delta, T_\infty)$ can be easily derived from Theorem 1, for instance that $\frac{\partial}{\partial\delta}\phi(\delta,\boldsymbol{T}) \to 0$ as $\delta\to-\infty$ while $\frac{\partial}{\partial\delta}\phi(\delta,\boldsymbol{T}) \to \frac{1}{2}$ as $\delta\to+\infty$, which have a clear physical interpretation thanks to (1.7).

The proof of Theorem 1 is given in Section 2, using renewal theory ideas. Besides identifying the free energy, we introduce a slightly modified version of the polymer measure $\mathbf{P}^T_{N,\delta}$ which can be given an explicit renewal theory interpretation. This provides a key tool to study the path behavior (see below).

One consequence of Theorem 1 is that any $\boldsymbol{T}$ such that $T_\infty = \infty$ yields *the same free energy* $\phi(\delta,\boldsymbol{T}) = \phi(\delta,\infty)$ as the classical homogeneous pinning model. However we are going to see that the actual path behavior of $\mathbf{P}^{T_N}_{N,\delta}$ as $N\to\infty$ depends strongly on the speed at which $T_N\to\infty$, a phenomenon which is not caught by the free energy.

1.3. **The scaling behavior.** Henceforth we focus on the case $\delta > 0$. We assume that $\boldsymbol{T} = (T_N)_{N\in\mathbb{N}}$ has been chosen such that $T_N\to\infty$ as $N\to\infty$. Then the free energy $\phi(\delta,\boldsymbol{T}) = \phi(\delta,\infty)$ is that of the homogeneous pinning model: in particular $\phi(\delta,\boldsymbol{T}) > 0$ for every $\delta > 0$. Since $\phi(\delta,\boldsymbol{T}) = 0$ for $\delta\leq 0$, by convexity and by formula (1.7) it follows that for $\delta > 0$ the typical paths of $\mathbf{P}^{T_N}_{N,\delta}$ touch the interfaces for large N a positive fraction of time, and it is customary to say that we are in a *localized regime*.

We now investigate more closely the path properties of $\mathbf{P}^{T_N}_{N,\delta}$. A natural question is: does the polymer visit infinitely many *different* interfaces, or does it stick to a finite number of them? And more precisely: what is the scaling behavior of S_N under $\mathbf{P}^{T_N}_{N,\delta}$ as $N\to\infty$?

The answer turns out to depend on the speed at which $T_N\to\infty$. Let c_δ be the positive constant defined as

$$c_\delta \;:=\; \phi(\delta,\infty) + \log\left(1+\sqrt{1-e^{-2\phi(\delta,\infty)}}\right) \;=\; \frac{\delta}{2} + \log\sqrt{2-e^{-\delta}}\,, \tag{1.12}$$

where the r.h.s. of (1.12) is obtained with the help of (1.11). Then, the behavior of the sequence $T_N - \frac{1}{c_\delta}\log N$ determines the scaling properties of the polymer measure. More precisely, we have the following result, where $\Longrightarrow$ denotes convergence in law and $\mathcal{N}(0,1)$ the standard Normal distribution.

Theorem 2. *Let $\delta > 0$ and $\boldsymbol{T} = (T_N)_{N\in\mathbb{N}}$ such that $T_N\to\infty$ as $N\to\infty$.*

(i) *If $T_N - \frac{\log N}{c_\delta} \to -\infty$ as $N\to\infty$, then under $\mathbf{P}^{T_N}_{N,\delta}$ as $N\to\infty$*

$$\frac{S_N}{C_\delta\left(e^{-\frac{c_\delta}{2}T_N}\,T_N\right)\sqrt{N}} \quad\Longrightarrow\quad \mathcal{N}(0,1)\,, \tag{1.13}$$

where $C_\delta := \sqrt{2\,e^\delta\,\phi'(\delta,\infty)\,\sqrt{1-e^{-2\phi(\delta,\infty)}}} = (1-e^{-\delta})\sqrt{\frac{2e^\delta}{2-e^{-\delta}}}$ is an explicit positive constant.

(ii) *If there exists* $\zeta \in \mathbb{R}$ *such that* $T_{N'} - \frac{\log N'}{c_\delta} \to \zeta$ *along a sub-sequence* N', *then under* $\mathbf{P}_{N',\delta}^{T_{N'}}$ *as* $N' \to \infty$

$$\frac{S_{N'}}{T_{N'}} \implies S_\Gamma\,, \tag{1.14}$$

where Γ *is a random variable independent of the* $\{S_i\}_{i\geq 0}$ *and with a Poisson law of parameter* $t_{\delta,\zeta} := 2e^\delta\sqrt{1-e^{-2\phi(\delta,\infty)}}\phi'(\delta,\infty)\cdot e^{-c_\delta\zeta} = 2e^\delta\frac{(1-e^{-\delta})^2}{2-e^{-\delta}}\cdot e^{-c_\delta\zeta}$.

(iii) *If* $T_N - \frac{\log N}{c_\delta} \to +\infty$ *as* $N \to \infty$, *then the family of laws of* $\{S_N\}_{N\in\mathbb{N}}$ *under* $\mathbf{P}_{N,\delta}^{T_N}$ *is tight, i.e.,*

$$\lim_{L\to\infty}\sup_{N\in\mathbb{N}} \mathbf{P}_{N,\delta}^{T_N}\big(|S_N| > L\big) = 0\,. \tag{1.15}$$

Remark 1. It may appear strange that in point (ii) we have required that $T_{N'} - \frac{\log N'}{c_\delta} \to \zeta$ only along a sub-sequence N': however this is just because T_N takes integer values and therefore the full sequence $T_N - \frac{\log N}{c_\delta}$ cannot have a finite limit. In general, equation (1.14) implies that S_N/T_N is tight when the full sequence $|T_N - \frac{\log N}{c_\delta}|$ is bounded. □

The proof of Theorem 2 is distributed in Sections 3, 4 and 5. The crucial idea, described in §3.1, is to exploit the renewal theory description given in Section 2. Let us stress the intuitive content of this result. We set $\Delta_N := T_N - \frac{\log(N)}{c_\delta}$ and we anticipate that $e^{-c_\delta\Delta_N}$ is the typical number of different interfaces visited by the polymer of length N. With this in mind, we can give some more insight on Theorem 2.

- If $\Delta_N \to -\infty$, then the interfaces are departing slow enough so that it is worth for the polymer to visit infinitely many of them. Of course, this is also true when $T_N \equiv T < \infty$ for all $N \in \mathbb{N}$. This situation is not included in Theorem 2 for notational convenience, but a straightforward adaptation of our proof shows that in this case $S_N/(\mathcal{C}_T\sqrt{N}) \Longrightarrow \mathcal{N}(0,1)$ for a suitable $\mathcal{C}_T$ satisfying $\mathcal{C}_T \sim C_\delta\, e^{-\frac{c_\delta}{2}T}\, T$ as $T \to \infty$, thus matching perfectly with (1.13).

  We note that, independently of $(T_N)_N$ (such that $\Delta_N \to -\infty$), the limit law of S_N, properly rescaled, is always the standard Normal distribution. However the scaling constants $\big(e^{-\frac{c_\delta}{2}T_N}\,T_N\big)\sqrt{N}$ do depend on the sequence $(T_N)_N$ and in particular they are *sub-diffusive* as soon as $T_N \to \infty$. Also notice that, by varying T_N from $O(1)$ to the critical case $\frac{\log(N)}{c_\delta} + O(1)$, the scaling constants decrease smoothly from $\sqrt{N}$ to $\log N$.
- If $\Delta_N = O(1)$, then we are in the critical case when the polymer visits a finite number of different interfaces and therefore the scaling behavior of S_N is the same as T_N, i.e., $S_N \approx \log N$. The explicit form S_Γ of the scaling distribution has the following interpretation: the number Γ of different interfaces visited by the polymer is distributed according to a Poisson law and, conditionally on Γ, the polymer just performs Γ steps of a simple symmetric random walk on the interfaces.
- If $\Delta_N \to +\infty$, then the only interface visited by the polymer is the x-axes. The other interfaces are indeed too distant from the origin to be convenient for the polymer to visit them. Therefore, the model $\mathbf{P}_{N,\delta}^{T_N}$ becomes essentially the same as the classical homogeneous pinning model $\mathbf{P}_{N,\delta}^{\infty}$, where only the interface located at $S = 0$ is present. Since $\delta > 0$, we are in the localized regime for $\mathbf{P}_{N,\delta}^{\infty}$ and it

is well-known that $S_N = O(1)$. One could also determine the limit distribution of S_N, but we omit this for conciseness.

As already mentioned, the study of the path behavior in the delocalized regime $\delta < 0$ turns out to be rather different, both from a technical and a physical viewpoint, and will therefore be carried out in a future work.

2. A renewal theory path to the free energy

This section is devoted to proving Theorem 1. We also provide a renewal theory description for a slight modification of the polymer measure $\mathbf{P}^{\infty}_{N,\delta}$, which is the key tool in the next sections.

2.1. A slight modification.

We consider $\delta \in \mathbb{R}$ and $T \in 2\mathbb{N} \cup \{\infty\}$. It is convenient to introduce the *constrained partition function* $Z^{T,c}_{N,\delta}$, where only the trajectories $(S_i)_i$ that are pinned at an interface at their right extremity are taken into account, i.e.,

$$Z^{T,c}_{N,\delta} \;:=\; \mathbf{E}\Big(\exp\big(H^T_{N,\delta}(S)\big)\, \mathbf{1}_{\{S_N \in T\mathbb{Z}\}} \Big). \tag{2.1}$$

In order for the restriction on $\{S_N \in T\mathbb{Z}\}$ to be non-trivial, we work with $Z^{T,c}_{N,\delta}$ only for N even. This is the usual parity issue connected with the periodicity of the simple random walk: in fact $\mathbf{P}(S_N \in T\mathbb{Z}) = 0$ if N is odd (we recall that T is assumed to be even).

The reason for introducing $Z^{T,c}_{N,\delta}$ is that it is easier to handle than the original partition function, and at the same time it is not too different, as the following lemma shows.

Lemma 3. *The following relation holds for all $N \in \mathbb{N}, \delta \in \mathbb{R}, T \in 2\mathbb{N}$:*

$$e^{-|\delta|}\, Z^{T,c}_{2\lfloor N/2 \rfloor,\delta} \;\le\; Z^T_{N,\delta} \;\le\; \sqrt{(N+1)\, Z^{T,c}_{2N,\delta}}\,. \tag{2.2}$$

Proof. If N is even, then $2\lfloor N/2 \rfloor = N$ and the lower bound in (2.2) follows trivially from the definition (2.1) of $Z^{T,c}_{N,\delta}$. If N is odd, then $2\lfloor N/2 \rfloor = N-1$ and since

$$H^T_{N,\delta}(S) \;\ge\; H^T_{N-1,\delta}(S) \;-\; |\delta|\,,$$

the lower bound in (2.2) is proven in full generality.

To prove the upper bound, we observe that by the definition (2.1)

$$Z^{T,c}_{2N,\delta} \;\ge\; \mathbf{E}\Big(\exp\big(H^T_{2N,\delta}(S)\big)\, \mathbf{1}_{\{S_{2N}=0\}} \Big) \;=\; \sum_{k=-N}^{N} \mathbf{E}\Big(\exp\big(H^T_{2N,\delta}(S)\big)\, \mathbf{1}_{\{S_N=k\}}\, \mathbf{1}_{\{S_{2N}=0\}} \Big)\,,$$

and from the Markov property and the time-symmetry $i \mapsto N-i$ we have

$$Z^{T,c}_{2N,\delta} \;\ge\; \sum_{k=-N}^{N} \Big[\mathbf{E}\Big(\exp\big(H^T_{N,\delta}(S)\big)\, \mathbf{1}_{\{S_N=k\}} \Big) \Big]^2 .$$

Since $\mathbf{P}(S_N = k) > 0$ if and only if N and k have the same parity, there are only $N+1$ non-zero terms in the sum, and applying Jensen's inequality we get

$$Z^{T,c}_{2N,\delta} \;\ge\; \frac{1}{N+1} \Bigg[\sum_{k=-N}^{N} \mathbf{E}\Big(\exp\big(H^T_{N,\delta}(S)\big)\, \mathbf{1}_{\{S_N=k\}} \Big) \Bigg]^2 \;=\; \frac{1}{N+1} \Big[Z^T_{N,\delta} \Big]^2 ,$$

therefore the upper bound in (2.2) is proven and the proof is completed. □

As a direct consequence of Lemma 3, we observe that to prove the existence of the free energy, i.e., of the limit in (1.5), we can safely replace the original partition function $Z_{N,\delta}^{T_N}$ by the constrained one $Z_{N,\delta}^{T_N,c}$, restricting N to the even numbers. The next paragraphs are devoted to obtaining a more explicit expression of $Z_{N,\delta}^{T,c}$.

2.2. **The link with renewal theory.** We start with some definitions. For $T \in 2\mathbb{N} \cup \{\infty\}$, we set $\tau_0^T = 0$ and for $j \in \mathbb{N}$

$$\tau_j^T \;:=\; \inf\big\{ i \geq \tau_{j-1}^T + 1 :\; S_i \in T\mathbb{Z} \big\} \qquad \text{and} \qquad \varepsilon_j^T \;:=\; \frac{S_{\tau_j^T} - S_{\tau_{j-1}^T}}{T}\,, \tag{2.3}$$

where for $T = \infty$ we agree that $T\mathbb{Z} = \{0\}$. Notice that τ_j^T gives the j^{th} epoch at which S touches an interface, while ε_j^T tells whether the j^{th} interface touched is the same as the $(j-1)^{\text{th}}$ ($\varepsilon_j^T = 0$), or is the interface above ($\varepsilon_j^T = 1$) or below ($\varepsilon_j^T = -1$). Under the law $\mathbf{P}$ of the simple random walk, we define for $j = \{0, \pm 1\}$, $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ the quantities

$$q_T^j(n) \;:=\; \mathbf{P}\big(\tau_1^T = n\,,\, \varepsilon_1^T = j\big) \qquad \text{and} \qquad Q_T^j(\lambda) \;:=\; \sum_{n=1}^{\infty} e^{-\lambda n}\, q_T^j(n)\,. \tag{2.4}$$

Of course $Q_T^j(\lambda)$ may be (in fact, is) infinite for λ negative and large, and clearly $q_\infty^{\pm 1}(n) = 0$ for $n \geq 1$ and $Q_\infty^{\pm 1}(\lambda) = 0$ for $\lambda \geq 0$. Notice that $q_T^{-1} = q_T^1$ and $Q_T^{-1} = Q_T^1$, so that we can focus only on q_T^j, Q_T^j for $j \in \{0, 1\}$. We also set

$$\begin{aligned} q_T(n) \;&:=\; \sum_{j=0,\pm1} q_T^j(n) \;=\; q_T^0(n) \;+\; 2\, q_T^1(n) \;=\; \mathbf{P}\Big(\tau_1^T = n\Big) \\ Q_T(\lambda) \;&:=\; \sum_{j=0,\pm1} Q_T^j(\lambda) \;=\; Q_T^0(\lambda) \;+\; 2\, Q_T^1(\lambda) \;=\; \mathbf{E}\Big(e^{-\lambda\, \tau_1^T}\Big). \end{aligned} \tag{2.5}$$

Next we introduce

$$\mathcal{H} \;:=\; \big\{\mathbb{R} \times 2\mathbb{N}\big\} \;\cup\; \big\{\mathbb{R}^+ \times \{+\infty\}\big\} \tag{2.6}$$

and for $(\delta, T) \in \mathcal{H}$ we define the quantity $\lambda_{\delta,T}$ by the equation

$$Q_T\big(\lambda_{\delta,T}\big) \;=\; e^{-\delta}\,. \tag{2.7}$$

As we show in Appendix A, for $T < \infty$ the function $Q_T(\cdot)$ is analytic and decreasing on $(\lambda_0^T, +\infty)$, with $\lambda_0^T = -\frac{1}{2}\log\big(1 + (\tan\frac{\pi}{T})^2\big) < 0$, and such that $Q_T(\lambda) \to +\infty$ as $\lambda \downarrow \lambda_0^T$ and $Q_T(\lambda) \to 0$ as $\lambda \to +\infty$. In particular, equation (2.7) has exactly one solution for every $\delta \in \mathbb{R}$, so that $\lambda_{\delta,T}$ is well-defined. For $T = \infty$, $Q_T(.)$ is analytic and decreasing on $[0, \infty)$, $Q_T(0) = 1$ and $Q_T(\lambda) \to 0$ as $\lambda \to +\infty$, while $Q_T(\lambda) = \infty$ for $\lambda < 0$. This implies that equation (2.7) has exactly one solution $\lambda_{\delta,\infty}$ for every $\delta \geq 0$ and zero solution for $\delta < 0$. In the next paragraph we are going to show that when $\lambda_{\delta,T}$ exists, it is nothing but the free energy $\phi(\delta, T)$ (in agreement with Theorem 1).

We are finally ready to introduce, for $(\delta, T) \in \mathcal{H}$, the basic law $\mathcal{P}_{\delta,T}$, under which the sequence of vectors $\{(\xi_i, \varepsilon_i)\}_{i \geq 1}$, taking values in $\mathbb{N} \times \{\pm 1, 0\}$, is i.i.d. with marginal law

$$\mathcal{P}_{\delta,T}\big((\xi_1, \varepsilon_1) = (n, j)\big) \;:=\; e^{\delta}\, q_T^{|j|}(n)\, e^{-\lambda_{\delta,T}\, n}\,, \qquad n \in \mathbb{N},\; j \in \{\pm 1, 0\}\,. \tag{2.8}$$

Note that (2.7) ensures that this indeed is a probability law. Then we set $\tau_0 = 0$ and $\tau_n = \xi_1 + \cdots + \xi_n$, for $n \geq 1$. We denote by τ both the sequence of variables $\{\tau_n\}_{n \geq 0}$ and the corresponding random subset of $\mathbb{N} \cup \{0\}$ defined by $\tau = \bigcup_{n \geq 0}\{\tau_n\}$, so that expressions like $\{N \in \tau\}$ make sense. Notice that $\{\tau_n\}_{n \geq 0}$ under $\mathcal{P}_{\delta,T}$ is a classical *renewal process*, because the increments $\{\tau_n - \tau_{n-1}\}_{n\geq 1} = \{\xi_n\}_{n\geq 1}$ are i.i.d. positive random variables, with law

$$\mathcal{P}_{\delta,T}\big(\tau_1 = n\big) \;:=\; e^{\delta}\, q_T(n)\, e^{-\lambda_{\delta,T}\, n}\,, \qquad n \in \mathbb{N}\,. \tag{2.9}$$

Because of the periodicity of the simple random walk, $q_T(n) = 0$ for all odd $n \in \mathbb{N}$ and $q_T(n) > 0$ for all even $n \in \mathbb{N}$ (we recall that we only consider the case of even T). Therefore, the renewal process is *periodic with period 2*.

We now have all the ingredients to give an explicit expression of the partition function in terms of the jumps made by S between interfaces. This can be done for $(\delta, T) \in \mathcal{H}$ and for $Z_{N,\delta}^{T,c}$ (recall (2.1)) as follows. For $k, n \in \mathbb{N}$, $k \leq n$, we define the set

$$\mathcal{S}_{k,n} \;:=\; \big\{ t \in (\mathbb{N} \cup \{0\})^{k+1} :\; 0 = t_0 < t_1 < \ldots < t_k = n \big\}\,.$$

Then for $\lambda \in \mathbb{R}$ and N even we can write

$$\begin{aligned} Z_{N,\delta}^{T,c} \;&=\; \sum_{k=1}^{N} \sum_{\sigma \in \{-1,0,1\}^k} \sum_{t \in \mathcal{S}_{k,N}} \prod_{l=1}^{k} e^{\delta}\, q_T^{|\sigma_l|}(t_l - t_{l-1}) \\ &=\; e^{\lambda N} \sum_{k=1}^{N} \sum_{\sigma \in \{-1,0,1\}^k} \sum_{t \in \mathcal{S}_{k,N}} \prod_{l=1}^{k} e^{\delta}\, q_T^{|\sigma_l|}(t_l - t_{l-1})\, e^{-\lambda(t_l - t_{l-1})}. \end{aligned} \tag{2.10}$$

Then setting $\lambda = \lambda_{\delta,T}$ and recalling (2.8), we can rewrite (2.10) as

$$Z_{N,\delta}^{T,c} \;=\; e^{\lambda_{\delta,T}\cdot N}\, \mathcal{P}_{\delta,T}\big(N \in \tau\big)\,. \tag{2.11}$$

We stress that this equation retains a crucial importance in our approach. In fact the behavior of $Z_{N,\delta}^{T,c}$ is reduced to the asymptotic properties of the renewal process τ.

The next step is to lift relation (2.11) from the constrained partition function to the *constrained polymer measure* $\mathbf{P}_{N,\delta}^{T,c}$, defined for N even as

$$\mathbf{P}_{N,\delta}^{T,c}(\,\cdot\,) \;:=\; \mathbf{P}_{N,\delta}^{T}\big(\,\cdot\,\big|\, S_N \in T\mathbb{Z}\,\big)\,.$$

Recalling the definition (1.6) of $L_{N,T}$, for $(\delta, T) \in \mathcal{H}$, for $k \leq N$, $t \in \mathcal{S}_{k,N}$ and $\sigma \in \{\pm 1, 0\}^k$, in analogy to (2.10) we can write

$$\begin{aligned} &\mathbf{P}_{N,\delta}^{T,c}\Big(L_{N,T} = k,\; (\tau_i^T, \varepsilon_i^T) = (t_i, \sigma_i),\; 1 \leq i \leq k\Big) \\ &\qquad =\; \frac{e^{\lambda_{\delta,T}\, N}}{Z_{N,\delta}^{T,c}} \prod_{l=1}^{k} e^{\delta}\; q_T^{|\sigma_l|}(t_l - t_{l-1})\; e^{-\lambda_{\delta,T}\,(t_l - t_{l-1})}. \end{aligned} \tag{2.12}$$

Therefore from (2.8) and (2.11) we obtain

$$\begin{aligned} &\mathbf{P}_{N,\delta}^{T,c}\Big(L_{N,T} = k,\; (\tau_i^T, \varepsilon_i^T) = (t_i, \sigma_i),\; 1 \leq i \leq k\Big) \\ &\qquad =\; \mathcal{P}_{\delta,T}\Big(L_N = k,\; (\tau_i, \varepsilon_i) = (t_i, \sigma_i),\; 1 \leq i \leq k \,\Big|\, N \in \tau\Big)\,, \end{aligned} \tag{2.13}$$

where $L_N := \sup\{j \geq 1 : \tau_j \leq N\}$ in analogy with (1.6). Thus the process $\{(\tau_i^T, \varepsilon_i^T)\}_i$ under $\mathbf{P}_{N,\delta}^{T,c}$ is distributed like $\{(\tau_i, \varepsilon_i)\}_i$ under the explicit law $\mathcal{P}_{\delta,T}$, conditioned on the event $\{N \in \tau\}$. The crucial point is that $\{\tau_i\}_i$ under $\mathcal{P}_{\delta,T}$ is a genuine *renewal process*. This fact is the key to the path results that we prove in the next section, because we will show that the constrained law $\mathbf{P}_{N,\delta}^{T,c}$ is not too different from the original law $\mathbf{P}_{N,\delta}^{T}$.

2.3. **Proof of Theorem 1.** Thanks to Lemma 3, to prove Theorem 1 it suffices to show that for every sequence $(T_N)_N$ such that $T_N \to T_\infty$ as $N \to \infty$ we have

$$\lim_{N\to\infty,\, N \text{ even}} \frac{1}{N} \log Z_{\delta,N}^{T_N,c} \;=\; \begin{cases} \big(Q_{T_\infty}\big)^{-1}(e^{-\delta}) & \text{if } T_\infty < \infty \\ \big(Q_{T_\infty}\big)^{-1}(e^{-\delta} \wedge 1) & \text{if } T_\infty = \infty \end{cases}, \tag{2.14}$$

where we recall that $Q_T(\cdot)$ was introduced in (1.9). Recall also that for $(\delta, T) \in \mathcal{H}$ we have $\big(Q_T\big)^{-1}(e^{-\delta}) = \lambda_{\delta,T}$ (see (2.7)).

Consider first the case when $T_\infty < \infty$, i.e., $T_\infty \in \mathbb{N}$. Then the sequence $(T_N)_N$ takes eventually the constant value $T_N = T_\infty$ and thanks to (2.11) and (2.7) we can write

$$\frac{1}{N} \log Z_{\delta,N}^{T_\infty,c} \;=\; (Q_{T_\infty})^{-1}(e^{-\delta}) \;+\; \frac{1}{N} \log \mathcal{P}_{\delta,T_\infty}\big(N \in \tau\big)\,. \tag{2.15}$$

Therefore it remains to show that the last term in the r.h.s. vanishes as $N \to \infty$, N even, and we are done (as a by-product, we also show that $\lambda_{\delta,T_\infty}$ coincides with the free energy $\phi(\delta, T_\infty)$). We recall that the process $\tau = \{\tau_n\}_n$ under $\mathcal{P}_{\delta,T_\infty}$ is a classical renewal process with step-mean

$$m(\delta, T_\infty) \;:=\; \mathcal{E}_{\delta,T_\infty}\big(\tau_1\big) \;<\; +\infty\,. \tag{2.16}$$

The fact that $m(\delta, T_\infty) < +\infty$ is easily checked by (2.9), because by construction $\lambda_{\delta,T_\infty} > \lambda_0^{T_\infty}$, cf. (2.5), (2.7) and the following lines. Since the renewal process $\big(\{\tau_n\}_n, \mathcal{P}_{\delta,T_\infty}\big)$ has period 2, the Renewal Theorem yields

$$\lim_{N\to\infty,\, N \text{ even}} \mathcal{P}_{\delta,T_\infty}\big(N \in \tau\big) \;=\; \frac{2}{m(\delta, T_\infty)} \;>\; 0\,, \tag{2.17}$$

and looking back to (2.15) we see that (2.14) is proven.

Next we consider the case when $T_\infty = +\infty$, that is $T_N \to +\infty$ as $N \to \infty$. We can rewrite equation (2.15) as

$$\frac{1}{N} \log Z_{\delta,N}^{T_N,c} \;=\; (Q_{T_N})^{-1}(e^{-\delta}) \;+\; \frac{1}{N} \log \mathcal{P}_{\delta,T_N}\big(N \in \tau\big)\,. \tag{2.18}$$

We start considering the first term in the r.h.s. of (2.18), by proving the following lemma.

Lemma 4. *For every $\delta \in \mathbb{R}$*

$$\lim_{T\to\infty,\, T\in 2\mathbb{N}} \big(Q_T\big)^{-1}(e^{-\delta}) \;=\; \big(Q_\infty\big)^{-1}(e^{-\delta} \wedge 1)\,. \tag{2.19}$$

Proof. To this purpose, we observe that as $T \to \infty$ the variable τ_1^T, defined in (2.3) converges a.s. toward $\tau_1^\infty := \inf\{i > 0 :\ S_i = 0\}$, i.e., the first return to zero of the simple random walk. Accordingly, by dominated convergence (or by direct verification), $Q_T(\lambda)$ converges as $T \to \infty$, *for every* $\lambda \in [0, +\infty)$, toward $Q_\infty(\lambda) = 1 - \sqrt{1 - e^{-2\lambda}}$. Since $Q_\infty(\cdot)$ is strictly decreasing, it is easily checked that also the inverse functions converge, i.e., for every $y \in (0, 1]$ we have $(Q_T)^{-1}(y) \to (Q_\infty)^{-1}(y)$ as $T \to \infty$, so that (2.19) is checked for $\delta \geq 0$. On the other hand, when $\delta < 0$ we have $\lambda_0^T < (Q_T)^{-1}(e^{-\delta}) < 0$, because as we already mentioned $Q_T(\cdot)$ is decreasing and $Q_T(\lambda) \to \infty$ as $\lambda \downarrow \lambda_0^T$ and $Q_T(0) = 1$. Moreover, λ_0^T vanishes as $T \to \infty$ (see (A.6)) and consequently $(Q_T)^{-1}(e^{-\delta}) \to 0$ as $T \to \infty$. Hence (2.19) holds also for $\delta < 0$. □

Using Lemma 4 and the fact that $\mathcal{P}_{\delta,T_N}\big(N \in \tau\big) \leq 1$, by (2.18) we obtain

$$\limsup_{N\to\infty,\, N \text{ even}} \frac{1}{N} \log Z_{\delta,N}^{T_N,c} \;\leq\; \big(Q_\infty\big)^{-1}(e^{-\delta} \wedge 1)\,,$$

hence to complete the proof of (2.14) it remains to show that for every $\delta \in \mathbb{R}$

$$\liminf_{N\to\infty,\, N \text{ even}} \frac{1}{N} \log Z_{\delta,N}^{T_N,c} \;\geq\; \big(Q_\infty\big)^{-1}(e^{-\delta} \wedge 1)\,. \tag{2.20}$$

We start considering the case when $\delta \leq 0$, hence $\big(Q_\infty\big)^{-1}(e^{-\delta} \wedge 1) = 0$. We give a very rough lower bound on $Z_{\delta,N}^{T_N,c}$, namely for N even we can write

$$Z_{\delta,N}^{T_N,c} \;\geq\; \mathbf{E}\Big(\exp\big(H_{N,\delta}^{T_N}(S)\big)\, \mathbf{1}_{\{S_i \notin T_N\mathbb{Z},\, \forall 1\leq i\leq N-1\}} \mathbf{1}_{\{S_N=0\}}\Big) \;=\; e^{\delta} \cdot q_{T_N}^0(N)\,, \tag{2.21}$$

where we recall that $q_{T_N}^0(N) = \mathbf{P}\big(\tau_1^{T_N} = N; S_N = 0\big)$ was defined in (2.4). (If N is odd, the same formula holds just replacing N by $N-1$, and the following considerations are easily adapted.) So we are left with showing that $q_{T_N}^0(N)$ does not decay exponentially fast as $N \to \infty$: by the explicit formula (A.7) we have

$$q_{T_N}^0(N) \;\geq\; \frac{2}{T_N} \cos^{N-2}\left(\frac{\pi}{T_N}\right) \sin^2\left(\frac{\pi}{T_N}\right).$$

At this stage, by using the fact that $\sin^2(x) \sim x^2$ as $x \to 0$ we can assert that for N large enough $\sin(\pi/T_N) \geq \pi/(2T_N)$ and since by assumption $T_N \leq N$ we obtain

$$\mathbf{P}_1\big(\tau_1^{T_N} = N-1; S_{N-1} = 0\big) \geq \frac{\pi^2}{2N^3} e^{(N-2)\log\cos\left(\frac{\pi}{T_N}\right)},$$

which by (2.21) shows that (2.20) holds (note that the r.h.s. of (2.20) is zero for $\delta \leq 0$).

Finally, we have to prove that equation (2.20) holds true for $\delta > 0$. By (2.18) and Lemma 4 it suffices to show that

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathcal{P}_{\delta,T_N}\big(N \in \tau\big) \;=\; 0\,. \tag{2.22}$$

This is not straightforward, because the law $\mathcal{P}_{\delta,T_N}$ changes with N and therefore some uniformity is needed. Let us be more precise: by the Renewal Theorem, see (2.17), *for fixed* T we have that, as $n \to \infty$ along the even numbers,

$$\mathcal{P}_{\delta,T}\big(n \in \tau\big) \;\longrightarrow\; \frac{2}{m(\delta,T)}\,,$$

where $m(\delta, T)$ was introduced in (2.16). At the same time, as $T \to \infty$ we have

$$m(\delta,T) \longrightarrow m(\delta,\infty)\,,$$

as we prove in Lemma 6 below. Since $T_N \to \infty$ as $N \to \infty$, the last two equations suggest that for N large $\mathcal{P}_{\delta,T_N}\big(N \in \tau\big)$ should be close to $2/m(\delta,\infty)$. To show that this is indeed the case, we are going to apply Theorem 2 in [14], which is a uniform version of the Renewal Theorem. First recall that, by Lemma 4, $\lambda_{\delta,T} \to \lambda_{\delta,\infty} > 0$ as $T \to \infty$, $T \in 2\mathbb{N}$, and moreover $\lambda_{\delta,T} > 0$ for every $T \in 2\mathbb{N}$, hence there exist $C_1, C_2 > 0$ such that $C_1 \leq \lambda_{\delta,T} \leq C_2$ for every $T \in 2\mathbb{N}$. We are ready to verify the following two conditions:

(1) when $\delta > 0$ is fixed and T varies in $2\mathbb{N}$, the family of renewal process $\big(\{\tau_n\}_n, \mathcal{P}_{\delta,T}\big)$ restricted to the even numbers is uniformly aperiodic, in the sense of Definition 1 in [14], because $\mathcal{P}_{\delta,T}(\tau_1 = 2) = e^{\delta} q_T(2)\, e^{-2\lambda_{\delta,T}} \geq (e^{\delta}/2) \cdot e^{-2C_2} > 0$ for all $T \in 2\mathbb{N}$;

(2) when $\delta > 0$ is fixed and T varies in $2\mathbb{N}$, the family of renewal process $\big(\{\tau_n\}_n, \mathcal{P}_{\delta,T}\big)$ have uniformly summable tails, in the sense of Definition 2 in [14], because

$$\mathcal{P}_{\delta,T}\big(\tau_1 \geq t\big) \;\leq\; \sum_{r=t}^{\infty} e^{-C_1 r} \;=\; \frac{e^{-C_1 t}}{1 - e^{-C_1}}\,.$$

We can therefore apply Theorem 2 in [14], which yields the following Lemma. This implies (2.22) and therefore the proof of Theorem 1 is completed. $\square$

Lemma 5. *Fix $\delta > 0$. Then for every $\varepsilon > 0$ there exist $N_0 \in \mathbb{N}$ such that for every $T \in 2\mathbb{N}$ and for all $N \geq N_0$, N even, we have*

$$\left| \mathcal{P}_{\delta,T}\big(N \in \tau\big) \; - \; \frac{2}{m(\delta,\infty)} \right| \leq \varepsilon \,.$$

Lemma 6. *For all $\delta > 0$ and $k \in \mathbb{N}$*

$$\lim_{T\to\infty} \mathcal{E}_{\delta,T}\big((\tau_1)^k\big) \;=\; \mathcal{E}_{\delta,\infty}\big((\tau_1)^k\big) \,. \tag{2.23}$$

Proof. By Lemma 4 we know that for $\delta > 0$ we have $\lambda_{\delta,T} \to \lambda_{\delta,\infty} > 0$ as $T \to \infty$, $T \in 2\mathbb{N}$. Thus, by writing

$$\mathcal{E}_{\delta,T}\big((\tau_1)^k\big) \;=\; e^{\delta} \sum_{n=1}^{\infty} n^k \, q_T(n) \, e^{-\lambda_{\delta,T}\, n} \,,$$

it suffices to apply the Dominated Convergence Theorem (since $q_T(n) \leq 1$). $\square$

Remark 2. Now that we have proven that the free energy $\phi(\delta,T)$ indeed equals the r.h.s. of (2.14), we can restate Lemma 4 in the following way:

$$\lim_{T\to\infty} \phi(\delta,T) \;=\; \phi(\delta,\infty) \qquad\qquad \forall \delta \in \mathbb{R} \,. \tag{2.24}$$

Remark 3. For $(\delta,T) \in \mathcal{H}$ we know that $\lambda_{\delta,T} = \phi(\delta,T)$. Consequently, we will use $\phi(\delta,T)$ instead of $\lambda_{\delta,T}$ in what follows. $\square$

3. Proof of Theorem 2 (i)

This section is devoted to the proof of part (i) of Theorem 2. We recall that $\delta > 0$ is fixed and that $T_N - \frac{1}{c_\delta} \log N \to -\infty$ as $N \to \infty$, where c_δ is defined in (1.12).

We recall that $(\tau_i^T, \varepsilon_i^T)_{i\geq 1}$ defined in (2.3) under $\mathbf{P}_{N,\delta}^{T_N}$ represents the jump process of the polymer between the interfaces, whereas $(\tau_i, \varepsilon_i)_{i\geq 1}$ introduced in (2.8) under the law $\mathcal{P}_{\delta,T_N}$ represents an auxiliary renewal process. For $N \geq 1$ we set

$$Y_N^T = \sum_{i=1}^{N} \varepsilon_i^T \qquad \text{and recall from (1.6)} \qquad L_{N,T} = \sup\{j \geq 1 : \tau_j^T \leq N\} \,, \tag{3.1}$$

and

$$Y_N = \sum_{i=1}^{N} \varepsilon_i \qquad \text{and} \qquad L_N = \sup\{j \geq 1 : \tau_j \leq N\} \,. \tag{3.2}$$

3.1. **General strategy.** Let us describe the strategy of our proof. The aim is to determine the asymptotic behavior of S_N under $\mathbf{P}^{T_N}_{N,\delta}$ as $N \to \infty$. The starting point is given by the following considerations:

- by definition we have $S_N = T \cdot Y^T_{L_{N,T}} + O(T)$, hence the behavior of S_N can be recovered from that of $L_{N,T}$ and $\{Y^T_n\}_n$;
- it turns out that the *free* polymer measure $\mathbf{P}^{T_N}_{N,\delta}$ is not too different from the *constrained* one $\mathbf{P}^{T_N,c}_{N,\delta} = \mathbf{P}^{T_N}_{N,\delta}(\,\cdot\,|S_N \in T_N\mathbb{Z})$, which in turn is closely linked to the law $\mathcal{P}_{\delta,T_N}$ introduced in §2.2, cf. in particular (2.13).

For these reasons, the first part of the proof of Theorem 2 consists in determining the asymptotic behavior of $\{Y_n\}_n$ and L_N under $\mathcal{P}_{\delta,T_N}$. This is carried out in §3.3 (Step 1) and §3.4 (Step 2) below, exploiting ideas and techniques from random walks and renewal theory. The second part of the proof is devoted to showing that the law $\mathcal{P}_{\delta,T_N}$ can indeed be replaced by $\mathbf{P}^{T_N,c}_{N,\delta}$, see §3.5 (Step 3), and finally by $\mathbf{P}^{T_N}_{N,\delta}$, see §3.6 (Step 4).

Let us give a closer (heuristic) look at the core of the proof. *For fixed* T, the process $\{Y_n\}_n$ under $\mathcal{P}_{\delta,T}$ is just a symmetric random walk on $\mathbb{Z}$ with step law

$$\mathcal{P}_{\delta,T}(Y_1 = j) \;=\; \mathcal{P}_{\delta,T}(\varepsilon_1 = j) \;=\; e^{\delta}\, Q^{|j|}_T(\phi(\delta,T)) \qquad j \in \{\pm 1, 0\}\,,$$

see equations (2.8) and (2.4), (2.5). In particular the Central Limit Theorem yields

$$Y_N \;\approx\; C_T\sqrt{N} \qquad \text{under } \mathcal{P}_{\delta,T} \text{ as } N \to \infty\,, \tag{3.3}$$

where $C_T = \sqrt{2e^{\delta}Q^1_T(\phi(\delta,T))}$ is the standard deviation of Y_1.

Of course we are interested in the case when $T = T_N$ is not fixed anymore but varies with N, more precisely $T_N \to \infty$ as $N \to \infty$. Then it is easy to see that $C_{T_N} \to 0$. However, if it happens that $C_{T_N}\sqrt{N} \to \infty$ as $N \to \infty$, one may hope that equation (3.3) still holds with T replaced by T_N. This is indeed true, as we are going to show. To determine the asymptotic behavior of C_T, the following lemma is useful.

Lemma 7. *Fix* $\delta > 0$. *Then as* $T \to \infty$

$$Q^1_T(\phi(\delta,T)) \;=\; \sqrt{1 - e^{-2\phi(\delta,\infty)}}\; e^{-c_\delta T}\,(1 + o(1))\,, \tag{3.4}$$

where $c_\delta = \phi(\delta,\infty) + \log(1 + \sqrt{1 - e^{-2\phi(\delta,\infty)}})$ *(recall* (1.12)*).*

This shows that the condition $C_{T_N}\sqrt{N} \to \infty$ as $N \to \infty$ is equivalent to $T_N - \frac{\log N}{c_\delta} \to -\infty$, *which is exactly the hypothesis of part (i) of Theorem 2.* As we mentioned, in this case we show that (3.3) still holds, so that

$$Y_N \;\approx\; C_{T_N}\sqrt{N} \;\approx\; C^* e^{-\frac{c_\delta}{2}T_N}\sqrt{N} \qquad \text{under } \mathcal{P}_{\delta,T_N} \text{ as } N \to \infty\,, \tag{3.5}$$

with $C^* = \sqrt{2e^{\delta}\sqrt{1 - e^{-2\phi(\delta,\infty)}}}$.

Now let us come back to S_N. By definition we have $S_N = T_N\, Y^{T_N}_{L_{N,T_N}} + O(T_N)$ and from equation (1.7) we get $L_{N,T_N} \approx cN$, with $c = \phi'(\delta,\infty) > 0$. Moreover, as we already mentioned, the law $\mathcal{P}_{\delta,T_N}$ can be replaced by the original polymer measure $\mathbf{P}^{T_N}_{N,\delta}$ without changing the asymptotic behavior. Together with (3.5), these considerations yield

$$S_N \;\approx\; T_N \cdot Y^{T_N}_{cN} \;\approx\; C_\delta\,(e^{-\frac{c_\delta}{2}T_N}\,T_N)\,\sqrt{N} \qquad \text{under } \mathbf{P}^{T_N}_{N,\delta} \text{ as } N \to \infty\,,$$

where $C_\delta := C^* \sqrt{c} = \sqrt{2e^\delta \phi'(\delta,\infty)\sqrt{1-e^{-2\phi(\delta,\infty)}}}$. Notice that this matches exactly with the result of Theorem 2.

Proof of Lemma 7. We can rewrite the second relation in (A.3) as

$$Q_T^1(\lambda) \;=\; \sqrt{1-e^{-2\lambda}} \cdot e^{-\widetilde{c}_\lambda T} \cdot \frac{1}{1+\left(\frac{1-\sqrt{1-e^{-2\lambda}}}{1+\sqrt{1-e^{-2\lambda}}}\right)^T}\,, \tag{3.6}$$

where $\widetilde{c}_\lambda \;:=\; \lambda + \log(1+\sqrt{1-e^{-2\lambda}})$. We have to replace λ by $\phi(\delta,T)$ in this relation and study the asymptotic behavior as $T \to \infty$.

Observe that $\phi(\delta,T)$ and $\phi(\delta,\infty)$ are both strictly positive, since $\delta > 0$, and moreover $\phi(\delta,T) \to \phi(\delta,\infty)$ as $T \to \infty$ (see Remark 2). This easily implies that the last factor in the r.h.s. of (3.6) is $1+o(1)$, hence as $T \to \infty$

$$Q_T^1(\phi(\delta,T)) \;=\; \sqrt{1-e^{-2\phi(\delta,\infty)}}\, e^{-\widetilde{c}_{\phi(\delta,T)} T}\,(1+o(1))\,.$$

To prove (3.4) it remains to show that $\widetilde{c}_{\phi(\delta,T)} T = c_\delta T + o(1)$ as $T \to \infty$. Since $c_\delta = \widetilde{c}_{\phi(\delta,\infty)}$, this follows once we show that $|\phi(\delta,T) - \phi(\delta,\infty)| = o(\frac{1}{T})$.

To this purpose, we fix $\varepsilon > 0$ such that $\phi(\delta,T) \geq \varepsilon$ for every T. By equation (A.4), there exists $\kappa = \kappa_\varepsilon > 0$ such that, *uniformly for* $\lambda \in [\varepsilon,\infty)$,

$$Q_T(\lambda) \;=\; 1 - \sqrt{1-e^{-2\lambda}} + O(e^{-\kappa T}) \qquad (T \to \infty)\,.$$

Recalling that $Q_\infty(\lambda) = 1 - \sqrt{1-e^{-2\lambda}}$ and that $e^{-\delta} = Q_T(\phi(\delta,T)) = Q_\infty(\phi(\delta,\infty))$ by Theorem 1, we obtain

$$Q_\infty(\phi(\delta,T)) - Q_\infty(\phi(\delta,\infty)) \;=\; O(e^{-\kappa T}) \qquad (T \to \infty)\,.$$

Since $Q_\infty(\lambda)$ is continuously differentiable with non-zero derivative for $\lambda > 0$, it follows that $\phi(\delta,T) - \phi(\delta,\infty) = O(e^{-\kappa T})$, and the proof is completed. $\square$

3.2. **Preparation.** We start the proof of Theorem 2 by rephrasing equation (1.13), which is our goal, in a slightly different form. We recall that $T_N - \frac{1}{c_\delta}\log N \to -\infty$ as $N \to \infty$, or equivalently $e^{-c_\delta T_N} N \to \infty$, and that by construction $|S_N - Y^{T_N}_{L_{N,T_N}} \cdot T_N| \leq T_N$. Therefore equation (1.13) is equivalent to the following: for all $x \in \mathbb{R}$

$$\lim_{N\to\infty} \mathbf{P}^{T_N}_{N,\delta}\left(\frac{Y^{T_N}_{L_{N,T_N}}}{C_\delta \sqrt{e^{-c_\delta T_N} N}} \leq x \right) \;=\; P(\mathcal{N}(0,1) \leq x)\,, \tag{3.7}$$

where

$$C_\delta = \sqrt{2\, e^\delta\, \phi'(\delta,\infty)\, \sqrt{1-e^{-2\phi(\delta,\infty)}}}\,. \tag{3.8}$$

Recall the definition (2.9) of the renewal process $(\tau, \mathcal{P}_{\delta,T})$. For $\delta > 0$ and $T \in 2\mathbb{N} \cup \{+\infty\}$, we set

$$s_T \;:=\; \frac{1}{\mathcal{E}_{\delta,T}(\tau_1)} \in (0,\infty)\,. \tag{3.9}$$

Differentiating the relation $Q_T(\phi(\delta,T)) = e^{-\delta}$ one obtains $\phi'(\delta,T) = -e^{-\delta}/Q_T'(\phi(\delta,T))$, and by direct computation

$$\mathcal{E}_{\delta,T}(\tau_1) \;=\; e^\delta \sum_{n\in\mathbb{N}} n\, q_T(n)\, e^{-\phi(\delta,T) n} \;=\; -e^\delta\, Q_T'(\phi(\delta,T)) \;=\; \frac{1}{\phi'(\delta,T)}\,, \qquad \forall T \in 2\mathbb{N}\,. \tag{3.10}$$

In particular, $\phi'(\delta,\infty) = s_\infty$. Recalling Lemma 7 and setting $Q^1_{T_N} := Q^1_{T_N}(\phi(\delta,T_N))$ for conciseness, we can finally restate (3.7) as

$$\lim_{N\to\infty} \mathbf{P}^{T_N}_{N,\delta}\left(\frac{Y^{T_N}_{L_{N,T_N}}}{\sqrt{s_\infty}\sqrt{2e^\delta Q^1_{T_N} N}} \le x\right) \;=\; P(\mathcal{N}(0,1)\le x)\,, \tag{3.11}$$

which is exactly what we are going to prove. This will be achieved in four steps. We stress that the assumption $T_N - \frac{1}{c_\delta}\log N \to -\infty$ as $N\to\infty$ is equivalent to $Q^1_{T_N}\cdot N\to\infty$.

3.3. **Step 1.** In this step we consider the auxiliary renewal process of law $\mathcal{P}_{\delta,T_N}$ and we prove that for $x\in\mathbb{R}$

$$\lim_{N\to\infty} \mathcal{P}_{\delta,T_N}\left(\frac{Y_N}{\sqrt{2e^\delta Q^1_{T_N} N}} \le x\right) \;=\; P(\mathcal{N}(0,1)\le x)\,. \tag{3.12}$$

Under the law $\mathcal{P}_{\delta,T_N}$, $(\varepsilon_1,\dots,\varepsilon_N)$ are symmetric i.i.d. random variables taking values $-1,0,1$. Therefore, they satisfy

$$\mathcal{E}_{\delta,T_N}(|\varepsilon_1|^3) = \mathcal{E}_{\delta,T_N}((\varepsilon_1)^2) = 2e^\delta Q^1_{T_N}, \tag{3.13}$$

and we can apply the Berry-Esséen Theorem which gives

$$\left|\mathcal{P}_{\delta,T_N}\left(\frac{Y_N}{\alpha_\delta(N,T_N)}\le x\right) - P(\mathcal{N}(0,1)\le x)\right| \;\le\; \frac{3\,\mathcal{E}_{\delta,T_N}(|\varepsilon_1|^3)}{\mathcal{E}_{\delta,T_N}(\varepsilon_1^2)^{\frac32}\sqrt{N}} \;=\; \frac{3}{\sqrt{2e^\delta Q^1_{T_N} N}}\,. \tag{3.14}$$

Since $Q^1_{T_N}\cdot N\to\infty$ by assumption, equation (3.12) is proved.

3.4. **Step 2.** In this step we prove that for $x\in\mathbb{R}$

$$\lim_{N\to\infty} \mathcal{P}_{\delta,T_N}\left(\frac{Y_{L_N}}{\sqrt{s_\infty}\sqrt{2e^\delta Q^1_{T_N} N}} \le x\right) \;=\; P(\mathcal{N}(0,1)\le x)\,. \tag{3.15}$$

The idea is to show that $L_N \approx s_\infty\cdot N$ and then to apply (3.12). We need the following

Lemma 8. *For every $\varepsilon>0$ there exists $T_0 = T_0(\varepsilon)\in\mathbb{N}$ such that*

$$\lim_{N\to\infty}\sup_{T\ge T_0} \mathcal{P}_{\delta,T}\left(\left|\frac{L_N}{N} - s_\infty\right| > \varepsilon\right) \;=\; 0\,. \tag{3.16}$$

Proof. Lemma 6 yields $s_T\to s_\infty$ as $T\to\infty$ (we recall the definition (3.9)). Therefore we fix $T_0 = T_0(\varepsilon)$ such that $|s_\infty - s_T|\le\frac{\varepsilon}{2}$ for $T\ge T_0$ and consequently

$$\mathcal{P}_{\delta,T}\left(\left|\frac{L_N}{N}-s_\infty\right|>\varepsilon\right) \;\le\; \mathcal{P}_{\delta,T}\left(\left|\frac{L_N}{N}-s_T\right|>\frac{\varepsilon}{2}\right).$$

Setting $\xi_i = \tau_i - \tau_{i-1}$ and $\widetilde{\xi}_i = \xi_i - \frac{1}{s_T}$, by Chebychev's inequality we get

$$\begin{aligned}\mathcal{P}_{\delta,T}\left(\frac{L_N}{N} > s_T+\varepsilon\right) \;&=\; \mathcal{P}_{\delta,T}\big(\tau_{\lfloor (s_T+\varepsilon)N\rfloor}\le N\big) \;=\; \mathcal{P}_{\delta,T}\left(-\widetilde{\xi}_1 - \cdots - \widetilde{\xi}_{\lfloor (s_T+\varepsilon)N\rfloor} \ge \frac{\varepsilon N}{s_T}\right)\\ &\le\; \frac{s_T^2\,(s_T+\varepsilon)\,\mathcal{E}_{\delta,T}\big(\widetilde{\xi}_1^2\big)}{\varepsilon^2 N}\,.\end{aligned}$$

By Lemma 6, both the sequences $T \mapsto s_T$ and $T \mapsto \mathcal{E}_{\delta,T}\big(\tilde{\xi}_1^2\big)$ are bounded and therefore the r.h.s. above vanishes as $N \to \infty$, uniformly in T. The event $\{\frac{L_N}{N} < s_T - \varepsilon\}$ is dealt with analogous arguments and the proof is completed. $\square$

We set

$$\frac{Y_{L_N}}{\sqrt{s_\infty}\sqrt{2e^\delta Q^1_{T_N} N}} \;=\; \frac{Y_{\lfloor s_\infty N \rfloor}}{\sqrt{s_\infty}\sqrt{2e^\delta Q^1_{T_N} N}} \;+\; \frac{Y_{L_N} - Y_{\lfloor s_\infty N \rfloor}}{\sqrt{s_\infty}\sqrt{2e^\delta Q^1_{T_N} N}} \;=:\; V_N \;+\; G_N\,.$$

Step 1, see equation (3.12), entails directly that V_N converges in law towards $\mathcal{N}(0,1)$. Therefore, it remains to prove that G_N converges in probability to 0. For $\eta, \varepsilon > 0$ we write

$$\begin{aligned} \mathcal{P}_{\delta,T_N}\big(|G_N| > \eta\big) &\le \mathcal{P}_{\delta,T_N}\left(|G_N| > \eta, \left|\frac{L_N}{N} - s_\infty\right| \le \varepsilon\right) + \mathcal{P}_{\delta,T_N}\left(\left|\frac{L_N}{N} - s_\infty\right| > \varepsilon\right) \\ &\le \mathcal{P}_{\delta,T_N}\big(U_{\varepsilon,N} > \eta\big) \;+\; \mathcal{P}_{\delta,T_N}\left(\left|\frac{L_N}{N} - s_\infty\right| > \varepsilon\right), \end{aligned} \tag{3.17}$$

3.5. **Step 3.** This is the most delicate step, where we show that one can replace the free measure $\mathcal{P}_{\delta,T_N}$ by the constrained one $\mathcal{P}_{\delta,T_N}\big(\,\cdot\,\big|\,N \in \tau\big)$. More precisely, we prove that for $x \in \mathbb{R}$

$$\lim_{N\to\infty,\, N \text{ even}} \mathcal{P}_{\delta,T_N}\left(\frac{Y_{L_N}}{\sqrt{s_\infty}\sqrt{2e^\delta Q^1_{T_N} N}} \le x \,\middle|\, N \in \tau\right) \;=\; P(\mathcal{N}(0,1) \le x)\,. \tag{3.18}$$

We note that one can safely replace L_N with $L_{N-\lfloor\sqrt{T_N}\rfloor}$ in the l.h.s., because $Y_{L_{N-\lfloor\sqrt{T_N}\rfloor}}$ differs from Y_{L_N} at most by ± 1. The same is true for equation (3.15), that we rewrite for convenience:

$$\lim_{N\to\infty} \mathcal{P}_{\delta,T_N}\left(\frac{Y_{L_{N-\lfloor\sqrt{T_N}\rfloor}}}{\sqrt{s_\infty}\sqrt{2e^\delta Q^1_{T_N} N}} \le x\right) \;=\; P(\mathcal{N}(0,1) \le x)\,, \tag{3.19}$$

By summing over the locations of the last point t in τ before $N - \lfloor\sqrt{T_N}\rfloor$ and of the first point r in τ after $N - \lfloor\sqrt{T_N}\rfloor$, and using the Markov property, we obtain

$$\begin{aligned} &\mathcal{P}_{\delta,T_N}\left(\frac{Y_{L_{N-\lfloor\sqrt{T_N}\rfloor}}}{\sqrt{s_\infty}\sqrt{2e^\delta Q^1_{T_N} N}} \le x \,\middle|\, N \in \tau\right) \\ &\quad= \frac{1}{\mathcal{P}_{\delta,T_N}\big(N \in \tau\big)} \sum_{t=0}^{N-\lfloor\sqrt{T_N}\rfloor} \sum_{r=t+1}^{t+\lfloor\sqrt{T_N}\rfloor} \mathcal{P}_{\delta,T_N}\left(\frac{Y_{L_{N-\lfloor\sqrt{T_N}\rfloor}}}{\sqrt{s_\infty}\sqrt{2e^\delta Q^1_{T_N} N}} \le x\,,\, N - \lfloor\sqrt{T_N}\rfloor - t \in \tau\right) \\ &\qquad\qquad\qquad \cdot \mathcal{P}_{\delta,T_N}\big(\tau_1 = r\big) \cdot \mathcal{P}_{\delta,T_N}\Big(t + \lfloor\sqrt{T_N}\rfloor - r \in \tau\Big)\,. \end{aligned}$$

Introducing the function

$$\Theta_{\delta,N}(t) \;:=\; \frac{\sum_{r=t+1}^{t+\lfloor\sqrt{T_N}\rfloor} \mathcal{P}_{\delta,T_N}\big(\tau_1 = r\big) \cdot \mathcal{P}_{\delta,T_N}\big(t + \lfloor\sqrt{T_N}\rfloor - r \in \tau\big)}{\mathcal{P}_{\delta,T_N}\big(N \in \tau\big) \cdot \sum_{r=t+1}^{\infty} \mathcal{P}_{\delta,T_N}\big(\tau_1 = r\big)}\,,$$

we can write

$$
\begin{aligned}
&\mathcal{P}_{\delta,T_N}\left(\frac{Y_{L_{N-\lfloor\sqrt{T_N}\rfloor}}}{\sqrt{s_\infty}\sqrt{2e^\delta Q^1_{T_N} N}} \le x \,\middle|\, N\in\tau \right) \\
&\quad = \sum_{t=0}^{N-\lfloor\sqrt{T_N}\rfloor} \mathcal{P}_{\delta,T_N}\left(\frac{Y_{L_{N-\lfloor\sqrt{T_N}\rfloor}}}{\sqrt{s_\infty}\sqrt{2e^\delta Q^1_{T_N} N}} \le x\,,\ N-\lfloor\sqrt{T_N}\rfloor - t\in\tau \right)\cdot \mathcal{P}_{\delta,T_N}\big(\tau_1 > t\big)\cdot \Theta_{\delta,N}(t)\,.
\end{aligned}
\tag{3.20}
$$

Notice that if we set $\Theta_{\delta,N}(t)\equiv 1$, the r.h.s. of the last relation becomes the l.h.s. of (3.19). In fact $\Theta_{\delta,N}(t)$ is nothing but the Radon-Nikodym derivative of the conditioned law $\mathcal{P}_{\delta,T_N}\big(\,\cdot\,\big|\,N\in\tau\big)$ with respect to the free one $\mathcal{P}_{\delta,T_N}$. We are going to show that $\Theta_{\delta,N}(t)\to 1$ as $N\to\infty$, *uniformly in the values of* t that have the same parity as $\lfloor\sqrt{T_N}\rfloor$ (otherwise $\Theta_{\delta,N}(t)=0$). If we succeed in this, equation (3.18) will follow from (3.19).

Let us set $K_N(n) := \mathcal{P}_{\delta,T_N}(\tau_1 = n)$ and $u_N(n) := \mathcal{P}_{\delta,T_N}(n\in\tau)$, so that we can rewrite $\Theta_{\delta,N}(t)$ as

$$
\Theta_{\delta,N}(t) \;:=\; \frac{\sum_{r=t+1}^{t+\lfloor\sqrt{T_N}\rfloor} K_N(r)\cdot u_N(t+\lfloor\sqrt{T_N}\rfloor - r)}{u_N(N)\cdot \sum_{r=t+1}^{\infty} K_N(r)}\,. \tag{3.21}
$$

We recall that

$$
K_N(n) \;=\; e^\delta\, e^{-\phi(\delta,T_N)\cdot n}\, q_{T_N}(n)\,,
$$

see (2.9), and $q_T(\cdot)$ is defined in (2.4). We are going to show the following: *for every* $\varepsilon>0$ *there exists* $N_0=N_0(\varepsilon)$ *such that for every* $N\ge N_0$ *and for all the value of* $t\le N-\lfloor\sqrt{T_N}\rfloor$ *that have the same parity as* $\lfloor\sqrt{T_N}\rfloor$ *we have*

$$
1-\varepsilon \;\le\; \Theta_{\delta,N}(t) \;\le\; 1+\varepsilon\,. \tag{3.22}
$$

Then the proof of this step will be completed. We first need a preliminary lemma.

Lemma 9. *For every* $\eta>0$ *there exists* $N_1=N_1(\eta)$ *such that for every* $N\ge N_1$ *and for all* $0\le t\le N-\lfloor\sqrt{T_N}\rfloor$ *we have*

$$
\sum_{r=t+\lfloor\sqrt{T_N}\rfloor/2}^{\infty} K_N(r) \;\le\; \eta\cdot\left(\sum_{r=t+1}^{t+\lfloor\sqrt{T_N}\rfloor/2} K_N(r)\right). \tag{3.23}
$$

Proof. We first observe that, by the explicit formulas in (A.7) the following upper bound holds for every $T,n\in\mathbb{N}$ with $n\ge 2$:

$$
\max\left\{q_T^0(n),\, 2\,q_T^1(n)\right\} \;\le\; \frac{2}{T}\sum_{\nu=1}^{\lfloor (T-1)/2\rfloor} \cos^{n-2}\left(\frac{\pi\nu}{T}\right)\sin^2\left(\frac{\pi\nu}{T}\right).
$$

We can bound the l.h.s. of (3.23) as

$$
\sum_{r=t+\lfloor\sqrt{T_N}\rfloor/2}^{\infty} K_N(r) \;\le\; e^\delta\, e^{-\phi(\delta,T_N)\cdot(t+\frac{\lfloor\sqrt{T_N}\rfloor}{2})} \sum_{r=t+\lfloor\sqrt{T_N}\rfloor/2}^{\infty} q_{T_N}(r)\,,
$$

and since $q_T(r) = q_T^0(r) + 2q_T^1(r)$ we have

$$\begin{aligned}\sum_{r=t+\lfloor\sqrt{T_N}\rfloor/2}^{\infty} q_{T_N}(r) \;&\le\; 2 \sum_{r=t+\lfloor\sqrt{T_N}\rfloor/2}^{\infty} \left(\frac{2}{T_N} \sum_{\nu=1}^{\lfloor (T_N-1)/2\rfloor} \cos^{r-2}\left(\frac{\pi\nu}{T_N}\right) \sin^2\left(\frac{\pi\nu}{T_N}\right)\right) \\ &=\; \frac{4}{T_N} \sum_{\nu=1}^{\lfloor (T_N-1)/2\rfloor} \frac{\big(\cos\big(\frac{\pi\nu}{T_N}\big)\big)^{t-2+\lfloor\sqrt{T_N}\rfloor/2}}{1-\cos\big(\frac{\pi\nu}{T_N}\big)} \sin^2\left(\frac{\pi\nu}{T_N}\right) \\ &\le\; \frac{4}{T_N}\cdot\frac{T_N}{2}\left(\cos\left(\frac{\pi}{T_N}\right)\right)^{t-2+\lfloor\sqrt{T_N}\rfloor/2} \cdot 2\,,\end{aligned}$$

where we have used that $\sin^2 x/(1-\cos x) = 1+\cos x \le 2$ for $x \in (0, \frac{\pi}{2}]$. Therefore

$$\sum_{r=t+\lfloor\sqrt{T_N}\rfloor/2}^{\infty} K_N(r) \;\le\; 4\, e^{\delta}\, e^{-\phi(\delta,T_N)\cdot(t+\frac{\lfloor\sqrt{T_N}\rfloor}{2})} \left(\cos\left(\frac{\pi}{T_N}\right)\right)^{t-2+\lfloor\sqrt{T_N}\rfloor/2}. \tag{3.24}$$

Next we bound from below the r.h.s. of (3.23):

$$\sum_{r=t+1}^{t+\lfloor\sqrt{T_N}\rfloor/2} K_N(r) \;\ge\; e^{\delta}\, e^{-\phi(\delta,T_N)\cdot(t+2)} \Big(q^0_{T_N}(t+1) + q^0_{T_N}(t+2)\Big)\,.$$

One of the two numbers $t+1, t+2$ is even, call it ℓ: then we can apply equation (A.7) to get

$$q^0_{T_N}(\ell) \;=\; \frac{2}{T_N}\sum_{\nu=1}^{\lfloor (T_N-1)/2\rfloor} \cos^{\ell-2}\left(\frac{\pi\nu}{T_N}\right)\sin^2\left(\frac{\pi\nu}{T_N}\right) \;\ge\; \frac{2}{T_N}\cos^{\ell-2}\left(\frac{\pi}{T_N}\right)\sin^2\left(\frac{\pi}{T_N}\right),$$

hence

$$\sum_{r=t+1}^{t+\lfloor\sqrt{T_N}\rfloor/2} K_N(r) \;\ge\; e^{\delta}\, e^{-\phi(\delta,T_N)\cdot(t+2)} \frac{2}{T_N}\cos^{t}\left(\frac{\pi}{T_N}\right)\sin^2\left(\frac{\pi}{T_N}\right). \tag{3.25}$$

The ratio of the r.h.s. of equations (3.24) and (3.25) equals

$$2\,T_N\, e^{-\phi(\delta,T_N)\cdot(\lfloor\sqrt{T_N}\rfloor/2-2)} \frac{\big(\cos\big(\frac{\pi}{T_N}\big)\big)^{\lfloor\sqrt{T_N}\rfloor/2-2}}{\sin^2\big(\frac{\pi}{T_N}\big)} \;\le\; \frac{8}{\pi^2}\,(T_N)^3\, e^{-\phi(\delta,T_N)\cdot(\lfloor\sqrt{T_N}\rfloor/2-2)}\,.$$

Since the r.h.s. does not depend on t anymore and vanishes as $N\to\infty$, the proof is completed. □

Let us come back to the proof of (3.22). We first observe that thanks to Lemma 5, for every $\eta > 0$ there exists $N_2 = N_2(\eta)$ and such that for all $N \in \mathbb{N}$ and for all $r \ge N_2$, r even, we have

$$(1-\eta)\,2s_\infty \;\le\; u_N(r) \;\le\; (1+\eta)\,2s_\infty$$

(s_∞ is defined in (3.9)). Henceforth we assume that t has the same parity as $\lfloor\sqrt{T_N}\rfloor$. Then if N is large, such that $\lfloor\sqrt{T_N}\rfloor/2 \ge N_2$, we can bound $\Theta_{\delta,N}(t)$ (recall (3.21)) by

$$\Theta_{\delta,N}(t) \;\le\; \frac{(1+\eta)\,2s_\infty \sum_{r=t+1}^{t+\lfloor\sqrt{T_N}\rfloor/2} K_N(r) \;+\; \sum_{t+\lfloor\sqrt{T_N}\rfloor/2+1}^{t+\lfloor\sqrt{T_N}\rfloor} K_N(r)}{(1-\eta)\,2s_\infty \sum_{r=t+1}^{t+\lfloor\sqrt{T_N}\rfloor/2} K_N(r)}\,,$$

and if $N \geq N_1$ we can apply Lemma 9 to obtain

$$\Theta_{\delta,N}(t) \;\leq\; \frac{1+\eta+\eta/(2s_\infty)}{1-\eta} \;\leq\; 1+\varepsilon\,,$$

provided η is chosen sufficiently small. Therefore the upper bound in (3.22) is proven. The lower bound is analogous: for large N we have

$$\Theta_{\delta,N}(t) \;\geq\; \frac{(1-\eta)\,2s_\infty \sum_{r=t+1}^{t+\lfloor\sqrt{T_N}\rfloor/2} K_N(r)}{(1+\eta)\,2s_\infty \sum_{r=t+1}^{t+\lfloor\sqrt{T_N}\rfloor/2} K_N(r) \;+\; \sum_{t+\lfloor\sqrt{T_N}\rfloor/2+1}^{t+\lfloor\sqrt{T_N}\rfloor} K_N(r)}\,,$$

and applying again Lemma 9 we finally obtain

$$\Theta_{\delta,N}(t) \;\geq\; \frac{1-\eta}{1+\eta+\eta/(2s_\infty)} \;\geq\; 1-\varepsilon\,,$$

provided η is small. Recalling (3.20) and the following lines, the step is completed.

3.6. **Step 4.** In this step we finally complete the proof of Theorem 2 (i), proving equation (3.11), that we rewrite for convenience: for every $x \in \mathbb{R}$

$$\lim_{N\to\infty} \mathbf{P}_{N,\delta}^{T_N}\left(\frac{Y_{L_{N,T_N}}^{T_N}}{\sqrt{s_\infty}\sqrt{2e^\delta Q_{T_N}^1 N}} \leq x \right) \;=\; P(\mathcal{N}(0,1) \leq x)\,. \tag{3.26}$$

We start summing over the location $\mu_N := \tau_{L_{N,T_N}}^{T_N}$ of the last point in τ^{T_N} before N (we assume henceforth that N is even):

$$\mathbf{P}_{N,\delta}^{T_N}\left(\frac{Y_{L_{N,T_N}}^{T_N}}{\sqrt{s_\infty}\sqrt{2e^\delta Q_{T_N}^1 N}} \leq x \right) \;=\; \sum_{\ell=0}^{N} \mathbf{P}_{N,\delta}^{T_N}\left(\frac{Y_{L_{N,T_N}}^{T_N}}{\sqrt{s_\infty}\sqrt{2e^\delta Q_{T_N}^1 N}} \leq x \,\middle|\, \mu_N = N-\ell \right) \cdot \mathbf{P}_{N,\delta}^{T_N}\big(\mu_N = N-\ell\big)\,.$$

Of course only the terms with ℓ even are non-zero. We start showing that we can truncate the sum at a finite number of terms. To this purpose we estimate

$$\mathbf{P}_{N,\delta}^{T_N}\big(\mu_N = N-\ell\big) \;=\; \frac{\mathbf{E}\big(\exp\big(H_{N-\ell,\delta}^{T_N}(S)\big)\mathbf{1}_{\{N-\ell\in\tau\}}\big)\cdot \mathbf{P}\big(\tau_1 > \ell\big)}{\mathbf{E}\big(\exp\big(H_{N,\delta}^{T_N}(S)\big)\big)}\,.$$

We focus on the denominator: inserting the event $\{N-\ell \in \tau\}$ and using the Markov property yields

$$\mathbf{E}\big(\exp\big(H_{N,\delta}^{T_N}(S)\big)\big) \;\geq\; \mathbf{E}\big(\exp\big(H_{N-\ell,\delta}^{T_N}(S)\big)\mathbf{1}_{\{N-\ell\in\tau\}}\big)\cdot \mathbf{E}\Big(\exp\big(H_{\ell,\delta}^{T_N}(S)\big)\Big)\,,$$

hence

$$\mathbf{P}_{N,\delta}^{T_N}\big(\mu_N = N-\ell\big) \;\leq\; \frac{\mathbf{P}\big(\tau_1 > \ell\big)}{\mathbf{E}\big(\exp\big(H_{\ell,\delta}^{T_N}(S)\big)\big)} \;\leq\; \frac{1}{\mathbf{E}\big(\exp\big(H_{\ell,\delta}^{\infty}(S)\big)\big)} \;=\; \frac{1}{Z_{\ell,\delta}^{\infty}}\,,$$

where we have used the elementary fact that $\mathbf{E}\big(\exp\big(H_{\ell,\delta}^{T}(S)\big)\big) \geq \mathbf{E}\big(\exp\big(H_{\ell,\delta}^{\infty}(S)\big)\big)$ for every $T \in \mathbb{N}$, see (1.1) and (1.3). Notice that the r.h.s. above does not depend on N anymore and that $Z_{\ell,\delta}^{\infty} \asymp \exp(\phi(\delta,\infty)\cdot\ell)$ as $\ell\to\infty$, where $\asymp$ denotes equivalence in the

Laplace sense, cf. [12]. Since $\phi(\delta,\infty) > 0$ for $\delta > 0$, it follows that for every $\varepsilon > 0$ there exists $\ell_0 = \ell_0(\varepsilon)$ such that for every $N \in \mathbb{N}$ we have

$$\sum_{\ell=\ell_0+1}^{N} \mathbf{P}_{N,\delta}^{T_N}\big(\mu_N = N-\ell\big) \;\le\; \varepsilon\,. \tag{3.27}$$

As a consequence, we have

$$\begin{aligned}\Bigg| \mathbf{P}_{N,\delta}^{T_N}&\left(\frac{Y_{L_{N,T_N}}^{T_N}}{\sqrt{s_\infty}\sqrt{2e^\delta Q_{T_N}^1 N}} \le x\right)\\ &- \sum_{\ell=0}^{\ell_0} \mathbf{P}_{N,\delta}^{T_N}\left(\frac{Y_{L_{N,T_N}}^{T_N}}{\sqrt{s_\infty}\sqrt{2e^\delta Q_{T_N}^1 N}} \le x \,\middle|\, \mu_N = N-\ell\right)\cdot \mathbf{P}_{N,\delta}^{T_N}\big(\mu_N = N-\ell\big)\Bigg| \;\le\; \varepsilon\,.\end{aligned}$$

Therefore to complete the proof of (3.26) it remains to show that, for every fixed $\ell \in \mathbb{N}\cup\{0\}$,

$$\lim_{N\to\infty} \mathbf{P}_{N,\delta}^{T_N}\left(\frac{Y_{L_{N,T_N}}^{T_N}}{\sqrt{s_\infty}\sqrt{2e^\delta Q_{T_N}^1 N}} \le x \,\middle|\, \mu_N = N-\ell\right) \;=\; P(\mathcal{N}(0,1)\le x)\,. \tag{3.28}$$

However this is easy. In fact on the event $\{\mu_N = N-\ell\}$ we have $Y_{L_{N,T_N}}^{T_N} = Y_{L_{N-\ell,T_N}}^{T_N}$ and by the Markov property we get

$$\mathbf{P}_{N,\delta}^{T_N}\left(\frac{Y_{L_{N,T_N}}^{T_N}}{\sqrt{s_\infty}\sqrt{2e^\delta Q_{T_N}^1 N}} \le x \,\middle|\, \mu_N = N-\ell\right) \;=\; \mathbf{P}_{N,\delta}^{T_N}\left(\frac{Y_{L_{N-\ell,T_N}}^{T_N}}{\sqrt{s_\infty}\sqrt{2e^\delta Q_{T_N}^1 N}} \le x \,\middle|\, N-\ell\in\tau\right)$$

However, arguing as in §2.2 (see in particular (2.13)), we have that

$$\mathbf{P}_{N,\delta}^{T_N}\left(\frac{Y_{L_{N-\ell,T_N}}^{T_N}}{\sqrt{s_\infty}\sqrt{2e^\delta Q_{T_N}^1 N}} \le x \,\middle|\, N-\ell\in\tau\right) \;=\; \mathcal{P}_{\delta,T_N}\left(\frac{Y_{L_{N-\ell}}}{\sqrt{s_\infty}\sqrt{2e^\delta Q_{T_N}^1 N}} \le x \,\middle|\, N-\ell\in\tau\right).$$

Therefore (3.28) follows easily from (3.18). □

4. Proof of Theorem 2 (ii)

This section is devoted to the proof of part (ii) of Theorem 2, which in a sense is the *critical regime*. We stress that $\delta > 0$ is fixed throughout the section. The assumption in part (ii) is that the sequence $(T_N)_N$ is such that $T_{N'} - \frac{\log N'}{c_\delta} \to \zeta$ along a sub-sequence N', where $\zeta \in \mathbb{R}$ (the reason for considering only a sub-sequence is explained in Remark 1). However, for notational convenience, in this section we drop the sub-sequence and we assume that for some $\zeta \in \mathbb{R}$ as $N\to\infty$

$$T_N - \frac{\log N}{c_\delta} \;\longrightarrow\; \zeta \qquad \text{or equivalently} \qquad Q_{T_N}^1 \cdot N \;\longrightarrow\; \sqrt{1-e^{-2\phi(\delta,\infty)}}\,e^{-c_\delta\zeta}\,, \tag{4.1}$$

where we have used Lemma 7 and we recall the shorthand $Q_{T_N}^1 := Q_{T_N}^1(\phi(\delta,T_N))$ introduced in the previous section.

We recall that the variables $(\xi_i, \varepsilon_i, \tau_i)_{i \geq 1}$ are defined under the law $\mathcal{P}_{\delta, T_N}$ (see (2.8)). We now introduce the successive epochs $(\theta_i)_{i \geq 0}$ at which the jump process changes of interface, by setting $\theta_0 = 0$ and for $j \geq 1$

$$\theta_j \;:=\; \inf\big\{ m > \theta_{j-1} : \exists i \in \mathbb{N} \text{ such that } \tau_i = m \text{ and } |\varepsilon_i| = 1 \big\} . \tag{4.2}$$

The number of these jumps occurring before time N is given by

$$L'_N \;:=\; \sup\{j \geq 0 : \theta_j \leq N\} \;=\; \#\{i \leq L_N : |\varepsilon_i| = 1\} . \tag{4.3}$$

Notice that $\theta \subseteq \tau$, where as usual we identify $\theta = \{\theta_n\}_n$ with a (random) subset of $\mathbb{N} \cup \{0\}$.

We split the proof in three steps.

4.1. **Step 1.** We start proving that under $\mathcal{P}_{\delta, T_N}$ the variable L'_N converges in law towards a Poisson law of parameter $t_{\delta,\zeta}$ with $t_{\delta,\zeta} := 2e^{\delta}\sqrt{1 - e^{-2\phi(\delta,\infty)}}\,\phi'(\delta,\infty) \cdot e^{-c_\delta \zeta}$, i.e.,

$$\lim_{N\to\infty} \mathcal{P}_{\delta, T_N}\big(L'_N = j\big) \;=\; e^{-t_{\delta,\zeta}}\, \frac{(t_{\delta,\zeta})^j}{j!} \qquad \forall j \in \mathbb{N} \cup \{0\} . \tag{4.4}$$

We note that $\{|\varepsilon_i|\}_{i \geq 1}$ under $\mathcal{P}_{\delta, T_N}$ is a sequence of i.i.d. Bernoulli trials with success probability given by

$$p_{T_N} \;:=\; \mathcal{P}_{\delta, T_N}(|\varepsilon_1| = 1) \;=\; 2e^{\delta} Q^1_{T_N} . \tag{4.5}$$

We also set

$$\Delta \;:=\; \inf\{i \geq 1 : |\varepsilon_i| = 1\} .$$

Notice that $(\theta_j - \theta_{j-1})_{j \geq 1}$ are i.i.d. random variables. Moreover we can write

$$\theta_1 \;=\; \sum_{j=1}^{\Delta} \xi_j .$$

We now study the asymptotic behavior of θ_j and by (4.3) we derive that of L'_N. The building blocks are given in the following Lemma.

Lemma 10. *The following convergences in law hold as $N \to \infty$ under $\mathcal{P}_{\delta, T_N}$:*

$$\frac{\xi_\Delta}{N} \;\Longrightarrow\; 0 , \qquad \frac{\Delta - 1}{N} \;\Longrightarrow\; \mathrm{Exp}\big(v_\delta\big) , \qquad \frac{1}{\Delta - 1} \sum_{j=1}^{\Delta - 1} \xi_j \;\Longrightarrow\; \mathcal{E}_{\delta,\infty}(\xi_1) , \tag{4.6}$$

where $v_{\delta,\zeta} := 2e^{\delta}\sqrt{1 - e^{-2\phi(\delta,\infty)}}\, e^{-c_\delta \zeta}$ and $\mathrm{Exp}(\lambda)$ denotes the Exponential law of parameter λ, i.e., $P(\mathrm{Exp}(\lambda) \in \mathrm{d}x) = \lambda\, e^{-\lambda x}\, \mathbf{1}_{\{x \geq 0\}}\, \mathrm{d}x$.

Proof. For the first relation, it suffices to show that $\mathcal{E}_{\delta, T_N}(\xi_\Delta / N)$ vanishes as $N \to \infty$. By definition, the variable ξ_Δ gives the length of a jump conditioned to occur between two different interfaces, namely, ξ_Δ has the same law as ξ_1 conditionally on the event $\{|\varepsilon_1| = 1\}$. This leads to the following formula (see (2.8)):

$$\mathcal{E}_{\delta, T_N}\Big(\frac{\xi_\Delta}{N}\Big) \;=\; \frac{1}{Q^1_{T_N} N} \sum_{n=1}^{\infty} n\, q^1_{T_N}(n)\, e^{-\phi(\delta, T_N)\, n} . \tag{4.7}$$

By (4.1) $Q^1_{T_N} N \to c' > 0$ as $N \to \infty$ and for every fixed $n \geq 1$ we observe that plainly $q^1_{T_N}(n) \to 0$ as $N \to \infty$ (in fact $q^1_T(n) = 0$ for $T > n$). Since $\phi(\delta, T_N) \to \phi(\delta, \infty) > 0$ as $N \to \infty$, see Remark 2, by Dominated Convergence the r.h.s. of (4.7) vanishes as $N \to \infty$.

For the second relation in (4.6), note that the variable Δ has a Geometric law of parameter p_{T_N}, i.e., for all $j \in \mathbb{N}$

$$\mathcal{P}_{\delta,T_N}(\Delta = j) \;=\; (1-p_{T_N})^{j-1} p_{T_N}\,.$$

Since $N \cdot p_{T_N} \;\to\; v_{\delta,\zeta} \;=\; 2e^{\delta}\sqrt{1-e^{-2\phi(\delta,\infty)}}\,e^{-c_\delta \zeta}$ as $N \to \infty$, see (4.5) and (4.1), it is well-known (and easy to check) that Δ/N converges to an Exponential law of parameter v_δ, and of course the same is true for $(\Delta-1)/N$.

Next we focus on the third relation in (4.6). Since $\mathcal{P}_{\delta,T_N}\big(\Delta \le \sqrt{N}\big) \to 0$ as $N \to \infty$ by the result just proved, it suffices to consider for $\varepsilon > 0$ the quantity

$$\begin{aligned}
&\mathcal{P}_{\delta,T_N}\left(\left|\frac{1}{\Delta-1}\sum_{j=1}^{\Delta-1}\xi_j - \mathcal{E}_{\delta,\infty}(\xi_1)\right| > \varepsilon,\, \Delta > \sqrt{N}\right) \\
&\quad = \sum_{l=\lceil\sqrt{N}\rceil}^{\infty} \mathcal{P}_{\delta,T_N}(\Delta = l)\; \mathcal{P}_{\delta,T_N}\left(\left|\frac{1}{l-1}\sum_{j=1}^{l-1}\xi_j - \mathcal{E}_{\delta,\infty}(\xi_1)\right| > \varepsilon \,\middle|\, \Delta = l\right).
\end{aligned} \tag{4.8}$$

To evaluate the last term, we notice that under $\mathcal{P}_{\delta,T_N}\big(\,\cdot\,\big|\Delta = l\big)$ the variables $\xi_1, \ldots, \xi_{l-1}$ are i.i.d. with marginal law simply given by the law of ξ_1 conditionally on the event $\{\varepsilon_1 = 0\}$ (which means that the jump occurs at the same interface). Denoting for simplicity by $\mathcal{P}^0_{\delta,T_N}$ this law, we have for $n \ge 1$,

$$\mathcal{P}^0_{\delta,T_N}(\xi_1 = n) = \frac{1}{1-2e^{\delta}Q^1_{T_N}}\, q^0_{T_N}(n)\, e^{\delta}\, e^{-\phi(\delta,T_N)\,n}. \tag{4.9}$$

By (4.1) we have $Q^1_{T_N} \to 0$ as $n \to \infty$. Moreover $q^0_{T_N}(n) \to q_\infty(n)$ by definition and $\phi(\delta, T_N) \to \phi(\delta,\infty) > 0$ by Remark 2. These considerations yield by Dominated Convergence $\mathcal{E}^0_{\delta,T_N}(\xi_1) \to \mathcal{E}_{\delta,\infty}(\xi_1)$ and $\mathrm{Var}^0_{\delta,T_N}(\xi_1) \to \mathrm{Var}_{\delta,\infty}(\xi_1)$ as $N \to \infty$. In particular, in the r.h.s. of (4.8) we can replace $\mathcal{E}_{\delta,\infty}(\xi_1)$ by $\mathcal{E}^0_{\delta,T_N}(\xi_1)$ and ε by (say) $\varepsilon/2$ and we get an upper bound for large N. Applying Chebychev's inequality we obtain

$$\mathcal{P}_{\delta,T_N}\left(\left|\frac{1}{l-1}\sum_{j=1}^{l-1}\xi_j - \mathcal{E}^0_{\delta,T_N}(\xi_1)\right| > \frac{\varepsilon}{2}\,\middle|\, \Delta = l\right) \;\le\; \frac{4\,\mathrm{Var}^0_{\delta,T_N}(\xi_1)}{\varepsilon^2\,(l-1)}\,. \tag{4.10}$$

This shows that the r.h.s. of (4.8) vanishes as $N \to \infty$ and this completes the proof. $\square$

By writing

$$\frac{\theta_1}{N} \;=\; \frac{\Delta-1}{N}\cdot\frac{1}{\Delta-1}\sum_{j=1}^{\Delta-1}\xi_j \;+\; \frac{\xi_\Delta}{N}$$

and applying Lemma 10 we can easily conclude that θ_1/N converges in law to an Exponential distribution of parameter $t_{\delta,\zeta}$ given by

$$t_{\delta,\zeta} \;:=\; v_{\delta,\zeta}/\mathcal{E}_{\delta,\infty}(\xi_1) \;=\; 2e^{\delta}\sqrt{1-e^{-2\phi(\delta,\infty)}}\,e^{-c_\delta\zeta}\cdot \phi'(\delta,\infty)\,,$$

having used (3.10). By independence, for every fixed $j \in \mathbb{N}$ the variable θ_j/N converges to a Gamma law with parameters $(j, t_{\delta,\zeta})$, hence by (4.3) the variable L'_N converges to a Poisson law of parameter $t_{\delta,\zeta}$. This completes the step.

4.2. **Step 2.** In this step we want to prove that under the law $\mathcal{P}_{\delta,T_N}(\,\cdot\,|\,N \in \tau)$, with $N \in 2\mathbb{N}$, the quantity L'_N still converges toward a Poisson distribution of parameter $t_{\delta,\zeta}$, i.e.,

$$\lim_{N\to\infty,\, N \text{ even}} \mathcal{P}_{\delta,T_N}\big(L'_N = j \,\big|\, N \in \tau\big) \;=\; e^{-t_{\delta,\zeta}} \frac{(t_{\delta,\zeta})^j}{j!} \qquad \forall j \in \mathbb{N}\cup\{0\}\,. \tag{4.11}$$

We start elaborating a bit on (4.4). Fix $L \in \mathbb{N}$ and write, by the renewal property,

$$\mathcal{P}_{\delta,T_N}\big(\tau \cap (N-L,N] = \emptyset\big) \;=\; \sum_{r=0}^{N-L} \sum_{s=N+1}^{\infty} u_N(r)\cdot K_N(s-r)\,,$$

where we recall the definitions $u_N(n) := \mathcal{P}_{\delta,T_N}(n \in \tau)$ and $K_N(n) := \mathcal{P}_{\delta,T_N}(\tau_1 = n)$. Since $u_N(r) \le 1$ and $K_N(n) \le e^{\delta}\, e^{-\phi(\delta,T_N)\cdot n}$, see (2.9), and since $\phi(\delta,T_N) \to \phi(\delta,\infty) > 0$ as $N \to \infty$, see Remark 2, it follows that

$$\mathcal{P}_{\delta,T_N}\big(\tau \cap (N-L,N] = \emptyset\big) \;\le\; e^{\delta} \sum_{r=0}^{N-L} \sum_{s=N+1}^{\infty} e^{-\phi(\delta,T_N)\cdot(s-r)} \;\le\; C\cdot e^{-C'\cdot L}\,, \tag{4.12}$$

where C, C' are suitable positive constants depending only on δ. This means that the probability of the event $\{\tau \cap (N-L,N] = \emptyset\}$ can be made arbitrarily small, *uniformly in* N, by taking L large. It is then easy to see that equation (4.4) yields the following: for all $\varepsilon > 0$ and $j \in \mathbb{N}\cup\{0\}$ there exist N_0, L_0 such that for all $N \ge N_0$ and $L \ge L_0$ we have

$$\mathcal{P}_{\delta,T_N}\big(L'_N = j \,\big|\, \tau \cap (N-L,N] \neq \emptyset\big) \;\in\; \bigg(e^{-t_{\delta,\zeta}} \frac{(t_{\delta,\zeta})^j}{j!} - \varepsilon,\; e^{-t_{\delta,\zeta}} \frac{(t_{\delta,\zeta})^j}{j!} + \varepsilon\bigg)\,. \tag{4.13}$$

Next we show that equation (4.11) follows from (4.13). The idea is that conditioning on the event $\{\tau \cap (N-L,N] \neq \emptyset\}$, i.e., that there is a renewal epoch in $(N-L,N]$, is the same as conditioning on $\{N-i \in \tau\}$ for some $i = 0,\ldots,L-1$, and the latter is essentially independent of i. More precisely, we have the following lemma.

Lemma 11. *For every $i \in 2\mathbb{N}\cup\{0\}$, the following relation holds as $N \to \infty$, with $N \in 2\mathbb{N}$:*

$$\mathcal{P}_{\delta,T_N}\big(L'_N = j \,\big|\, N \in \tau\big) \;=\; \mathcal{P}_{\delta,T_N}\big(L'_{N-i} = j \,\big|\, N-i \in \tau\big) \;+\; \varepsilon_i(N), \tag{4.14}$$

where $\varepsilon_i(N) \to 0$ as $N \to \infty$.

Proof. Notice that $\{L'_N = j\} = \{\theta_j \le N,\, \theta_{j+1} > N\}$. First we restrict the expectation on the event $\{\theta_j \le N - \sqrt{N}\}$, which has almost full probability. In fact for fixed $i \in 2\mathbb{N}\cup\{0\}$

$$\mathcal{P}_{\delta,T_N}\big(L'_{N-i} = j,\, \theta_j > N - \sqrt{N} \,\big|\, N-i \in \tau\big) \;\le\; \frac{\mathcal{P}_{\delta,T_N}\big(N - \sqrt{N} < \theta_j \le N\big)}{\mathcal{P}_{\delta,T_N}(N-i \in \tau)} \;=\; o(1)\,, \tag{4.15}$$

as $N \to \infty$, $N \in 2\mathbb{N}$, because θ_j/N converges as $N \to \infty$ to a atom-free law (in fact a Gamma) by Step 1 and, by Lemma 5, $\mathcal{P}_{\delta,T_N}(N-i \in \tau) \to 2/m(\delta,\infty) > 0$ as $N \to \infty$. Specializing (4.15) to $i = 0$ we can therefore write as $N \to \infty$, $N \in 2\mathbb{N}$,

$$\begin{aligned}
\mathcal{P}_{\delta,T_N}\big(L'_N = j \,\big|\, N \in \tau\big) \;&=\; \mathcal{P}_{\delta,T_N}\big(L'_N = j,\, \theta_j \le N - \sqrt{N} \,\big|\, N \in \tau\big) \;+\; o(1) \\
&=\; \mathcal{P}_{\delta,T_N}\big(\theta_j \le N - \sqrt{N},\, \theta_{j+1} > N \,\big|\, N \in \tau\big) \;+\; o(1)\,.
\end{aligned}$$

The renewal property then yields

$$\mathcal{P}_{\delta,T_N}\big(L'_N = j \,\big|\, N \in \tau\big) \;=\; \sum_{r=1}^{\lfloor N-\sqrt{N}\rfloor} \mathcal{P}_{\delta,T_N}(\theta_j = r) \cdot \frac{\mathcal{P}_{\delta,T_N}(\theta_1 > N-r, N-r\in\tau)}{\mathcal{P}_{\delta,T_N}(N\in\tau)} \;+\; o(1)\,. \tag{4.16}$$

We now study the term $\mathcal{P}_{\delta,T_N}(\theta_1 > l, l \in \tau)$. We have

$$\begin{aligned}\mathcal{P}_{\delta,T_N}(\theta_1 > l, l\in\tau) \;&=\; \sum_{k=1}^{l} \sum_{0=:t_0<t_1<\ldots<t_k=l} \prod_{j=1}^{k} e^{\delta}\, q^0_{T_N}(t_j - t_{j-1})\, e^{-\phi(\delta,T_N)(t_j-t_{j-1})} \\ &=\; e^{\nu_N\cdot l}\cdot \sum_{0=:t_0<t_1<\ldots<t_k=l} \prod_{j=1}^{k} \widetilde{K}^0_N(t_j-t_{j-1})\,,\end{aligned} \tag{4.17}$$

where we have set for $n\in\mathbb{N}$

$$\widetilde{K}^0_N(n) \;:=\; e^{\delta}\, q^0_{T_N}(n)\, e^{-(\phi(\delta,T_N)+\nu_N)\cdot n}\,,$$

and we fix $\nu_N < 0$ such that $\sum_{n\in\mathbb{N}} \widetilde{K}^0_N(n) = 1$, i.e., $Q^0_{T_N}(\phi(\delta,T_N)+\nu_N) = e^{-\delta}$, which is always possible because $Q^0_T(\lambda)$ diverges as $\lambda \downarrow \lambda^0_T$, see Appendix A. Denoting by $\widetilde{\mathcal{P}}^0_{\delta,T_N}$ the global law of τ, when the step distribution is $\widetilde{K}^0_N(n)$, we can rewrite (4.17) with $l = N-r$ as

$$\mathcal{P}_{\delta,T_N}(\theta_1 > N-r,\, N-r\in\tau) \;=\; e^{\nu_N\cdot(N-r)}\cdot \widetilde{\mathcal{P}}^0_{\delta,T_N}(N-r\in\tau)\,. \tag{4.18}$$

Plainly, as $N\to\infty$ we have $q^0_{T_N}(n) \to q_\infty(n)$ for every $n\in\mathbb{N}$, where we recall that $q_\infty(n)$ is the return time distribution for the simple random walk, cf. §2.2. Hence $\nu_N \to 0$ and $\widetilde{K}^0_N(n) \to \mathcal{P}_{\delta,\infty}(n\in\tau)$ as $N\to\infty$. Then a slight modification of Lemma 5 shows that, for any fixed $r\in 2\mathbb{N}$, $\widetilde{\mathcal{P}}^0_{\delta,T_N}(N-r\in\tau) \to 2/m(\delta,\infty) > 0$ as $N\to\infty$. Then in equation (4.18) we can replace N by $N-i$, any fixed $i\in 2\mathbb{N}$, by paying $o(1)$: more precisely, as $N\to\infty$, with $N\in 2\mathbb{N}$,

$$\mathcal{P}_{\delta,T_N}(\theta_1 > N-r,\, N-r\in\tau) \;=\; \mathcal{P}_{\delta,T_N}(\theta_1 > N-i-r,\, N-i-r\in\tau) \;+\; o(1)\,.$$

Coming back to (4.16) and replacing also $\mathcal{P}_{\delta,T_N}(N\in\tau)$ by $\mathcal{P}_{\delta,T_N}(N-i\in\tau)$, we can write

$$\begin{aligned}\mathcal{P}_{\delta,T_N}\big(L'_N = j \,\big|\, N\in\tau\big) \;&=\; \mathcal{P}_{\delta,T_N}\big(L'_{N-i} = j,\, \theta_j \le N-\sqrt{N} \,\big|\, N-i\in\tau\big) \;+\; o(1) \\ &=\; \mathcal{P}_{\delta,T_N}\big(L'_{N-i} = j \,\big|\, N-i\in\tau\big) \;+\; o(1)\,,\end{aligned}$$

where the second equality follows by (4.15). The proof is completed. □

Let us come back to (4.13). We write the event $\{\tau\cap(N-L,N] \neq \emptyset\}$ as a disjoint union

$$\{\tau\cap(N-L,N]\neq\emptyset\} \;=\; \bigcup_{i=0}^{L-1} \mathcal{A}_i\,, \qquad \mathcal{A}_i \;:=\; \{N-i\in\tau,\, N-k\notin\tau \text{ for } 0\le k<i\}\,, \tag{4.19}$$

i.e., $N-i$ is the last renewal epoch before N. Then we can write the l.h.s. of (4.13) as

$$\mathcal{P}_{\delta,T_N}\big(L'_N = j,\; \tau\cap(N-L,N]\neq\emptyset\big) \;=\; \sum_{i=0}^{L-1} \mathcal{P}_{\delta,T_N}\big(L'_N = j \,\big|\, \mathcal{A}_i\big)\cdot \mathcal{P}_{\delta,T_N}\big(\mathcal{A}_i\big)\,. \tag{4.20}$$

Notice that $\mathcal{P}_{\delta,T_N}\big(L'_N = j \,\big|\, \mathcal{A}_i\big) = \mathcal{P}_{\delta,T_N}\big(L'_{N-i} = j \,\big|\, \mathcal{A}_i\big)$, because $L'_N = L'_{N-i}$ on the event $\mathcal{A}_i$. The next basic fact is that, by the renewal property, we have

$$\mathcal{P}_{\delta,T_N}\big(L'_{N-i} = j \,\big|\, \mathcal{A}_i\big) \;=\; \mathcal{P}_{\delta,T_N}\big(L'_{N-i} = j \,\big|\, N-i\in\tau\big)\,,$$

because the event $\{L'_{N-i} = j\}$ depends only on $\tau \cap [0, N-i]$. Therefore we can apply Lemma 11 and rewrite (4.20) as

$$\begin{aligned}\mathcal{P}_{\delta,T_N}\big(L'_N = j,\ \tau \cap (N-L, N] \neq \emptyset\big) \;&=\; \mathcal{P}_{\delta,T_N}\big(L'_N = j \,\big|\, N \in \tau\big)\left(\sum_{i=0}^{L-1} \mathcal{P}_{\delta,T_N}\big(\mathcal{A}_i\big)\right) \;+\; o(1)\\ &=\; \mathcal{P}_{\delta,T_N}\big(L'_N = j \,\big|\, N \in \tau\big) \cdot \mathcal{P}_{\delta,T_N}\big(\tau \cap (N-L, N] \neq \emptyset\big) \;+\; o(1)\,. \qquad (4.21)\end{aligned}$$

However, by (4.12) the term $\mathcal{P}_{\delta,T_N}\big(\tau \cap (N-L, N] \neq \emptyset\big)$ is as close to one as we wish, by taking L large. Combining (4.13) with (4.21), this means that for every $j \in \mathcal{N} \cup \{0\}$ and for N sufficiently large we have

$$\mathcal{P}_{\delta,T_N}\big(L'_N = j \,\big|\, N \in \tau\big) \;\in\; \left(e^{-t_{\delta,\zeta}}\,\frac{(t_{\delta,\zeta})^j}{j!} - 2\varepsilon,\; e^{-t_{\delta,\zeta}}\,\frac{(t_{\delta,\zeta})^j}{j!} + 2\varepsilon\right).$$

Since ε is arbitrary, (4.11) is proven and the step is completed.

4.3. **Step 3.** In this last step it remains to prove that for all $\varepsilon > 0$ and all $j \in \mathbb{N} \cup \{0\}$,

$$\lim_{N\to\infty} \mathbf{P}^{T_N}_{N,\delta}\left(\frac{S_N}{T_N} \in [j-\varepsilon, j+\varepsilon]\right) \;=\; \mathbf{P}(S_\Gamma = j), \qquad (4.22)$$

where Γ is a random variable independent of the $\{S_i\}_{i\geq 0}$ and with a Poisson law of parameter $t_{\delta,\zeta}$.

Let $\varepsilon > 0$ and set

$$V_\varepsilon(N) := \left|\mathbf{P}^{T_N}_{N,\delta}\left(\frac{S_N}{T_N} \in [j-\varepsilon, j+\varepsilon]\right) - \mathbf{P}(S_\Gamma = j)\right|. \qquad (4.23)$$

Our goal is to prove that for all $\eta > 0$ we have $V_\varepsilon(N) \leq \eta$ when N is large enough. We let $\mathcal{V}(N, l)$ be the set $\tau^{T_N} \cap [N-l, N]$ and it is useful to recall the result obtained in (3.27), i.e., there exists $\ell_0 = \ell_0(\eta)$ such that for every $N \geq \ell_0$ we have $\mathbf{P}^{T_N}_{N,\delta}\big(\mathcal{V}(N, \ell_0) = \emptyset\big) \leq \eta/4$. Therefore, with N large enough we obtain

$$V_\varepsilon(\delta) \;\leq\; \frac{\eta}{2} + \left|\mathbf{P}^{T_N}_{N,\delta}\left(\frac{S_N}{T_N} \in [j-\varepsilon, j+\varepsilon],\, \mathcal{V}(N,\ell_0) \neq \emptyset\right) - \mathbf{P}(S_\Gamma = j)\,\mathbf{P}^{T_N}_{N,\delta}\big(\mathcal{V}(N,\ell_0) \neq \emptyset\big)\right|.$$

With some abuse of notation, we still denote by θ_j and L'_N the variables on the S space defined by (4.2) and (4.3) with τ_i replaced by $\tau_i^{T_N}$ and ε_i by $\varepsilon_i^{T_N}$ (in particular $L'_N := \#\{i \leq L_{N,T_N} : |\varepsilon_i^{T_N}| = 1\}$). Then notice that on the event $\mathcal{V}(N, \ell_0)$ we have $|S_N - S_{\theta_{L'_N}}| \leq \ell_0$. Moreover, for all $N \geq 1$ we have $S_{\theta_{L'_N}}/T_N \in \mathbb{Z}$, therefore, assuming that ε has been chosen small enough, we obtain for N large enough

$$\mathbf{P}^{T_N}_{N,\delta}\left(\frac{S_N}{T_N} \in [j-\varepsilon, j+\varepsilon],\ \mathcal{V}(N,\ell_0) \neq \emptyset\right) \;=\; \mathbf{P}^{T_N}_{N,\delta}\left(\frac{S_{\theta_{L'_N}}}{T_N} = j,\ \mathcal{V}(N,\ell_0) \neq \emptyset\right). \qquad (4.24)$$

We can rewrite the r.h.s. of (4.24) by using, for $i \in \{0, \ldots, \ell_0\}$, the sets $\mathcal{A}_i$ introduced in (4.19). This gives

$$\mathbf{P}^{T_N}_{N,\delta}\left(\frac{S_{\theta_{L'_N}}}{T_N} = j,\ \mathcal{V}(N,\ell_0) \neq \emptyset\right) \;=\; \sum_{i=0}^{l_0} \mathbf{P}^{T_N}_{N,\delta}\left(\frac{S_{\theta_{L'_{N-i}}}}{T_N} = j \,\middle|\, \mathcal{A}_i\right) \mathbf{P}^{T_N}_{N,\delta}(\mathcal{A}_i). \qquad (4.25)$$

At this stage, the Markov property and equation (2.13) give

$$\mathbf{P}^{T_N}_{N,\delta}(\cdot \,|\, \mathcal{A}_i) \;=\; \mathbf{P}^{T_N}_{N,\delta}(\cdot \,|\, N-i \in \tau) \;=\; \mathcal{P}_{\delta,T_N}(\cdot \,|\, N-i \in \tau)\,,$$

hence we can rewrite (4.25) as

$$\mathbf{P}^{T_N}_{N,\delta}\left(\frac{S_{\theta_{L'_N}}}{T_N} = j,\ \mathcal{V}(N,\ell_0) \neq \emptyset\right) \;=\; \sum_{i=0}^{l_0} \mathcal{P}_{\delta,T_N}\left(\frac{S_{\theta_{L'_{N-i}}}}{T_N} = j \,\middle|\, N-i \in \tau\right) \mathbf{P}^{T_N}_{N,\delta}(\mathcal{A}_i)\,. \tag{4.26}$$

Thus, the proof of this step will be completed if we can show that for all $i \in \{0, \ldots, \ell_0\}$

$$\lim_{N\to\infty} \mathcal{P}_{\delta,T_N}\left(\frac{S_{\theta_{L'_{N-i}}}}{T_N} = j \,\middle|\, N-i \in \tau\right) \;=\; \mathbf{P}(S_\Gamma = j)\,.$$

This is proved once we show that, for all $(v,j) \in \mathbb{N} \cup \{0\} \times \mathbb{Z}$,

$$\lim_{N\to\infty} \mathcal{P}_{\delta,T_N}\left(L'_{N-i} = v,\ \frac{S_{\theta_v}}{T_N} = j \,\middle|\, N-i \in \tau\right) \;=\; \mathbf{P}(\Gamma = v)\,\mathbf{P}(S_v = j)\,. \tag{4.27}$$

We can rewrite the l.h.s. of (4.27) as

$$\mathcal{P}_{\delta,T_N}\left(\frac{S_{\theta_v}}{T_N} = j \,\middle|\, L'_{N-i} = v,\ N-i \in \tau\right) \cdot \mathcal{P}_{\delta,T_N}\big(L'_{N-i} = v \,\big|\, N-i \in \tau\big) \tag{4.28}$$

and it is easy to figure out that the process $(S_{\theta_n}/T_N)_{n\in\mathbb{N}}$ is just the symmetric simple random walk on $\mathbb{Z}$ and is independent of (L'_{N-i}, τ). Therefore, the first factor in (4.28) equals $\mathbf{P}(S_j = v)$ and then Lemma 11 and equation (4.11) are sufficient to complete the proof. □

5. Proof of Theorem 2 (iii)

In this section we prove part (iii) of Theorem 2. The parameter $\delta > 0$ is fixed throughout the section and the assumption is that the sequence $(T_N)_N$ is such that $T_N - \frac{\log N}{c_\delta} \to +\infty$, or equivalently

$$Q^1_{T_N} \cdot N \;\longrightarrow\; 0 \qquad\qquad (N \to \infty)\,, \tag{5.1}$$

where we have used Lemma 7 and we recall the shorthand $Q^1_{T_N} := Q^1_{T_N}(\phi(\delta, T_N))$ introduced in Section 3. The goal is to prove equation (1.15), i.e., that the law of S_N under $\mathbf{P}^{T_N}_{N,\delta}$ is tight.

In analogy with the previous sections, we start working under the law $\mathcal{P}_{\delta,T_N}$. We show that the polymer of length N does not visit any interface other than the one located at $S = 0$, i.e., (recalling (4.2) and (4.3)) $L'_N = 0$. Notice in fact that

$$\{L'_N \geq 1\} \;=\; \{\theta_1 \leq N\} \;=\; \bigcup_{i=1}^{L_N} \{|\varepsilon_i = 1|\} \;\subseteq\; \bigcup_{i=1}^{N/2} \{|\varepsilon_i = 1|\}\,,$$

because plainly $L_N \leq N/2$ (recall (3.2)), hence the inclusion bound yields

$$\mathcal{P}_{\delta,T_N}\big(L'_N \geq 1\big) \;\leq\; \frac{N}{2} \cdot \mathcal{P}_{\delta,T_N}\big(|\varepsilon_1| = 1\big) \;=\; e^{\delta}\, N\, Q^1_{T_N} \;\longrightarrow\; 0 \qquad\quad (N \to \infty)\,, \tag{5.2}$$

where we have used (4.5) and (5.1). With the same abuse of notation as in the previous section, we denote by L'_N also the variable on the S spaced defined by $L'_N := \#\{i \leq L_{N,T_N} : |\varepsilon^{T_N}_i| = 1\}$, so that applying (2.13) we get as $N \to \infty$ with $N \in 2\mathbb{N}$

$$\mathbf{P}^{T_N}_{N,\delta}\big(L'_N \geq 1 \,\big|\, S_N \in T_N\mathbb{Z}\big) \;=\; \mathcal{P}_{\delta,T_N}\big(L'_N \geq 1 \,\big|\, N \in \tau\big) \;\leq\; \frac{\mathcal{P}_{\delta,T_N}\big(L'_N \geq 1\big)}{\mathcal{P}_{\delta,T_N}\big(N \in \tau\big)} \;\longrightarrow\; 0\,, \tag{5.3}$$

having applied (5.2) and Lemma 5.

Now set $|S_n|^* := \max_{0 \le k \le n} |S_k|$ and observe that equation (5.3) can be rephrased as

$$\mathbf{P}_{N,\delta}^{T_N}\big(|S_N|^* \ge T_N \,\big|\, N \in \tau^{T_N}\big) \;\longrightarrow\; 0 \qquad (N \to \infty,\, N \in 2\mathbb{N})\,.$$

We want to remove the conditioning on $N \in \tau^{T_N}$. To this purpose, we let $\mu_N := \tau^{T_N}_{L_{N,T_N}}$ denote the location of the last point of $\tau^{T_N} \cap [0,N]$. Let us recall equation (3.27), which holds whenever $\delta > 0$ and hence can be applied here: for every $\varepsilon > 0$ there exists ℓ_0 such that $\mathbf{P}_{N,\delta}^{T_N}\big(\mu_N < N - \ell_0\big) < \varepsilon$, for every $N \in \mathbb{N}$. Therefore

$$\left| \mathbf{P}_{N,\delta}^{T_N}\big(|S_N|^* \ge T_N\big) \;-\; \sum_{\ell=0}^{\ell_0} \mathbf{P}_{N,\delta}^{T_N}\big(|S_N|^* \ge T_N \,\big|\, \mu_N = N - \ell\big)\, \mathbf{P}_{N,\delta}^{T_N}\big(\mu_N = N-\ell\big) \right| \;\le\; \varepsilon\,. \tag{5.4}$$

However on the event $\{\mu_N = N - \ell\}$ we have $|S_N|^* \ge T_N$ if and only if $|S_{N-\ell}|^* \ge T_N$. Moreover $\{|S_{N-\ell}|^* \ge T_N\} = \{L'_{N-\ell} \ge 1\}$, hence, using the Markov property and (2.13), for ℓ even we get

$$\begin{aligned} \mathbf{P}_{N,\delta}^{T_N}\big(|S_N|^* \ge T_N \,\big|\, \mu_N = N-\ell\big) \;&=\; \mathbf{P}_{N,\delta}^{T_N}\big(L'_{N-\ell} \ge 1 \,\big|\, N-\ell \in \tau^{T_N}\big) \\ &=\; \mathcal{P}_{\delta,T_N}\big(L'_{N-\ell} \ge 1 \,\big|\, N - \ell \in \tau\big) \;\longrightarrow\; 0 \qquad (N \to \infty,\, N \in 2\mathbb{N})\,. \end{aligned}$$

Then equation (5.4) yields, for N sufficiently large,

$$\mathbf{P}_{N,\delta}^{T_N}\big(|S_N|^* \ge T_N\big) \;\le\; 2\varepsilon\,.$$

We can finally prove that S_N is tight. Denoting by ξ_N a quantity such that $|\xi_N| \le 2\varepsilon$ for N large, we have

$$\mathbf{P}_{N,\delta}^{T_N}\big(|S_N| \ge L\big) \;=\; \mathbf{P}_{N,\delta}^{T_N}\big(|S_N| \ge L,\, |S_N|^* < T_N\big) \;+\; \xi_N \;\le\; \mathbf{P}_{N,\delta}^{T_N}\big(\mu_N \le N - L\big) \;+\; \xi_N\,,$$

where the inequality follows by the inclusion bound, since $\{|S_N| \ge L,\, |S_N|^* < T_N\} \subseteq \{\mu_N \le N - L\}$. Then, again by (3.27), if $L \ge \ell_0$ we have for large N

$$\mathbf{P}_{N,\delta}^{T_N}\big(|S_N| \ge L\big) \;\le\; 3\varepsilon\,.$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$\lim_{L \to \infty} \sup_{N \in \mathbb{N}} \mathbf{P}_{N,\delta}^{T_N}\big(|S_N| \ge L\big) \;=\; 0\,,$$

hence (1.15) is proven and the proof is completed. □

Appendix A. Computing $Q_T^i(\lambda)$

The computation of $Q_T^1(\lambda)$ and $Q_T^2(\lambda)$, defined in (2.4), is a classical problem, cf. [10, Ch. XIV]. For completeness, here we are going to derive an explicit formula for $Q_T^1(\lambda)$ and $Q_T^2(\lambda)$, using a simple martingale argument. We assume that $T \in \mathbb{N}$ (i.e., $T < \infty$).

For $\mu \in \mathbb{C}$ and $n \in \mathbb{N}$ we set

$$M_n \;:=\; \frac{e^{\mu S_n}}{(\cosh \mu)^n}$$

and we observe that the $\{M_n\}_{n \ge 0}$ under $\mathbf{P}$ is a $\mathbb{C}$-valued martingale (i.e., its real and imaginary parts are $\mathbb{R}$-valued martingales) with respect to the natural filtration of the simple random walk $\{S_i\}_i$. We will be only interested in the special cases when $\mu \in \mathbb{R}$ or $\mu \in (-\frac{\pi}{2}i, \frac{\pi}{2}i)$, so that in any case $\cosh \mu \in \mathbb{R}^+$ and therefore the expression $\log \cosh \mu$ is well-defined with no need of further specifications.

We denote by $\mathbf{P}_1$ the law $\mathbf{P}(\,\cdot\,|\,S_1 = 1\,)$ and note that $\{M_n\}_{n\geq 1}$ is a martingale under $\mathbf{P}_1$. Moreover both τ_1^T and $|\varepsilon_1^T|$ have the same law under $\mathbf{P}$ and $\mathbf{P}_1$. The Optimal Stopping Theorem yields $\mathbf{E}_1(S_{\tau_1^T}) = \mathbf{E}_1(S_1)$, i.e.,

$$e^{\mu T}\mathbf{E}_1\bigg(\frac{1}{(\cosh\mu)^{\tau_1^T}}\,\mathbf{1}_{\{|\varepsilon_1^T|=1\}}\bigg) \;+\; \mathbf{E}_1\bigg(\frac{1}{(\cosh\mu)^{\tau_1^T}}\,\mathbf{1}_{\{|\varepsilon_1^T|=0\}}\bigg) \;=\; \frac{e^{\mu}}{\cosh\mu}$$

Setting for short $Q_T^i := Q_T^i(\log\cosh\mu)$, we can rewrite this relation as

$$2\,e^{\mu T}\,Q_T^1 \;+\; Q_T^0 \;=\; \frac{e^{\mu}}{\cosh\mu}\,.$$

The analogous relation with μ replaced by $-\mu$ leads to the following couple of equations:

$$\begin{aligned} 2\,\cosh(\mu T)\,Q_T^1 \;+\; Q_T^0 \;&=\; 1\\ 2\,\sinh(\mu T)\,Q_T^1 \;&=\; \tanh\mu\,, \end{aligned}$$

which yields the solutions

$$Q_T^0(\log\cosh\mu) \;=\; 1 - \frac{\tanh(\mu)}{\tanh(\mu T)}\,, \qquad\qquad Q_T^1(\log\cosh\mu) \;=\; \frac{\tanh(\mu)}{2\sinh(\mu T)}\,, \tag{A.1}$$

and for $Q_T(\cdot) := Q_T^0(\cdot) + 2Q_T^1(\cdot)$ we have

$$Q_T(\log\cosh\mu) \;=\; 1 - \tanh(\mu)\cdot\frac{\cosh(\mu T)-1}{\sinh(\mu T)}\,. \tag{A.2}$$

Setting $\lambda = \log\cosh\mu$, i.e., $\mu = \lambda + \log\big(1+\sqrt{1-e^{-2\lambda}}\big)$, we finally obtain

$$\begin{aligned} Q_T^0(\lambda) \;&=\; 1 - \sqrt{1-e^{-2\lambda}}\cdot\frac{\big(1+\sqrt{1-e^{-2\lambda}}\big)^T + \big(1-\sqrt{1-e^{-2\lambda}}\big)^T}{\big(1+\sqrt{1-e^{-2\lambda}}\big)^T - \big(1-\sqrt{1-e^{-2\lambda}}\big)^T}\\ Q_T^1(\lambda) \;&=\; \frac{\sqrt{1-e^{-2\lambda}}\cdot e^{-\lambda T}}{\big(1+\sqrt{1-e^{-2\lambda}}\big)^T - \big(1-\sqrt{1-e^{-2\lambda}}\big)^T}\,, \end{aligned} \tag{A.3}$$

and therefore

$$Q_T(\lambda) \;=\; 1 \;-\; \sqrt{1-e^{-2\lambda}}\cdot\frac{\big(1+\sqrt{1-e^{-2\lambda}}\big)^T + \big(1-\sqrt{1-e^{-2\lambda}}\big)^T - 2e^{-\lambda T}}{\big(1+\sqrt{1-e^{-2\lambda}}\big)^T - \big(1-\sqrt{1-e^{-2\lambda}}\big)^T}\,. \tag{A.4}$$

Notice that when $\lambda < 0$ we have $\mu = \lambda + \log\big(1+\sqrt{1-e^{-2\lambda}}\big) = i\arctan\sqrt{e^{-2\lambda}-1}$, hence we can write more explicitly

$$\begin{aligned} Q_T^0(\lambda) \;=\; 1 - \frac{\sqrt{e^{-2\lambda}-1}}{\tan\big(T\arctan\sqrt{e^{-2\lambda}-1}\big)}\,, \quad Q_T^1(\lambda) \;&=\; \frac{\sqrt{e^{-2\lambda}-1}}{2\sin\big(T\arctan\sqrt{e^{-2\lambda}-1}\big)}\\ Q_T(\lambda) \;=\; 1 + \sqrt{e^{-2\lambda}-1}\cdot\frac{1-\cos\big(T\arctan\sqrt{e^{-2\lambda}-1}\big)}{\sin\big(T\arctan\sqrt{e^{-2\lambda}-1}\big)}\,. \end{aligned} \tag{A.5}$$

Of course these formulas break down if $|\lambda|$ is too large. This happens at the first negative zero of the denominator $\lambda = \lambda_0^T$, where $\big(T\arctan\sqrt{e^{-2\lambda_0^T}-1}\big) = \pi$, i.e.,

$$\lambda_0^T \;:=\; -\frac{1}{2}\,\log\bigg(1+\Big(\tan\frac{\pi}{T}\Big)^2\bigg)\,, \tag{A.6}$$

and note that as $\lambda \downarrow \lambda_o^T$ both $Q_T^0(\lambda)$ and $Q_T^1(\lambda)$ diverge (they have a pole). Also note that taking the limit $\lambda \to 0$ in (A.3) or (A.5) we get

$$Q_T^0(0) \;=\; 1 - \frac{1}{T}\,, \qquad\qquad Q_T^1(0) \;=\; \frac{1}{2T}\,.$$

We conclude by noting that also the probabilities $q^j(n)$ introduced in (2.4) can be given an explicit formula. More precisely, by equation (5.8) in Chapter XIV of [10] we have $\forall n \geq 2$

$$\begin{aligned} q_T^0(n) \;&=\; \left(\frac{2}{T} \sum_{\nu=1}^{\lfloor (T-1)/2 \rfloor} \cos^{n-2}\left(\frac{\pi\nu}{T}\right) \sin^2\left(\frac{\pi\nu}{T}\right) \right) \cdot \mathbf{1}_{\{n \text{ is even}\}} \\ q_T^1(n) \;&=\; \left(\frac{1}{T} \sum_{\nu=1}^{\lfloor (T-1)/2 \rfloor} (-1)^{\nu+1} \cos^{n-2}\left(\frac{\pi\nu}{T}\right) \sin^2\left(\frac{\pi\nu}{T}\right) \right) \cdot \mathbf{1}_{\{n-T \text{ is even}\}} \end{aligned} \; . \qquad \text{(A.7)}$$

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