ON THE MOMENTS OF THE $(2+1)$-DIMENSIONAL DIRECTED POLYMER AND STOCHASTIC HEAT EQUATION IN THE CRITICAL WINDOW

FRANCESCO CARAVENNA, RONGFENG SUN, AND NIKOS ZYGOURAS

ABSTRACT. The partition function of the directed polymer model on $\mathbb{Z}^{2+1}$ undergoes a phase transition in a suitable continuum and weak disorder limit. In this paper, we focus on a window around the critical point. Exploiting local renewal theorems, we compute the limiting third moment of the space-averaged partition function, showing that it is uniformly bounded. This implies that the rescaled partition functions, viewed as a generalized random field on $\mathbb{R}^2$, have non-trivial subsequential limits, and each such limit has the same explicit covariance structure. We obtain analogous results for the stochastic heat equation on $\mathbb{R}^2$, extending previous work by Bertini and Cancrini [BC98].

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1. Introduction and results

We set $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We write $a_n \sim b_n$ to mean $\lim_{n\to\infty} a_n/b_n = 1$. We denote by $C_b(\mathbb{R}^d)$ (resp. $C_c(\mathbb{R}^d)$) the space of continuous and bounded (resp. compactly supported) real functions defined on $\mathbb{R}^d$, with norm $|\phi|_\infty := \sup_{x\in\mathbb{R}^d} |\phi(x)|$.

1.1. Directed polymer in random environment. One of the simplest, yet also most interesting models of disordered system is the directed polymer model in random environment on $\mathbb{Z}^{d+1}$, which has been the subject of the recent monograph by Comets [Com17].

Let $S = (S_n)_{n\in\mathbb{N}_0}$ be the simple symmetric random walk on $\mathbb{Z}^d$. The random environment (or disorder) is a collection $\omega = (\omega_{n,x})_{(n,x)\in\mathbb{N}\times\mathbb{Z}^d}$ of i.i.d. random variables. We use P and E, resp. $\mathbb{P}$ and $\mathbb{E}$, to denote probability and expectation for S, resp. for ω. We assume that

$$\mathbb{E}[\omega_{n,x}] = 0\,, \qquad \mathbb{V}\mathrm{ar}[\omega_{n,x}] = 1\,, \qquad \lambda(\beta) := \log \mathbb{E}[e^{\beta\omega_{n,x}}] \in \mathbb{R} \quad \text{for small } \beta > 0. \tag{1.1}$$

Given ω, polymer length $N \in \mathbb{N}$, and inverse temperature (or disorder strength) $\beta > 0$, the polymer measure P_N^β is then defined via a Gibbs change of measure for S:

$$\mathrm{P}_N^\beta(S) := \frac{e^{\sum_{n=1}^{N-1}(\beta\omega_{n,S_n} - \lambda(\beta))}}{Z_N^\beta}\, \mathrm{P}(S)\,, \tag{1.2}$$

where Z_N^β is the normalization constant, called *partition function*:

$$Z_N^\beta := \mathrm{E}\left[e^{\sum_{n=1}^{N-1}(\beta\omega_{n,S_n} - \lambda(\beta))}\right]. \tag{1.3}$$

(We stop the sum at $N-1$ instead of N, which is immaterial, for later notational convenience.) Note that Z_N^β is a random variable, as a function of ω.

We use P_z and E_z to denote probability and expectation for the random walk starting at $S_0 = z \in \mathbb{Z}^d$. We denote by $Z_N^\beta(z)$ the corresponding partition function:

$$Z_N^\beta(z) := \mathrm{E}_z\left[e^{\sum_{n=1}^{N-1}(\beta\omega_{n,S_n} - \lambda(\beta))}\right]. \tag{1.4}$$

We investigate the behavior as $N \to \infty$ of the diffusively rescaled random field

$$\left\{ Z_{Nt}^{\beta_N}\big(x\sqrt{N}\big) \;:\quad t > 0\,,\ x \in \mathbb{R}^d \right\}, \tag{1.5}$$

for suitable $\beta = \beta_N$, where we agree that $Z_N^\beta(z) := Z_{\lfloor N\rfloor}^\beta(\lfloor z\rfloor)$ for non-integer N, z.

In dimension $d = 1$, Alberts, Khanin and Quastel [AKQ14] showed that for $\beta_N = \hat\beta N^{-1/4}$, the random field (1.5) converges in distribution to the Wiener chaos solution $u(t,x)$ of the one-dimensional stochastic heat equation (SHE)

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta u(t,x) + \hat\beta\, \dot W(t,x)\, u(t,x)\,, \qquad u(0,x) \equiv 1\,, \tag{1.6}$$

where $\dot W$ is space-time white noise on $\mathbb{R}\times\mathbb{R}$. The existence of such an intermediate disorder regime is a general phenomenon among models that are so-called *disorder relevant*, see [CSZ17a], and the directed polymer in dimension $d = 1$ is one such example.

A natural question is whether an intermediate disorder regime also exists for the directed polymer in dimension $d = 2$. We gave an affirmative answer in [CSZ17b], although the problem turns out to be much more subtle than $d = 1$. The standard Wiener chaos approach fails, because the model in $d = 2$ is so-called marginally relevant, or critical. We will further elaborate on this later. Let us recall the results from [CSZ17b], which provide the starting point of this paper.

Henceforth we focus on $d = 2$, so $S = (S_n)_{n \in \mathbb{N}_0}$ is the simple random walk on $\mathbb{Z}^2$. Let

$$\mathbb{Z}^k_{\text{even}} := \{(z_1, \ldots, z_k) \in \mathbb{Z}^k : z_1 + \cdots + z_k \text{ is even}\}. \tag{1.7}$$

Due to periodicity, if we take $S_0 \in \mathbb{Z}^2_{\text{even}}$, then $(n, S_n) \in \mathbb{Z}^3_{\text{even}}$ for all $n \in \mathbb{N}$. The transition probability kernel of S will be denoted by

$$q_n(x) := \mathrm{P}(S_n = x \,|\, S_0 = 0) = \big(g_{n/2}(x) + o(\tfrac{1}{n})\big)\, 2\, \mathbb{1}_{\{(n,x) \in \mathbb{Z}^3_{\text{even}}\}} \qquad \text{as } n \to \infty, \tag{1.8}$$

by the local central limit theorem, where $g_u(\cdot)$ is the standard Gaussian density on $\mathbb{R}^2$:

$$g_u(x) := \frac{1}{2\pi u}\, e^{-\frac{|x|^2}{2u}}, \qquad u > 0, \; x \in \mathbb{R}^2. \tag{1.9}$$

For notational convenience, we will drop the conditioning in (1.8) when the random walk starts from zero. The multiplicative factor 2 in (1.8) is due to periodicity, while the Gaussian density $g_{n/2}(x)$ is due to the fact that at time n, the walk has covariance matrix $\frac{n}{2} I$.

The overlap (expected number of encounters) of two independent simple symmetric random walks S and S' on $\mathbb{Z}^2$ is defined by

$$R_N := \sum_{n=1}^N \mathrm{P}(S_n = S'_n) = \sum_{n=1}^N \sum_{x \in \mathbb{Z}^2} q_n(x)^2 = \sum_{n=1}^N q_{2n}(0) = \frac{\log N}{\pi}\big(1 + o(1)\big) \tag{1.10}$$

where the asymptotic behavior follows from (1.8). It was shown in [CSZ17b] that the correct choice of the disorder strength is $\beta = \beta_N = \hat\beta / \sqrt{R_N}$. More precisely, denoting by W_1 a standard normal, we have the following convergence in distribution:

$$Z_N^{\beta_N} \xrightarrow[N \to \infty]{d} \begin{cases} \exp\Big(\sigma_{\hat\beta} W_1 - \frac{1}{2}\sigma^2_{\hat\beta}\Big) & \text{if } \hat\beta < 1 \\ 0 & \text{if } \hat\beta \geqslant 1 \end{cases}, \qquad \text{where} \quad \sigma^2_{\hat\beta} := \log \tfrac{1}{1 - \hat\beta^2}. \tag{1.11}$$

This establishes a weak to strong disorder phase transition in $\hat\beta$ (with critical point $\hat\beta_c = 1$), similar to what was known for the directed polymer model in $\mathbb{Z}^{d+1}$ with $d \geqslant 3$ [Com17]. It was also proved in [CSZ17b, Theorem 2.13] that for $\hat\beta < 1$, after centering and rescaling, the random field of partition functions (1.5) converges to the solution of the SHE with *additive* space-time white noise, known as Edwards-Wilkinson fluctuation. Similar results have been recently obtained in [GRZ17] for the SHE with multiplicative noise.

The behavior at the critical point $\hat\beta = \hat\beta_c$, i.e. $\beta_N = 1/\sqrt{R_N}$, is quite subtle. For each $x \in \mathbb{R}^2$ and $t > 0$, the partition function $Z^{\beta_N}_{Nt}(x\sqrt{N})$ converges to zero in distribution as $N \to \infty$, by (1.11), while its expectation is identically one, see (1.4), and its second moment diverges. This suggests that the random field $x \mapsto Z^{\beta_N}_{Nt}(x\sqrt{N})$ becomes *rough* as $N \to \infty$, so we should look at it as a *random distribution on* $\mathbb{R}^2$ (actually a random measure, see below). We thus average the field in space and define

$$Z^{\beta_N}_{Nt}(\phi) := \frac{1}{N} \sum_{x \in \frac{1}{\sqrt{N}}\mathbb{Z}^2} \phi(x)\, Z^{\beta_N}_{Nt}\big(x\sqrt{N}\big), \qquad \text{for} \quad \phi \in C_c(\mathbb{R}^2). \tag{1.12}$$

The first moment of $Z^{\beta_N}_{Nt}(\phi)$ is easily computed by Riemann sum approximation:

$$\lim_{N \to \infty} \mathbb{E}\big[Z^{\beta_N}_{Nt}(\phi)\big] = \lim_{N \to \infty} \frac{1}{N} \sum_{x \in \frac{1}{\sqrt{N}}\mathbb{Z}^2} \phi(x) = \int_{\mathbb{R}^2} \phi(x)\, \mathrm{d}x. \tag{1.13}$$

Our main result is the sharp asymptotic evaluation of the *second and third moments*. These will yield important information on the convergence of the generalized random field (1.12).

Let us first specify our choice of $\beta = \beta_N$. Recalling that $\lambda(\cdot)$ is the log-moment generating function of the disorder ω, see (1.1), we fix β_N such that

$$\sigma_N^2 := e^{\lambda(2\beta_N)-2\lambda(\beta_N)} - 1 \underset{N\to\infty}{=} \frac{1}{R_N}\left(1 + \frac{\vartheta}{\log N}\big(1+o(1)\big)\right), \quad \text{for some } \vartheta \in \mathbb{R}. \tag{1.14}$$

Since $\lambda(t) \sim \frac{1}{2}t^2$ as $t \to 0$, we have $\beta_N \sim 1/\sqrt{R_N}$, so we are indeed exploring a window around the critical point $\hat{\beta}_c = 1$. Let us recall the *Euler-Mascheroni constant*:

$$\gamma := -\int_0^\infty e^{-u} \log u \, \mathrm{d}u \simeq 0.577\,. \tag{1.15}$$

Remark 1.1. *The asymptotic behavior in* (1.10) *can be refined as follows:*

$$R_N = \tfrac{\log N}{\pi} + \tfrac{\alpha}{\pi} + o(1) \qquad \textit{where} \qquad \alpha := \gamma + \log 16 - \pi\,, \tag{1.16}$$

see [CSZ18, Proposition 3.2]. *This leads to an equivalent reformulation of* (1.14)*:*

$$\sigma_N^2 = \tfrac{\pi}{\log N}\big(1 + \tfrac{\vartheta-\alpha}{\log N}(1+o(1))\big)\,.$$

It is possible to express this condition in terms of β_N (see [CSZ18, Appendix A.4]*):*

$$\beta_N^2 = \tfrac{\pi}{\log N} - \tfrac{\kappa_3\,\pi^{3/2}}{(\log N)^{3/2}} + \tfrac{\pi(\vartheta-\alpha)+\pi^2(\frac{3}{2}\kappa_3^2-\frac{1}{2}-\frac{7}{12}\kappa_4)}{(\log N)^2}\big(1+o(1)\big)\,, \tag{1.17}$$

where κ_3, κ_4 are the disorder cumulants, i.e. $\lambda(t) = \frac{1}{2}t^2 + \frac{\kappa_3}{3!}t^3 + \frac{\kappa_4}{4!}t^4 + O(t^5)$ as $t \to 0$.

We define the following special function:

$$G_\vartheta(w) := \int_0^\infty \frac{e^{(\vartheta-\gamma)s}\, s\, w^{s-1}}{\Gamma(s+1)}\, \mathrm{d}s\,, \qquad w \in (0,\infty)\,. \tag{1.18}$$

We now state our first result, where we compute the second moment of $Z_{Nt}^{\beta_N}(\phi)$.

Theorem 1.2 (Second moment). *Let $\phi \in C_c(\mathbb{R}^2)$, $t > 0$, $\vartheta \in \mathbb{R}$. Let β_N satisfy* (1.14)*. Then*

$$\lim_{N\to\infty} \mathbb{V}\mathrm{ar}\left[Z_{Nt}^{\beta_N}(\phi)\right] = \int_{\mathbb{R}^2\times\mathbb{R}^2} \phi(z)\,\phi(z')\,K_{t,\vartheta}(z-z')\,\mathrm{d}z\,\mathrm{d}z'\,, \tag{1.19}$$

where the covariance kernel $K_{t,\vartheta}(\cdot)$ is given by

$$K_{t,\vartheta}(x) := \pi \int\limits_{0<u<v<t} g_u(x)\,G_\vartheta(v-u)\,\mathrm{d}u\,\mathrm{d}v\,. \tag{1.20}$$

The same covariance kernel $K_{t,\vartheta}$ was derived by different methods by Bertini and Cancrini [BC98] for the 2d Stochastic Heat Equation, see Subsection 1.2. It is not difficult to see that

$$K_{t,\vartheta}(x) \sim C_t \log \tfrac{1}{|x|}, \quad \text{as} \quad |x| \to 0, \tag{1.21}$$

with $C_t \in (0,\infty)$, and hence the integral in (1.19) is finite.

Remark 1.3 (Scaling covariance). *It is easily checked from* (1.20) *that for any $t > 0$,*

$$K_{t,\vartheta}(x) = K_{1,\vartheta_t}(x/\sqrt{t}) \qquad \textit{with} \qquad \vartheta_t := \vartheta + \log t. \tag{1.22}$$

This is also clear because we can write $Z_{Nt}^{\beta_N}(\phi) = Z_M^{\beta_N}(\phi_t)$ with $M := Nt$ and $\phi_t(x) := t\,\phi(\sqrt{t}x)$, see (1.12)*, and note that β_N can be expressed as β_M, provided ϑ is replaced by $\vartheta_t = \vartheta + \log t$ (just set N equal to Nt in* (1.14)*, and recall* (1.16)*).*

The starting point of the proof of Theorem 1.2 is a polynomial chaos expansion of the partition function. The variance computation can then be cast in a *renewal theory framework*, which is the cornerstone of our approach (see Subsection 1.3 for an outline). This allows us to capture the much more challenging *third moment* of the field. Let us extend the function $G_\vartheta(w)$ in (1.18) with a spatial component, recalling (1.9):

$$G_\vartheta(w,x) := G_\vartheta(w)\, g_{w/4}(x)\,, \qquad w>0\,,\ x\in\mathbb{R}^2\,. \tag{1.23}$$

We can now state the main result of this paper.

Theorem 1.4 (Third moment). *Let $\phi\in C_c(\mathbb{R}^2)$, $t>0$, $\vartheta\in\mathbb{R}$. Let β_N satisfy (1.14). Then*

$$\lim_{N\to\infty}\mathbb{E}\Big[\big(Z^{\beta_N}_{Nt}(\phi)-\mathbb{E}[Z^{\beta_N}_{Nt}(\phi)]\big)^3\Big] = \int\limits_{(\mathbb{R}^2)^3}\phi(z)\,\phi(z')\,\phi(z'')\,M_{t,\vartheta}(z,z',z'')\,\mathrm{d}z\,\mathrm{d}z'\,\mathrm{d}z'' < \infty\,, \tag{1.24}$$

where the kernel $M_{t,\vartheta}(\cdot)$ is given by

$$M_{t,\vartheta}(z,z',z'') := \sum_{m=2}^{\infty} 2^{m-1}\,\pi^m\,\big\{\mathcal{I}^{(m)}_{t,\vartheta}(z,z',z'')+\mathcal{I}^{(m)}_{t,\vartheta}(z',z'',z)+\mathcal{I}^{(m)}_{t,\vartheta}(z'',z,z')\big\}\,, \tag{1.25}$$

with $\mathcal{I}^{(m)}_{t,\vartheta}(\cdot)$ defined as follows:

$$\begin{aligned}\mathcal{I}^{(m)}_{t,\vartheta}(z,z',z'') := \int\!\!\cdots\!\!\int\limits_{\substack{0<a_1<b_1<\ldots<a_m<b_m<t\\ x_1,y_1,\ldots,x_m,y_m\in\mathbb{R}^2}} &\mathrm{d}\vec{a}\,\mathrm{d}\vec{b}\,\mathrm{d}\vec{x}\,\mathrm{d}\vec{y}\ \ g_{\frac{a_1}{2}}(x_1-z)\,g_{\frac{a_1}{2}}(x_1-z')\,g_{\frac{a_2}{2}}(x_2-z'')\\ &\cdot G_\vartheta(b_1-a_1,y_1-x_1)\,g_{\frac{a_2-b_1}{2}}(x_2-y_1)\,G_\vartheta(b_2-a_2,y_2-x_2)\\ &\cdot\prod_{i=3}^{m} g_{\frac{a_i-b_{i-2}}{2}}(x_i-y_{i-2})\,g_{\frac{a_i-b_{i-1}}{2}}(x_i-y_{i-1})\,G_\vartheta(b_i-a_i,y_i-x_i)\,.\end{aligned} \tag{1.26}$$

The expression (1.26) reflects a key combinatorial structure which emerges from our renewal framework. Establishing the convergence of the series in (1.25) is highly non-trivial, which shows how delicate things become in the critical window.

We remark that relation (1.24) holds also for the mixed centered third moment with different test functions $\phi^{(1)},\phi^{(2)},\phi^{(3)}\in C_c(\mathbb{R}^2)$, with the same kernel $M_{t,\vartheta}(z,z',z'')$. Note that this kernel is invariant under *any* permutation of its variables, because $\mathcal{I}^{(m)}_{t,\vartheta}(z,z',z'')$ is symmetric in z and z' (but not in z'', hence the need of symmetrization in (1.25)).

Let us finally come back to the convergence of the random field $Z^{\beta_N}_{Nt}(x\sqrt{N})$ of diffusively rescaled partition functions. By averaging with respect to a test function, as in (1.12), we regard this field as a *random measure on* $\mathbb{R}^2$. More explicitly, if we define

$$\mathscr{Z}^{\beta_N}_{Nt}(\mathrm{d}x) := \frac{1}{N}\sum_{y\in\frac{1}{\sqrt{N}}\mathbb{Z}^2} Z^{\beta_N}_{Nt}\big(y\sqrt{N}\big)\,\delta_y(\mathrm{d}x)\,, \tag{1.27}$$

we can write $Z^{\beta_N}_{Nt}(\phi)=\int_{\mathbb{R}^2}\phi(x)\,\mathscr{Z}^{\beta_N}_{Nt}(\mathrm{d}x)$, see (1.12). Note that $(\mathscr{Z}^{\beta_N}_{Nt})_{N\in\mathbb{N}}$ is a sequence of random variables taking values in $\mathcal{M}(\mathbb{R}^2)$, the Polish space of locally finite measures on $\mathbb{R}^2$ with the vague topology (i.e. $\nu_n\to\nu$ in $\mathcal{M}(\mathbb{R}^2)$ if and only if $\int\phi\,\mathrm{d}\nu_n\to\int\phi\,\mathrm{d}\nu$ for any $\phi\in C_c(\mathbb{R}^2)$). We can make the following remarks.

- The convergence of the first moment (1.13) implies tightness of $(\mathscr{Z}_{Nt}^{\beta_N})_{N\in\mathbb{N}}$, see [K97, Lemma 14.15]. This yields the existence of weak subsequential limits:

$$\mathscr{Z}_{Nt}^{\beta_N}(\mathrm{d}x) \xrightarrow{d} \boldsymbol{\mathcal{Z}}(\mathrm{d}x) \qquad \text{as } N\to\infty \text{ along a subsequence}\,,$$

where the limit $\boldsymbol{\mathcal{Z}}(\mathrm{d}x) = \boldsymbol{\mathcal{Z}}_{t,\vartheta}(\mathrm{d}x)$ can in principle depend on the subsequence.

- The convergence of the second moment (1.19) implies uniform integrability of $Z_{Nt}^{\beta_N}(\phi)$. It follows that any subsequential limit $\boldsymbol{\mathcal{Z}}(\mathrm{d}x)$ has mean measure given by Lebesgue measure: $\mathbb{E}\big[\int_{\mathbb{R}^2}\phi(x)\,\boldsymbol{\mathcal{Z}}(\mathrm{d}x)\big] = \int \phi(x)\,\mathrm{d}x$. Moreover, by (1.19) and Fatou's Lemma,

$$\mathbb{V}\mathrm{ar}\bigg[\int_{\mathbb{R}^2}\phi(x)\,\boldsymbol{\mathcal{Z}}(\mathrm{d}x)\bigg] \leqslant \int_{\mathbb{R}^2\times\mathbb{R}^2}\phi(z)\,\phi(z')\,K_{t,\vartheta}(z-z')\,\mathrm{d}z\,\mathrm{d}z' \;<\; \infty\,. \tag{1.28}$$

However, this does not rule out that the variance in (1.28) might actually vanish, in which case the limit $\boldsymbol{\mathcal{Z}}(\mathrm{d}x)$ would just be the trivial Lebesgue measure.

- *The convergence of the third moment* (1.24) *rules out this triviality.* Indeed, (1.24) implies that $\mathbb{E}[|Z_{Nt}^{\beta_N}(\phi)|^3] \leqslant \mathbb{E}[Z_{Nt}^{\beta_N}(|\phi|)^3]$ is bounded, so the squares $Z_{Nt}^{\beta_N}(\phi)^2$ are uniformly integrable and the inequality in (1.28) is actually an equality.

We can combine the previous considerations in the following result.

Theorem 1.5. *Let $t>0$, $\vartheta\in\mathbb{R}$. Let β_N satisfy* (1.14)*. The random measures $(\mathscr{Z}_{Nt}^{\beta_N}(\mathrm{d}x))_{N\in\mathbb{N}}$ in* (1.27) *admit weak subsequential limits $\boldsymbol{\mathcal{Z}}_{t,\vartheta}(\mathrm{d}x)$, and any such limit satisfies*

$$\mathbb{E}\bigg[\int_{\mathbb{R}^2}\phi(x)\,\boldsymbol{\mathcal{Z}}_{t,\vartheta}(\mathrm{d}x)\bigg] \;=\; \int\phi(x)\,\mathrm{d}x \tag{1.29}$$

$$\mathbb{V}\mathrm{ar}\bigg[\int_{\mathbb{R}^2}\phi(x)\,\boldsymbol{\mathcal{Z}}_{t,\vartheta}(\mathrm{d}x)\bigg] \;=\; \int_{\mathbb{R}^2\times\mathbb{R}^2}\phi(z)\,\phi(z')\,K_{t,\vartheta}(z-z')\,\mathrm{d}z\,\mathrm{d}z' \tag{1.30}$$

$$\mathbb{E}\bigg[\bigg|\int_{\mathbb{R}^2}\phi(x)\,\boldsymbol{\mathcal{Z}}_{t,\vartheta}(\mathrm{d}x)\bigg|^3\bigg] \;<\; \infty\,. \tag{1.31}$$

In particular, every weak subsequential limit $\boldsymbol{\mathcal{Z}}_{t,\vartheta}(\mathrm{d}x)$ is a random measure with the same covariance structure. It is natural to conjecture that the whole sequence $(\mathscr{Z}_{Nt}^{\beta_N}(\mathrm{d}x))_{N\in\mathbb{N}}$ has a weak limit, but this remains to be proved.

We conclude with a remark on *intermittency*. As the asymptotics behavior (1.21) suggests, when we fix the starting point of the partition function instead of averaging over it, i.e. we consider $Z_N^{\beta_N}$ defined in (1.3), the second moment blows up like $\log N$. More precisely, in [CSZ18, Proposition A.1] we have shown that as $N\to\infty$

$$\mathbb{E}\big[(Z_N^{\beta_N})^2\big] \sim c\,(\log N)\,, \qquad \text{with} \qquad c = \textstyle\int_0^1 G_\vartheta(t)\,\mathrm{d}t\,. \tag{1.32}$$

This is a signature of intermittency, because it shows that $\mathbb{E}\big[(Z_N^{\beta_N})^2\big] \gg \mathbb{E}[Z_N^{\beta_N}]^2 = 1$. It also implies that for any $q\geqslant 2$ we have the bound

$$\mathbb{E}\big[(Z_N^{\beta_N})^q\big] \geqslant c'\,(\log N)^{q-1}\,. \tag{1.33}$$

Indeed, since $\mathbb{E}[Z_N^{\beta_N}]=1$, we can introduce the size-biased probability $\mathbb{P}^*(A) := \mathbb{E}[\mathbb{1}_A\,Z_N^{\beta_N}]$ and note that $\mathbb{E}\big[(Z_N^{\beta_N})^q\big] = \mathbb{E}^*\big[(Z_N^{\beta_N})^{q-1}\big] \geqslant \mathbb{E}^*\big[Z_N^{\beta_N}\big]^{q-1} = \mathbb{E}\big[(Z_N^{\beta_N})^2\big]^{q-1}$ by Jensen.

Remark 1.6. *We formulated our results only for the directed polymer on $\mathbb{Z}^{2+1}$, but our techniques carry through for other marginally relevant directed polymer type models, such as* *the disordered pinning model with tail exponent* $1/2$*, and the directed polymer on* $\mathbb{Z}^{1+1}$ *with Cauchy tails (see* [CSZ17b]*).*

1.2. The $2d$ stochastic heat equation. An analogue of Theorem 1.2 for the stochastic heat equation (SHE) in $\mathbb{R}^2$ was proved by Bertini and Cancrini in [BC98], although they did not obtain the analogue of Theorem 1.4. We formulate these results next.

The SHE as written in (1.6) is ill-posed due to the product $\dot{W} \cdot u$. To make sense of it, we mollify the space-time white noise $\dot{W}$ in the space variable. Let $j \in C_c^\infty(\mathbb{R}^2)$ be a probability density on $\mathbb{R}^2$ with $j(x) = j(-x)$, and let

$$J := j * j.$$

For $\varepsilon > 0$, let $j_\varepsilon(x) := \varepsilon^{-2} j(x/\varepsilon)$. Then the space-mollified noise $\dot{W}^\varepsilon$ is defined by $\dot{W}^\varepsilon(t,x) := \int_{\mathbb{R}^2} j_\varepsilon(x-y)\dot{W}(t,y)\mathrm{d}y$. We consider the mollified equation

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{1}{2}\Delta u^\varepsilon + \beta_\varepsilon\, u^\varepsilon\, \dot{W}^\varepsilon\,, \qquad u^\varepsilon(0,x) = 1 \ \ \forall\, x \in \mathbb{R}^2\,, \tag{1.34}$$

which admits a unique mild solution (with Ito integration).

It was shown in [CSZ17b] that if we rescale $\beta_\varepsilon := \hat{\beta}\sqrt{\frac{2\pi}{\log \varepsilon^{-1}}}$, then for any fixed $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^2$ the mollified solution $u^\varepsilon(t,x)$ converges in distribution as $\varepsilon \to 0$ to the same limit as in (1.11) for the directed polymer partition function, with $\hat{\beta}_c = 1$ being the critical point.

In [BC98], Bertini and Cancrini considered the critical window around $\hat{\beta}_c = 1$ given by

$$\beta_\varepsilon^2 = \frac{2\pi}{\log\frac{1}{\varepsilon}} + \frac{\varrho + o(1)}{(\log\frac{1}{\varepsilon})^2}\,, \qquad \text{with } \varrho \in \mathbb{R}\,. \tag{1.35}$$

This is comparable to our choice of β_N, see (1.14) and (1.17), if we make the identification $\varepsilon^2 = 1/N$ (note that the third cumulant $\kappa_3 = 0$ for Gaussian random variables). In this critical window, $u^\varepsilon(t,x)$ converges to 0 in distribution, while its expectation is constant:

$$\mathbb{E}\bigg[\int_{\mathbb{R}^2} \phi(x) u^\varepsilon(t,x)\mathrm{d}x\bigg] \;\equiv\; \int_{\mathbb{R}^2} \phi(x)\,\mathrm{d}x\,. \tag{1.36}$$

Bertini and Cancrini showed that when interpreted as a random distribution on $\mathbb{R}^2$, $u^\varepsilon(t,\cdot)$ admits subsequential weak limits, and they computed the limiting covariance. This is the analogue of our Theorem 1.2, which we now state explicitly. Let us set

$$u^\varepsilon(t,\phi) := \int_{\mathbb{R}^2} \phi(x)\, u^\varepsilon(t,x)\,\mathrm{d}x\,, \qquad \text{for } \phi \in C_c(\mathbb{R}^2)\,.$$

Theorem 1.7 ([BC98])**.** *Let* β_ε *be chosen as in* (1.35)*. Then, for any* $\phi \in C_c(\mathbb{R}^2)$*,*

$$\lim_{\varepsilon \to 0^+} \mathbb{V}\mathrm{ar}\big[u^\varepsilon(t,\phi)\big] \;=\; 2\int_{\mathbb{R}^2\times\mathbb{R}^2} \phi(z)\,\phi(z')\,K_{t,\vartheta}\big(\tfrac{z-z'}{\sqrt{2}}\big)\,\mathrm{d}z\,\mathrm{d}z'\,, \tag{1.37}$$

where $K_{t,\vartheta}$ *is defined as in Theorem 1.2, with*

$$\vartheta = \log 4 + 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} J(x)\log\frac{1}{|x-y|}\,J(y)\,\mathrm{d}x\mathrm{d}y - \gamma + \frac{\varrho}{\pi}. \tag{1.38}$$

In Section 8 we provide an independent proof of Theorem 1.7, which employs the renewal framework of this paper. Note that, by Feynman-Kac formula, the mollified solution $u^\varepsilon(t,\phi)$ can be interpreted as the partition function of a continuum directed polymer model.

Remark 1.8. *The covariance kernel in* (1.37) *coincides with the one in* [BC98, eq. (3.14)], *provided we identify the parameter* β *in* [BC98] *with* $e^{\vartheta-\gamma}$. *If we plug* $\beta = e^{\vartheta-\gamma}$ *into* [BC98, eq. (2.6)], *with* ϑ *given by* (1.38), *we obtain precisely* (1.35).

Our renewal framework leads to analogues of Theorems 1.4 and 1.5 for the SHE. For simplicity, we content ourselves with showing that the third moment is bounded, but the same techniques would allow to compute its sharp asymptotic behavior, as in (1.24)-(1.26).

Theorem 1.9. *Follow the same assumptions and notation as in Theorem 1.7. Then*

$$\sup_{\varepsilon>0} \mathrm{E}\Big[\big(u^\varepsilon(t,\phi) - \textstyle\int_{\mathbb{R}^2}\phi(x)\,\mathrm{d}x\big)^3\Big] < \infty.$$

If $\boldsymbol{u}_\vartheta(t,\cdot)$ *is any subsequential weak limit in* $\mathcal{M}(\mathbb{R}^2)$ *of* $u^\varepsilon(t,\cdot)$ *as* $\varepsilon \to 0^+$, *then* $\boldsymbol{u}_\vartheta(t,\cdot)$ *satisfies the analogues of* (1.29)–(1.31), *with* $K_{t,\vartheta}(z-z')$ *in* (1.30) *replaced by* $2K_{t,\vartheta}\big(\frac{z-z'}{\sqrt{2}}\big)$.

1.3. Outline of the proof strategy. We present the key ideas of our approach. First we compute the *second moment* of the partition function, sketching the proof of (1.32). Then we describe the combinatorial structure of the *third moment*, which leads to Theorem 1.4. This illustrates how *renewal theory* emerges in our problem.

Second moment. We start from a *polynomial chaos expansion* of the partition function Z_N^β, which arises from a binomial expansion of the exponential in (1.3) (see Subsection 2.1):

$$\begin{aligned} Z_N^\beta = 1 + \sum_{k\geqslant 1} \sum_{\substack{0<n_1<\ldots<n_k\leqslant N \\ x_1,\ldots,x_k\in\mathbb{Z}^2}} & q_{n_1}(x_1)\,\xi_{n_1,x_1} \cdot q_{n_2-n_1}(x_2-x_1)\,\xi_{n_2,x_2} \cdot \\ & \cdot \ldots \cdot q_{n_k-n_{k-1}}(x_k-x_{k-1})\,\xi_{n_k,x_k}, \end{aligned} \tag{1.39}$$

where we set $\xi_{n,x} = e^{\beta_N\omega_{n,x}-\lambda(\beta_N)} - 1$ for $n \in \mathbb{N}, x \in \mathbb{Z}^2$. Note that $\xi_{n,x}$ are i.i.d. with mean zero and variance $\sigma^2 = e^{\lambda(2\beta)-2\lambda(\beta)} - 1$, see (1.14). Then

$$\begin{aligned} \mathbb{V}\mathrm{ar}[Z_N^\beta] &= \sum_{k\geqslant 1} (\sigma^2)^k \sum_{\substack{0<n_1<\ldots<n_k\leqslant N \\ x_1,\ldots,x_k\in\mathbb{Z}^2}} q_{n_1}(x_1)^2 \cdot q_{n_2-n_1}(x_2-x_1)^2 \cdots q_{n_k-n_{k-1}}(x_k-x_{k-1})^2 \\ &= \sum_{k\geqslant 1} (\sigma^2)^k \sum_{0<n_1<\ldots<n_k\leqslant N} u_{n_1}^2 \cdot u_{n_2-n_1}^2 \cdots u_{n_k-n_{k-1}}^2, \end{aligned} \tag{1.40}$$

where we define

$$u_n^2 := \sum_{x\in\mathbb{Z}^2} q_n(x)^2 = q_{2n}(0) = \frac{1}{\pi n} + O\Big(\frac{1}{n^2}\Big). \tag{1.41}$$

Incidentally, (1.40) coincides with the variance of the partition function of the one-dimensional disordered pinning model based on the simple random walk on $\mathbb{Z}$ [CSZ18].

The key idea is to view the series of convolutions (1.40) through the lenses of *renewal theory*. The sequence u_n^2 is not summable, but we can normalize it to a probability on $\{1,\ldots,N\}$. We thus define a triangular array of independent random variables $(T_i^{(N)})_{i\in\mathbb{N}}$ by

$$\mathrm{P}\big(T_i^{(N)} = n\big) = \frac{1}{R_N}\, u_n^2\, \mathbb{1}_{\{1\leqslant n\leqslant N\}}, \qquad \text{where} \qquad R_N := \sum_{n=1}^N u_n^2. \tag{1.42}$$

We stress that $R_N = \frac{1}{\pi}\log N + O(1)$ is the same as in (1.10). If we fix β_N satisfying (1.14), and define the renewal process

$$\tau_k^{(N)} = T_1^{(N)} + \ldots + T_k^{(N)}, \tag{1.43}$$

we can rewrite (1.40) for $\beta = \beta_N$ as follows:

$$\mathbb{V}\mathrm{ar}\big[Z_N^{\beta_N}\big] = \sum_{k\geqslant 1} \big(\sigma_N^2\, R_N\big)^k\, \mathrm{P}\big(\tau_k^{(N)} \leqslant N\big) = \sum_{k\geqslant 1} e^{\vartheta \frac{k}{\log N} + O\big(\frac{k}{(\log N)^2}\big)}\, \mathrm{P}\big(\tau_k^{(N)} \leqslant N\big)\,. \quad (1.44)$$

This shows that $\mathbb{V}\mathrm{ar}\big[Z_N^{\beta_N}\big]$ can be interpreted as a (weighted) *renewal function* for $\tau_k^{(N)}$.

The renewal process $\tau_k^{(N)}$ is investigated in [CSZ18], where we proved that $(\tau^{(N)}_{\lfloor s\log N\rfloor}/N)_{s\geqslant 0}$ converges in law as $N\to\infty$ to a special Lévy process $Y = (Y_s)_{s\geqslant 0}$, called the *Dickman subordinator*, which admits an explicit density:

$$f_s(t) := \frac{\mathrm{P}(Y_s \in \mathrm{d}t)}{\mathrm{d}t} = \frac{e^{-\gamma s}\, s\, t^{s-1}}{\Gamma(s+1)} \qquad \text{for } t\in(0,1)\,. \quad (1.45)$$

Then $\mathrm{P}(\tau^{(N)}_{\lfloor s\log N\rfloor} \leqslant N) \to \mathrm{P}(Y_s \leqslant 1) = \int_0^1 f_s(t)\,\mathrm{d}t$, and by Riemann sum approximation

$$\lim_{N\to\infty} \frac{\mathbb{V}\mathrm{ar}\big[Z_N^{\beta_N}\big]}{\log N} = \int_0^1 \mathrm{d}t \left(\int_0^\infty \mathrm{d}s\, e^{\vartheta s}\, f_s(t)\right) = \int_0^1 \mathrm{d}t\, G_\vartheta(t)\,,$$

where $G_\vartheta(\cdot)$ is the same as in (1.18), which can now be interpreted as a renewal function for the Lévy process Y. This completes the derivation of (1.32).

Similar arguments can be applied to the partition function $Z_N^{\beta_N}(\phi)$ averaged over the starting point, to prove Theorem 1.2 using renewal theory.

Third moment. The proof of Theorem 1.4 is more challenging. In the second moment computation, the spatial variables $x_1,\ldots,x_k$ have been summed over to get (1.40), reducing the analysis to a one-dimensional renewal process. Such a reduction is not possible for Theorem 1.4. In addition to the "point-to-plane" partition functions (1.3)-(1.4), it will be important to consider *point-to-point partition functions*, where we also fix the endpoint S_N:

$$Z_N^\beta(0,y) := \mathrm{E}_0\left[e^{\sum_{n=1}^{N-1}(\beta\omega_{n,S_n}-\lambda(\beta))}\,\mathbb{1}_{\{S_N=y\}}\right].$$

We need to extend our renewal theory framework, enriching the process $\tau_k^{(N)}$ with a spatial component $S_k^{(N)}$ (see (2.13)-(2.14) below). This will yield the following analogue of (1.44):

$$\sigma_N^2\, \mathbb{E}\big[Z_M^{\beta_N}(0,y)^2\big] = \sum_{k\geqslant 1} \big(\sigma_N^2\, R_N\big)^k\, \mathrm{P}\big(\tau_k^{(N)} = M, S_k^{(N)} = y\big)\,, \quad (1.46)$$

which is now a *local* (weighted) renewal function for the random walk $(\tau_k^{(N)}, S_k^{(N)})_{k\geqslant 0}$. Its asymptotic behavior as $N\to\infty$ was determined in [CSZ18]:

$$\sigma_N^2\, \mathbb{E}\big[Z_M^{\beta_N}(0,y)^2\big] \sim \frac{\log N}{N^2}\, G_\vartheta\big(\tfrac{M}{N}, \tfrac{y}{\sqrt{N}}\big)\,, \quad (1.47)$$

where $G_\vartheta(t,z)$, defined in (1.23), is a continuum local renewal function.

We now explain how the second moment of the point-to-point partition function (1.46) enters in the third moment computation. We consider the partition function Z_N^β started at the origin, see (1.3), but everything extends to the averaged partition function $Z_N^\beta(\phi)$.

We compute $\mathbb{E}[(Z_N^\beta - 1)^3]$ using the expansion (1.39). This leads to a sum over *three sets of coordinates* (n_i^a, x_i^a), (n_j^b, x_j^b), (n_l^c, x_l^c), with associated random variables $\xi_{n,x}$, say

$$\mathbb{E}[(Z_N^\beta - 1)^3] = \sum_{\substack{k^a \geqslant 1 \\ k^b \geqslant 1 \\ k^c \geqslant 1}} \sum_{\substack{(n_i^a, x_i^a)_{i=1,\ldots,k^a} \\ (n_j^b, x_j^b)_{j=1,\ldots,k^b} \\ (n_l^c, x_l^c)_{l=1,\ldots,k^c}}} c_{N,\{(n_i^a,x_i^a),(n_j^b,x_j^b),(n_l^c,x_l^c)\}} \, \mathbb{E}\Bigg[\prod_{i,j,l} \xi_{n_i^a,x_i^a} \, \xi_{n_j^b,x_j^b} \, \xi_{n_l^c,x_l^c}\Bigg], \quad (1.48)$$

for suitable (explicit) coefficients $c_{N,\{\ldots\}}$. The basic observation is that if a coordinate, say (n_i^a, x_i^a), is distinct from all other coordinates, then it gives no contribution to (1.48), because the random variable ξ_{n_i,x_i} is independent of the other $\xi_{n,x}$'s and it has $\mathbb{E}[\xi_{n_i,x_i}] = 0$. This means that *the coordinates in* (1.48) *have to match*, necessarily in pairs or in triples.† We will show that triple matchings can be neglected, so we restrict to pairwise matchings.

Let $\boldsymbol{D} \subseteq \{1,\ldots,N\} \times \mathbb{Z}^2$ be the subset of space-time points given by the union of all coordinates (n_i^a, x_i^a), (n_j^b, x_j^b), (n_l^c, x_l^c) in (1.48). By the pairwise matching constraint, any index $(n,x) \in \boldsymbol{D}$ must appear exactly twice among the three sets of coordinates with labels a, b, c. So we can label each index in $\boldsymbol{D}$ as either ab, bc or ac, and we say that consecutive indexes with the same label form a *stretch*. This decomposition into stretches will lead to the integral representation (1.26) for the third moment, as we now explain.

Let us write $\boldsymbol{D} = \{(n_i, x_i) : \ i = 1,\ldots,r\}$ and consider the case when the first stretch has, say, label ab and length $k \leqslant r$ (this means that $(n_i, x_i) = (n_i^a, x_i^a) = (n_i^b, x_i^b)$ for $i = 1,\ldots,k$). The key observation is that, if we fix the last index $(n_k, x_k) = (M, y)$ and sum over the number k and the locations (n_i, x_i) of previous indexes inside the stretch, then we obtain an expression similar to (1.40), except that the last index is not summed but rather fixed to $(n_k, x_k) = (M, y)$ (see Section 5 for the details). But this turns out to be precisely the second moment (1.46) of the point-to-point partition function $Z_M^\beta(0, y)$.

In summary, when computing the third moment from (1.48), *the contribution of each stretch of pairwise matchings is given asymptotically by* (1.47). This is also the case when we consider the partition function $Z_N^{\beta_N}(\phi)$ averaged over the starting point.

We can finally explain qualitatively the structure of the kernel (1.25)-(1.26) in Theorem 1.4:

- the index m of the sum in (1.25) corresponds to the number of stretches;
- each stretch gives rise to a kernel $G_\vartheta(b_i - a_i, y_i - x_i)$ in (1.26), by (1.47);
- the switch from a stretch to the following consecutive stretch gives rise to the remaining kernels $g_{\frac{a_i - b_{i-2}}{2}}(x_i - y_{i-2}) \, g_{\frac{a_i - b_{i-1}}{2}}(x_i - y_{i-1})$ in (1.26).

We stress that the knowledge of precise asymptotic estimates such as (1.47) is crucial to compute the limiting expression (1.25)-(1.26) for the third moment.

We refer to Section 5 for a more detailed exposition of the combinatorial structure in the third moment calculation, which lies at the heart of the present paper.

1.4. Discussion. To put our results in perspective, we explain here some background. The key background notion is disorder relevance/irrelevance. The directed polymer is an example of a disordered system that arises as a disorder perturbation of an underlying pure model, the random walk S in this case. A fundamental question is whether the disorder perturbation, however small $\beta > 0$ is, changes the qualitative behavior of the pure model as

†Note that coordinates (n_i^α, x_i^α) with the same label $\alpha \in \{a, b, c\}$ are distinct, by $n_i^\alpha < n_{i+1}^\alpha$, see (1.39), hence more than triple matchings cannot occur.

$N \to \infty$. If the answer is affirmative, then disorder is said to be *relevant*; otherwise disorder is said to be *irrelevant*. For further background, see e.g. the monograph [G10].

For the directed polymer on $\mathbb{Z}^{d+1}$, the underlying random walk S is diffusive with $|S_N| \approx N^{1/2}$, while under the polymer measure P_N^β, it has been shown that for $d \geqslant 3$, there exists a critical value $\beta_c(d) > 0$ such that for $\beta < \beta_c(d)$, $|S_N| \approx N^{1/2}$ (see e.g. [CY06]); while for any $\beta > 0$ in $d = 1, 2$ and for $\beta > \beta_c(d)$ in $d \geqslant 3$, it is believed that $|S_N| \gg N^{1/2}$. Thus the directed polymer model should be disorder irrelevant in $d \geqslant 3$, disorder relevant in $d = 1$, while $d = 2$ turns out to be the critical dimension separating disorder relevance vs irrelevance, and disorder should be *marginally relevant*.

In [AKQ14], Alberts, Khanin and Quastel showed that on the intermediate disorder scale $\beta_N = \hat\beta/N^{1/4}$, the rescaled partition functions of the directed polymer on $\mathbb{Z}^{1+1}$ converges to the solution of the 1-dimensional SHE (1.6). We note that the idea of considering polymers with scaled temperature had already appeared in the physics literature [BD00, CDR10].

Inspired in particular by [AKQ14], we developed in [CSZ17a] a new perspective on disorder relevance vs irrelevance (see also [CSZ16]). The heuristic is that, if a model is disorder relevant, then under coarse graining and renormalization of space-time, the effective disorder strength of the coarse-grained model diverges. Therefore to compensate, it should be possible to choose the disorder strength $\beta_N \downarrow 0$ (known as weak disorder limit) as the lattice spacing $\delta := 1/N \downarrow 0$ (known as continuum limit) in such a way that we obtain a continuum disordered model. In particular, the partition function Z^ω_{N,β_N} should admit a non-trivial random limit for suitable choices of $\beta_N \downarrow 0$. In [CSZ17a], we formulated general criteria for the partition functions of a disordered system to have non-trivial continuum and weak disorder limits. These criteria were then verified for the disordered pinning model, a family of (possibly long-range) directed polymer on $\mathbb{Z}^{1+1}$, and the random field perturbation of the critical Ising model on $\mathbb{Z}^2$. However, the general framework developed in [CSZ17a] does not include models where disorder is only marginally relevant, such as the directed polymer on $\mathbb{Z}^{2+1}$, which led to our previous work [CSZ17b] and to our current work.

Disorder relevance/irrlevance is also closely linked to the classification of singular stochastic partial differential equations (SPDE), such as the SHE or the KPZ equation, into sub-critical, critical, or super-critical ones, which correspond respectively to disorder relevance, marginality and disorder irrelevance. For sub-critical singular SPDEs, a general solution theory called *regularity structures* has been developed in seminal work by Hairer in [H13, H14], and alternative approaches have been developed by Gubinelli, Imkeller, and Perkowski [GIP15], and also by Kupiainen [K14]. However, for critical singular SPDEs such as the SHE in $d = 2$, the only known results so far are: our previous work [CSZ17a], which established a phase transition in the intermediate disorder scale $\beta_\varepsilon = \hat\beta(2\pi/\log\frac{1}{\varepsilon})^{1/2}$ and identified the limit in distribution of the solution $u^\varepsilon(t,x)$ in the subcritical regime $\hat\beta < 1$; the work of Bertini and Cancrini [BC98], which computed the limiting covariance of the random field $u^\varepsilon(t,\cdot)$ at the critical point $\hat\beta = 1$; and our current work, which establishes the non-triviality of subsequential weak limits of the random field at the critical point $\hat\beta = 1$.

Let us mention some related work on the directed polymer model on the hierarchical lattice. In particular, for the marginally relevant case, Alberts, Clark and Kocić in [ACK17] established the existence of a phase transition, similar to [CSZ17a]. And more recently, Clark [Cla17] computed the moments of the partition function around a critical window for the case of bond disorder. The computations in the hierarchical lattice case employ the independence structure inherent in hierarchical models, which is not available on $\mathbb{Z}^d$.

Note added in publication. More recently, Gu, Quastel and Tsai [GQT19] *proved the existence of all moments for the 2-dimensional SHE in the critical window. They use different, functional analytic methods inspired by Dimock and Rajeev* [DR04].

1.5. Organization of the paper. In Section 2, we recall the polynomial chaos expansion for the partition functions and introduce the renewal framework, which are then used in Section 3 to prove Theorem 1.2 on the limiting second moment of the partition function. In Section 4, we derive a series expansion for the third moment of the averaged point-to-point partition functions, whose terms are separated into two groups: ones with so-called *triple intersections*, and ones with *no triple intersection*. Terms with no triple intersection is shown in Section 5 to converge to the desired limit, while terms with triple intersections are shown to be negligible in Section 7, using bounds developed in Section 6. Lastly, in Section 8, we prove Theorems 1.7 and 1.9 for the stochastic heat equation.

2. Polynomial chaos and renewal framework

In this section, we describe two key elements that form the basis of our analysis:

(1) *polynomial chaos expansions*, which represent the partition function as a multilinear polynomial of modified disorder random varibles, see Subsection 2.1.

(2) a *renewal theory framework*, which allows to relate the second moment of the partition function to suitable renewal functions, see Subsection 2.2.

We will use $\mathrm{P}_{a,x}$ and $\mathrm{E}_{a,x}$ to denote probability and expectation for the random walk S starting at time a from position $S_a = x \in \mathbb{Z}^2$, with the subscript omitted when $(a, x) = (0, 0)$. Recalling (1.7), we define the family of *point-to-point partition functions* by

$$Z_{a,b}^{\beta}(x,y) := \mathrm{E}_{a,x}\Big[e^{\sum_{n=a+1}^{b-1}(\beta\omega_{n,S_n}-\lambda(\beta))}\mathbb{1}_{\{S_b=y\}}\Big], \qquad (a,x),(b,y)\in\mathbb{Z}^3_{\mathrm{even}},\ a<b\,. \tag{2.1}$$

The original *point-to-plane partition function* $Z_N^{\beta}(x)$, see (1.4), can be recovered as follows:

$$Z_N^{\beta}(x) = \sum_{y\in\mathbb{Z}^2} Z_{0,N}^{\beta}(x,y)\,. \tag{2.2}$$

We note that the point-to-plane partition function has $\mathbb{E}[Z_{a,b}^{\beta}(x)] \equiv 1$, while for the point-to-point partition function we have

$$\mathbb{E}\big[Z_{a,b}^{\beta}(x,y)\big] = q_{a,b}(x,y) := q_{b-a}(y-x)\,, \tag{2.3}$$

the transition probability kernel defined in (1.8). We will need to average the partition functions $Z_{a,b}^{\beta}(x,y)$ over either x or y, or both, on the diffusive scale. More precisely, we define for $N \in \mathbb{N}$

$$Z_{a,b}^{N,\beta}(x,\psi) := \sum_{y\in\mathbb{Z}^2} Z_{a,b}^{\beta}(x,y)\,\psi\big(\tfrac{y}{\sqrt{N}}\big)\,, \qquad \psi\in C_b(\mathbb{R}^2)\,, \tag{2.4}$$

$$Z_{a,b}^{N,\beta}(\phi,y) := \sum_{x\in\mathbb{Z}^2} \phi\big(\tfrac{x}{\sqrt{N}}\big)\,Z_{a,b}^{\beta}(x,y)\,, \qquad \phi\in C_c(\mathbb{R}^2)\,, \tag{2.5}$$

$$Z_{a,b}^{N,\beta}(\phi,\psi) := \frac{1}{N}\sum_{x,y\in\mathbb{Z}^2} \phi\big(\tfrac{x}{\sqrt{N}}\big)\,Z_{a,b}^{\beta}(x,y)\,\psi\big(\tfrac{y}{\sqrt{N}}\big)\,, \qquad \phi\in C_c(\mathbb{R}^2)\,,\ \psi\in C_b(\mathbb{R}^2)\,. \tag{2.6}$$

The reason that the terminal function ψ is only required to be bounded and continuous, while the initial function ϕ is compactly supported is that, we would like to include the case $\psi \equiv 1$, which corresponds to the point-to-plane polymer partition function. On the other hand, the initial function ϕ plays the role of a test function used to average the partition function. (In general, the fact that at least one between ϕ and ψ is compactly supported ensures finiteness of the average (2.6).) Note that $Z^{\beta}_{Nt}(\phi)$ in (1.12) coincides with $Z^{N,\beta}_{0,Nt}(\phi,\psi)$ with $\psi \equiv 1$. From (2.3) we compute

$$\mathbb{E}\big[Z^{N,\beta}_{a,b}(x,\psi)\big] = q^N_{a,b}(x,\psi) := \sum_{y\in\mathbb{Z}^2} q_{b-a}(y-x)\,\psi\big(\tfrac{y}{\sqrt{N}}\big), \tag{2.7}$$

$$\mathbb{E}\big[Z^{N,\beta}_{a,b}(\phi,y)\big] = q^N_{a,b}(\phi,y) := \sum_{x\in\mathbb{Z}^2} \phi\big(\tfrac{x}{\sqrt{N}}\big) q_{b-a}(y-x), \tag{2.8}$$

$$\mathbb{E}\big[Z^{N,\beta}_{a,b}(\phi,\psi)\big] = q^N_{a,b}(\phi,\psi) := \frac{1}{N}\sum_{x,y\in\mathbb{Z}^2} \phi\big(\tfrac{x}{\sqrt{N}}\big)\, q_{b-a}(y-x)\,\psi\big(\tfrac{y}{\sqrt{N}}\big)\,. \tag{2.9}$$

Note that these expectations are of order 1 for $a = 0$ and $b = N$, because $q_N(y-x) \approx 1/N$ for $x, y = O(\sqrt{N})$, see (1.8)-(1.9). This explains the normalizations in (2.4)-(2.6).

2.1. Polynomial chaos expansion. Let us start by rewriting the point-to-point partition function from (2.1) as

$$Z^{\beta}_{a,b}(x,y) = \mathrm{E}_{a,x}\Big[e^{\sum_{n=a+1}^{b-1}(\beta\omega_{n,S_n}-\lambda(\beta))}\mathbb{1}_{\{S_b=y\}}\Big] = \mathrm{E}_{a,x}\Big[\prod_{a<n<b}\prod_{z\in\mathbb{Z}^2} e^{(\beta\omega_{n,z}-\lambda(\beta))\mathbb{1}_{\{S_n=z\}}}\,\mathbb{1}_{\{S_b=y\}}\Big].$$

Using the fact that $e^{x\mathbb{1}_{\{n\in\tau\}}} = 1 + (e^x - 1)\mathbb{1}_{\{n\in\tau\}}$ for $x \in \mathbb{R}$, we can write

$$\begin{gathered} Z^{\beta}_{a,b}(x,y) = \mathrm{E}_{a,x}\Bigg[\prod_{n=a+1}^{b-1}\prod_{z\in\mathbb{Z}^2}\big(1+\xi_{n,z}\,\mathbb{1}_{\{S_n=z\}}\big)\mathbb{1}_{S_b=y}\Bigg], \\ \text{where} \quad \xi_{n,z} := e^{\beta\omega_{n,z}-\lambda(\beta)} - 1\,. \end{gathered} \tag{2.10}$$

The random variables $\xi_{n,z}$ are i.i.d. with mean zero (thanks to the normalization by $\lambda(\beta)$) and with variance $\mathbb{V}\mathrm{ar}[\xi_{n,z}] = e^{\lambda(2\beta)-2\lambda(\beta)} - 1$. Recalling (2.3) and expanding the product, we obtain the following polynomial chaos expansion:

$$\begin{aligned} Z^{\beta}_{a,b}(x,y) = q_{a,b}(x,y) \;+ \sum_{k\geqslant 1}\; \sum_{\substack{a<n_1<\ldots<n_k<b \\ x_1,\ldots,x_k\in\mathbb{Z}^2}} & q_{a,n_1}(x,x_1)\,\xi_{n_1,x_1}\,\cdot \\ & \cdot\Bigg\{\prod_{j=2}^{k} q_{n_{j-1},n_j}(x_{j-1},x_j)\,\xi_{n_j,x_j}\Bigg\}\, q_{n_k,b}(x_k,y)\,, \end{aligned} \tag{2.11}$$

with the convention that the product equals 1 when $k = 1$. We have written $Z^{\beta}_{a,b}(x,y)$ as a multilinear polynomial of the random variables $\xi_{n,x}$.

Analogous expansions hold for the averaged point-to-point partition functions: by (2.6)

$$\begin{aligned} Z^{N,\beta}_{a,b}(\phi,\psi) = q^N_{a,b}(\phi,\psi) + \frac{1}{N}\sum_{k\geqslant 1}\; \sum_{\substack{a<n_1<\ldots<n_k<b \\ x_1,\ldots,x_k\in\mathbb{Z}^2}} & q^N_{a,n_1}(\phi,x_1)\,\xi_{n_1,x_1}\cdot \\ & \cdot\Bigg\{\prod_{j=2}^{k} q_{n_{j-1},n_j}(x_{j-1},x_j)\,\xi_{n_j,x_j}\Bigg\}\, q^N_{n_k,b}(x_k,\psi)\,. \end{aligned} \tag{2.12}$$

Similar expansions hold for $Z^{N,\beta}_{a,b}(x,\psi)$ and $Z^{N,\beta}_{a,b}(\phi,y)$, without the factor $\frac{1}{N}$.

2.2. Renewal theory framework. Given $N \in \mathbb{N}$, we define a sequence of i.i.d. random variables $\big((T_i^{(N)}, X_i^{(N)})\big)_{i\in\mathbb{N}}$ taking values in $\mathbb{N} \times \mathbb{Z}^2$, with marginal law

$$\mathrm{P}\big((T_i^{(N)}, X_i^{(N)}) = (n,x)\big) := \frac{q_n(x)^2}{R_N} \, \mathbb{1}_{\{1,\ldots,N\}}(n)\,, \tag{2.13}$$

where we recall that $q_n(x)$ is defined in (1.8) and $R_N = \sum_{n=1}^N \sum_{x\in\mathbb{Z}^2} q_n(x)^2$ is the replica overlap, see (1.10). We then define the corresponding random walk† on $\mathbb{N} \times \mathbb{Z}^2$

$$\big(\tau_k^{(N)}\,,\; S_k^{(N)}\big) := \Big(T_1^{(N)} + \ldots + T_k^{(N)}\,,\; X_1^{(N)} + \ldots + X_k^{(N)}\Big)\,, \qquad k \in \mathbb{N}\,. \tag{2.14}$$

Note that the first component $\tau_k^{(N)}$ is the renewal process that we introduced in Subsection 1.3, see (1.42)-(1.43).

We now describe the link with our model. We note that σ_N^2, see (1.14), is the variance of the random variables $\xi_{n,x} = e^{\beta\omega_{n,x}-\lambda(\beta)} - 1$ which appear in (2.11). Recalling (2.1) and (2.3), we introduce a crucial quantity $U_N(n,x)$, that will appear repeatedly in our analysis, which is a suitably rescaled second moment of the point-to-point partition function:

$$\begin{aligned} U_N(n,x) &:= \sigma_N^2\, \mathbb{E}\big[Z_{0,n}^{\beta_N}(0,x)^2\big] = \sigma_N^2 \left\{q_n(x)^2 + \mathbb{V}\mathrm{ar}\left[Z_{0,n}^{\beta_N}(0,x)\right]\right\}, \qquad n \geqslant 1\,, \\ U_N(0,x) &:= \delta_{x,0} = \mathbb{1}_{\{x=0\}}\,. \end{aligned} \tag{2.15}$$

By (2.11), we then have

$$\begin{aligned} U_N(n,x) = \sigma_N^2\, q_{0,n}(0,x)^2 \;+\; \sum_{k\geqslant 1} (\sigma_N^2)^{k+1} \sum_{\substack{0<n_1<\ldots<n_k<n \\ x_1,\ldots,x_k\in\mathbb{Z}^2}} & q_{0,n_1}(0,x_1)^2 \,\cdot \\ & \cdot \left\{\prod_{j=2}^k q_{n_{j-1},n_j}(x_{j-1},x_j)^2\right\} q_{n_k,n}(x_k,x)^2\,. \end{aligned} \tag{2.16}$$

Looking at (2.13)-(2.14), we have the following key probabilistic representation:

$$U_N(n,x) = \sum_{r\geqslant 1} (\lambda_N)^r\, \mathrm{P}\big(\tau_r^{(N)} = n\,,\, S_r^{(N)} = x\big)\,, \qquad \text{where} \quad \lambda_N := \sigma_N^2\, R_N\,. \tag{2.17}$$

It is also convenient to define

$$U_N(n) := \sum_{x\in\mathbb{Z}^2} U_N(n,x) = \sum_{r\geqslant 1} (\lambda_N)^r\, \mathrm{P}\big(\tau_r^{(N)} = n\big)\,. \tag{2.18}$$

Thus $U_N(n,x)$ and $U_N(n)$ can be viewed as (exponentially weighted) *local renewal functions.*

We investigated the asymptotic properties of the random walk $(\tau_k^{(N)}, S_k^{(N)})$ in [CSZ18]. In particular, introducing the rescaled process

$$\boldsymbol{Y}_s^{(N)} := (Y_s^{(N)}, V_s^{(N)}) := \left(\frac{\tau_{\lfloor s\log N\rfloor}^{(N)}}{N}, \frac{S_{\lfloor s\log N\rfloor}^{(N)}}{\sqrt{N}}\right), \qquad s \geqslant 0\,, \tag{2.19}$$

we proved in [CSZ18] that $\boldsymbol{Y}^{(N)}$ converges in distribution as $N \to \infty$ to the Lévy process $\boldsymbol{Y}$ on $[0,\infty) \times \mathbb{R}^2$ with Lévy measure

$$\boldsymbol{\nu}(\mathrm{d}t, \mathrm{d}x) = \frac{\mathbb{1}_{(0,1)}(t)}{t}\, g_{t/4}(x)\, \mathrm{d}t\, \mathrm{d}x,$$

†$S^{(N)}$ should not be confused with the random walk S in the definition of the directed polymer model.

where $g_u(x)$ is the standard Gaussian density on $\mathbb{R}^2$, see (1.9). Remarkably, the process $\boldsymbol{Y}$ admits an explicit density:

$$f_s(t,x) := \frac{\mathrm{P}(\boldsymbol{Y}_s \in (\mathrm{d}t, \mathrm{d}x))}{\mathrm{d}t\,\mathrm{d}x} = \frac{e^{-\gamma s}\, s\, t^{s-1}}{\Gamma(s+1)}\, g_{t/4}(x) \qquad \text{for } t \in (0,1)\,,\ x \in \mathbb{R}^2\,,$$

which leads to a corresponding explicit expression for the (weighted) local renewal function

$$G_\vartheta(t,x) \;:=\; \int_0^\infty e^{\vartheta s} f_s(t,x)\,\mathrm{d}s \;=\; \bigg(\int_0^\infty \frac{e^{(\vartheta-\gamma)s}\, s\, t^{s-1}}{\Gamma(s+1)}\,\mathrm{d}s\bigg)\, g_{t/4}(x) \;=\; G_\vartheta(t)\, g_{t/4}(x)\,,$$

where the functions $G_\vartheta(t)$ and $G_\vartheta(t,x)$ match with (1.18) and (1.23).

We showed in [CSZ18] that the sharp asymptotic behavior of $U_N(n,x)$ and $U_N(n)$ is captured by the functions $G_\vartheta(n,x)$ and $G_\vartheta(x)$. Note that for the weight λ_N in (2.17)-(2.18) we can write $\lambda_N = 1 + \frac{\vartheta}{\log N}(1+o(1))$ as $N \to \infty$, by our assumption (1.14). Then we can rephrase [CSZ18, Theorem 1.4 and Theorems 2.3-2.4] as follows.

Proposition 2.1. *Fix β_N such that (1.14) holds, for some $\vartheta \in \mathbb{R}$. Let $U_N(n)$ be defined as in (2.18). For any fixed $\delta > 0$, as $N \to \infty$ we have*

$$U_N(n) = \frac{\log N}{N}\big(G_\vartheta\big(\tfrac{n}{N}\big) + o(1)\big)\,, \qquad \textit{uniformly for}\ \ \delta N \leqslant n \leqslant N\,, \tag{2.20}$$

where G_ϑ is defined in (1.18). Moreover, there exists $C \in (0,\infty)$ such that for all $N \in \mathbb{N}$

$$U_N(n) \leqslant C\,\frac{\log N}{N}\, G_\vartheta\big(\tfrac{n}{N}\big)\,, \qquad \forall 1 \leqslant n \leqslant N\,. \tag{2.21}$$

Proposition 2.2. *Fix β_N such that (1.14) holds, for some $\vartheta \in \mathbb{R}$. Let $U_N(n,x)$ be defined as in (2.16)-(2.17). For any fixed $\delta > 0$, as $N \to \infty$ we have*

$$\begin{aligned} U_N(n,x) &= \frac{\log N}{N^2}\big(G_\vartheta\big(\tfrac{n}{N}, \tfrac{x}{\sqrt{N}}\big) + o(1)\big)\, 2\,\mathbb{1}_{\{(n,x)\in\mathbb{Z}^3_{\mathrm{even}}\}}\,, \\ &\qquad \textit{uniformly for}\ \ \delta N \leqslant n \leqslant N\,,\ |x| \leqslant \tfrac{1}{\delta}\sqrt{N}\,, \end{aligned} \tag{2.22}$$

where $G_\vartheta(t,x)$ is defined in (1.23). Moreover, there exists $C \in (0,\infty)$ such that for all $N \in \mathbb{N}$

$$\sum_{x\in\mathbb{Z}^2:\ |x|>M\sqrt{n}} \frac{U_N(n,x)}{U_N(n)} \leqslant \frac{C}{M^2}\,, \qquad \forall 1 \leqslant n \leqslant N\,,\ \forall M > 0\,. \tag{2.23}$$

We will also need the following asymptotic behavior on $G_\vartheta(t)$ from [CSZ18].

Proposition 2.3. *For every fixed $\vartheta \in \mathbb{R}$, we have that*

$$G_\vartheta(t) = \frac{1}{t\,(\log\frac{1}{t})^2} + \frac{2\vartheta + o(1)}{t\,(\log\frac{1}{t})^3} \qquad \textit{as} \quad t \to 0\,. \tag{2.24}$$

It follows that there exists $c_\vartheta \in (0,\infty)$ such that

$$G_\vartheta(t) \leqslant \hat{G}_\vartheta(t) := \frac{c_\vartheta}{t\,(2+\log\frac{1}{t})^2} = \frac{c_\vartheta}{t\,(\log\frac{e^2}{t})^2}\,, \qquad \forall t \in (0,1]\,. \tag{2.25}$$

By direct computation $\frac{\mathrm{d}}{\mathrm{d}t}\hat{G}(t) < 0$ for all $t \in (0,1)$, hence $\hat{G}_\vartheta(\cdot)$ is strictly decreasing.

3. Proof of Theorem 1.2

Recall the definition (1.9) of $g_t(x)$. Given a bounded function $\phi : \mathbb{R}^2 \to \mathbb{R}$, we define

$$\Phi_s(x) := (\phi * g_{s/2})(x) = \int_{\mathbb{R}^2} \phi(x-y)\, g_{s/2}(y)\, \mathrm{d}y\,, \qquad s>0\,,\ x \in \mathbb{R}^2\,. \tag{3.1}$$

The averaged partition function $Z_{Nt}^{\beta_N}(\phi)$ in Theorem 1.2, see (1.12), coincides with $Z_{0,Nt}^{N,\beta_N}(\phi,\psi)$ with $\psi \equiv 1$, see (2.6). By the expansion (2.12) with $\psi \equiv 1$, we obtain

$$\mathbb{V}\mathrm{ar}[Z_{Nt}^{\beta_N}(\phi)] = \frac{1}{N^2} \sum_{k\geqslant 1} (\sigma_N^2)^k \sum_{\substack{0<n_1<\ldots<n_k<Nt \\ x_1,\ldots,x_k \in \mathbb{Z}^2}} q_{0,n_1}^N(\phi,x_1)^2 \prod_{j=2}^{k} q_{n_{j-1},n_j}(x_{j-1},x_j)^2. \tag{3.2}$$

We isolate the term $k=1$, because given $(n_1,x_1)=(m,x)$ and $(n_k,x_k)=(n,y)$, the sum over $k \geqslant 2$ gives $\mathbb{E}[Z_{m,n}^{\beta_N}(x,y)^2] = U_N(n-m,y-x)/\sigma_N^2$, by (2.15)-(2.16). Therefore

$$\begin{aligned}
\mathbb{V}\mathrm{ar}[Z_{Nt}^{\beta_N}(\phi)] &= \frac{\sigma_N^2}{N^2} \sum_{\substack{0<n<Nt \\ x\in\mathbb{Z}^2}} q_{0,n}^N(\phi,x)^2 + \frac{\sigma_N^4}{N^2} \sum_{\substack{0<m<n<Nt \\ x,y\in\mathbb{Z}^2}} q_{0,m}^N(\phi,x)^2\, \mathbb{E}[Z_{m,n}^{\beta_N}(x,y)^2] \\
&= \frac{\sigma_N^2}{N^2} \sum_{\substack{0<n<Nt \\ x\in\mathbb{Z}^2}} q_{0,n}^N(\phi,x)^2 + \frac{\sigma_N^2}{N^2} \sum_{\substack{0<m<n<Nt \\ x\in\mathbb{Z}^2}} q_{0,m}^N(\phi,x)^2\, U_N(n-m),
\end{aligned} \tag{3.3}$$

where in the second equality we summed over $y \in \mathbb{Z}^2$ – this is the reason that only $U_N(n-m)$ appears instead of $U_N(n-m,y-x)$; recall (2.17) and (2.18).

We now let $N \to \infty$. We first show that the first term in the RHS of (3.3) vanishes as $O(\sigma_N^2) = O(\frac{1}{\log N})$, see (1.10) and (1.14). Note that for $v \in (0,1)$ and $x \in \mathbb{R}^2$ we have

$$\lim_{N\to\infty} q_{0,Nv}^N(\phi,\sqrt{N}x) = \Phi_v(x)\,, \qquad \sup_{m\in\mathbb{N},\, z\in\mathbb{Z}^2} q_{0,m}^N(\phi,z) \leqslant |\phi|_\infty < \infty\,, \tag{3.4}$$

see (2.8), (1.8) and (3.1). Then, by Riemann sum approximation, we have

$$\begin{aligned}
\frac{1}{N^2} \sum_{\substack{0<n<Nt \\ x\in\mathbb{Z}^2}} q_{0,n}^N(\phi,x)^2 \quad &\underset{N\to\infty}{\sim} \frac{1}{N^2} \sum_{\substack{0<n<Nt \\ x\in\mathbb{Z}^2}} \Phi_{\frac{n}{N}}\big(\tfrac{x}{\sqrt{N}}\big)^2 \\
&\xrightarrow[N\to\infty]{} \int_{(0,t)\times\mathbb{R}^2} \Phi_v(x)^2\, \mathrm{d}v\, \mathrm{d}x \in (0,\infty)\,.
\end{aligned}$$

Indeed, the approximation is uniform for $N\varepsilon < n < Nt$, with fixed $\varepsilon > 0$, while the contribution of $n \leqslant N\varepsilon$ is small, for $\varepsilon > 0$ small, by the uniform bound in (3.4).

It remains to focus on the second term in the RHS of (3.3). By (2.20)-(2.21) and (3.4), together with $\sigma_N^2 \sim \frac{\pi}{\log N}$, see (1.14) and (1.10), another Riemann sum approximation gives

$$\mathbb{V}\mathrm{ar}[Z_{Nt}^{\beta_N}(\phi)] \xrightarrow[N\to\infty]{} \pi \int_{\substack{0<u<v<t \\ x\in\mathbb{R}^2}} \Phi_u(x)^2\, G_\vartheta(v-u)\, \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}x\,, \tag{3.5}$$

Integrating out x, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} \Phi_u(x)^2\, \mathrm{d}x &= \int_{\mathbb{R}^2} \bigg(\int_{\mathbb{R}^2\times\mathbb{R}^2} \phi(z)\, \phi(z')\, g_{u/2}(x-z)\, g_{u/2}(x-z')\, \mathrm{d}z\, \mathrm{d}z' \bigg) \mathrm{d}x \\
&= \int_{\mathbb{R}^2\times\mathbb{R}^2} \phi(z)\, \phi(z')\, g_u(z-z')\, \mathrm{d}z\, \mathrm{d}z'\,,
\end{aligned}$$

which plugged into (3.5) proves (1.19). □

4. Expansion for the third moment

In this section, we give an expansion for the third moment of the partition function, which forms the basis of our proof of Theorem 1.4. We actually prove a more general version for the averaged *point-to-point* partition functions, which is of independent interest.

Theorem 4.1 (Third moment, averaged point-to-point). *Let $t > 0$, $\vartheta \in \mathbb{R}$ and β_N satisfy* (1.14)*. Fix a compactly supported $\phi \in C_c(\mathbb{R}^2)$ and a bounded $\psi \in C_b(\mathbb{R}^2)$. Then*

$$\lim_{N\to\infty} \mathbb{E}\Big[\big(Z^{N,\beta_N}_{0,Nt}(\phi,\psi) - \mathbb{E}\big[Z^{N,\beta_N}_{0,Nt}(\phi,\psi)\big]\big)^3\Big] = M_t(\phi,\psi) := 3\sum_{m=2}^{\infty} 2^{m-1}\,\pi^m\,\mathcal{I}^{(m)}_t(\phi,\psi) < \infty, \tag{4.1}$$

where we set $\Phi_s := \phi * g_{s/2}$ and $\Psi_s := \psi * g_{s/2}$, see (3.1)*, and define*

$$\begin{aligned}\mathcal{I}^{(m)}_t(\phi,\psi) := \int\!\!\cdots\!\!\int_{\substack{0<a_1<b_1<a_2<b_2<\ldots<a_m<b_m<t\\ x_1,y_1,x_2,y_2,\ldots,x_m,y_m\in\mathbb{R}^2}} & \Phi^2_{a_1}(x_1)\,\Phi_{a_2}(x_2)\,\cdot \\ & \cdot\; G_\vartheta(b_1-a_1,y_1-x_1)\,g_{\frac{a_2-b_1}{2}}(x_2-y_1)\,G_\vartheta(b_2-a_2,y_2-x_2)\,\cdot \\ & \cdot\;\prod_{i=3}^{m} g_{\frac{a_i-b_{i-2}}{2}}(x_i-y_{i-2})\,g_{\frac{a_i-b_{i-1}}{2}}(x_i-y_{i-1})\,G_\vartheta(b_i-a_i,y_i-x_i)\,\cdot \\ & \cdot\;\Psi_{t-b_{m-1}}(y_{m-1})\Psi^2_{t-b_m}(y_m)\;\mathrm{d}\vec a\,\mathrm{d}\vec b\,\mathrm{d}\vec x\,\mathrm{d}\vec y\,.\end{aligned} \tag{4.2}$$

We observe that Theorem 1.4 is a special case of Theorem 4.1: it suffices to take $\psi \equiv 1$ so that $Z^{N,\beta_N}_{0,Nt}(\phi,\psi) = Z^{\beta_N}_{Nt}(\phi)$, see (2.6) and (1.12), and it is easy to check that (4.1)-(4.2) match with (1.24)-(1.26), since $\Psi_s \equiv 1$.

It remains to prove Theorem 4.1. This will be reduced to Propositions 4.2 and 4.3 below. We exploit the multilinear expansion in (2.12) for the partition function, which leads to the following representation for the centered third moment (recall (2.7)-(2.9)):

$$\begin{aligned}&\mathrm{E}\Big[\big(Z^{N,\beta_N}_{s,t}(\phi,\psi) - \mathbb{E}[Z^{N,\beta_N}_{s,t}(\phi,\psi)]\big)^3\Big] \\ &= \sum_{\substack{\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}\subseteq\{s+1,\ldots,t-1\}\times\mathbb{Z}^2\\ |\boldsymbol{A}|\geqslant 1,\,|\boldsymbol{B}|\geqslant 1,\,|\boldsymbol{C}|\geqslant 1}} \frac{1}{N^3}\,q^N_{s,a_1}(\phi,x_1)\,q^N_{s,b_1}(\phi,y_1)\,\cdot\,q^N_{s,c_1}(\phi,z_1)\,\cdot \\ &\quad\cdot\;\mathbb{E}\Bigg[\xi_{A_1}\prod_{i=2}^{|\boldsymbol{A}|}\xi_{A_i}\,q(A_{i-1},A_i)\cdot\xi_{B_1}\prod_{j=2}^{|\boldsymbol{B}|}\xi_{B_j}\,q(B_{j-1},B_j)\cdot\xi_{C_1}\prod_{k=2}^{|\boldsymbol{C}|}\xi_{C_k}\,q(C_{k-1},C_k)\Bigg]\,\cdot \\ &\quad\cdot\;q^N_{a_{|\boldsymbol{A}|},t}(x_{|\boldsymbol{A}|},\psi)\,q^N_{b_{|\boldsymbol{B}|},t}(y_{|\boldsymbol{B}|},\psi)\,q^N_{c_{|\boldsymbol{C}|},t}(z_{|\boldsymbol{C}|},\psi)\,,\end{aligned} \tag{4.3}$$

where we agree that $\boldsymbol{A} = (A_1,\ldots,A_{|\boldsymbol{A}|})$ with $A_i = (a_i,x_i) \in \mathbb{Z}^3_{\mathrm{even}}$, and $\boldsymbol{B}$, $\boldsymbol{C}$ are defined similarly, with $B_j = (b_j,y_j)$, $C_k = (c_k,z_k)$, and we set for short

$$q(A_{i-1},A_i) := q_{a_i-a_{i-1}}(x_i-x_{i-1})\,.$$

(When $|\boldsymbol{A}| = 1$, the product $\prod_{i=2}^{|\boldsymbol{A}|}\ldots$ equals 1, by definition, and similarly for $\boldsymbol{B}$ and $\boldsymbol{C}$.)

We now split the sum in (4.3) into two parts:

$$\mathrm{E}\Big[\big(Z_{s,t}^{N,\beta_N}(\phi,\psi)-\mathbb{E}[Z_{s,t}^{N,\beta_N}(\phi,\psi)]\big)^3\Big] = M_{s,t}^{N,\mathrm{NT}}(\phi,\psi)+M_{s,t}^{N,\mathrm{T}}(\phi,\psi)\,, \tag{4.4}$$

defined as follows:

- $M_{s,t}^{N,\mathrm{NT}}(\phi,\psi)$ is the sum in (4.3) restricted to $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$ such that $\boldsymbol{A}\cap\boldsymbol{B}\cap\boldsymbol{C}=\varnothing$, which we call the case with *no triple intersections*;
- $M_{s,t}^{N,\mathrm{T}}(\phi,\psi)$ is the sum in (4.3) restricted to $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$ such that $\boldsymbol{A}\cap\boldsymbol{B}\cap\boldsymbol{C}\neq\varnothing$, which we call the case with *triple intersections*.

These parts are analyzed in the following propositions, which together imply Theorem 4.1.

Proposition 4.2 (Convergence with no triple intersections). *Let the assumptions of Theorem 4.1 hold. Then*

$$\lim_{N\to\infty} M_{0,Nt}^{N,\mathrm{NT}}(\phi,\psi) = M_t(\phi,\psi) = 3\sum_{m=2}^{\infty} 2^{m-1}\,\pi^m\,\mathcal{I}_t^{(m)}(\phi,\psi) < \infty. \tag{4.5}$$

Proposition 4.3 (Triple intersections are negligible). *Let the assumptions of Theorem 4.1 hold. Then*

$$\lim_{N\to\infty} M_{0,Nt}^{N,\mathrm{T}}(\phi,\psi) = 0. \tag{4.6}$$

Proposition 4.2 is proved in the next section. The proof of Proposition 4.3 will be given later, see Section 7.

5. Convergence without triple intersections

In this section, we prove Proposition 4.2 and several related results.

5.1. Proof of Proposition 4.2.

We first derive a representation for $M_{s,t}^{N,\mathrm{NT}}(\phi,\psi)$, which collects the terms in the expansion (4.3) with $\boldsymbol{A}\cap\boldsymbol{B}\cap\boldsymbol{C}=\varnothing$.

Denote $\boldsymbol{D} := \boldsymbol{A}\cup\boldsymbol{B}\cup\boldsymbol{C}\subset\{s+1,\ldots,t-1\}\times\mathbb{Z}^2$, with $\boldsymbol{D}=(D_1,\ldots,D_{|\boldsymbol{D}|})$ and $D_i=(d_i,w_i)$. Since $\mathbb{E}[\xi_z]=0$, the contributions to $M_{s,t}^{N,\mathrm{NT}}(\phi,\psi)$ come only from $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$ where the points in $\boldsymbol{A}\cup\boldsymbol{B}\cup\boldsymbol{C}$ pair up. In particular,

$$k := |\boldsymbol{D}| = \frac{1}{2}(|\boldsymbol{A}|+|\boldsymbol{B}|+|\boldsymbol{C}|) \geqslant 2,$$

and each point D_j belongs to exactly two of the three sets $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$, and hence we can associate a vector $\boldsymbol{\ell}=(\ell_1,\ldots,\ell_k)$ of labels $\ell_j\in\{AB,\,BC,\,AC\}$. Note that there is a one to one correspondence between $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C})$ and $(\boldsymbol{D},\boldsymbol{\ell})$. We also recall that $\xi_{n,z}=e^{\beta_N\omega(n,z)-\lambda(\beta_N)}-1$, hence $\sigma_N^2 = \mathbb{E}[\xi_z^2]$, see (1.14). From (4.3) we can then write

$$\begin{aligned} M_{s,t}^{N,\mathrm{NT}}(\phi,\psi) = \frac{1}{N^3} \sum_{k=2}^{\infty} \sigma_N^{2k} \sum_{\substack{\boldsymbol{D} \subseteq \{s+1,\ldots,t-1\}\times\mathbb{Z}^2 \\ |\boldsymbol{D}|=k\geqslant 2}} \sum_{\boldsymbol{\ell}\in\{AB,\,BC,\,AC\}^k} \\ q_{s,a_1}^N(\phi,x_1)\, q_{s,b_1}^N(\phi,y_1)\, q_{s,c_1}^N(\phi,z_1)\,\cdot \\ \cdot \prod_{i=2}^{|\boldsymbol{A}|} q(A_{i-1},A_i) \prod_{j=2}^{|\boldsymbol{B}|} q(B_{j-1},B_j) \prod_{m=2}^{|\boldsymbol{C}|} q(C_{m-1},C_m)\,\cdot \\ \cdot\, q^N_{a_{|\boldsymbol{A}|},t}(x_{|\boldsymbol{A}|},\psi)\, q^N_{b_{|\boldsymbol{B}|},t}(y_{|\boldsymbol{B}|},\psi)\, q^N_{c_{|\boldsymbol{C}|},t}(z_{|\boldsymbol{C}|},\psi)\,, \end{aligned} \tag{5.1}$$

where we agree that $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$ are implicitly determined by $(\boldsymbol{D},\boldsymbol{\ell})$.

We now make a combinatorial observation. The sequence $\boldsymbol{\ell} = (\ell_1,\ldots,\ell_k)$ consists of consecutive *stretches* $(\ell_1,\ldots,\ell_i)$, $(\ell_{i+1},\ldots,\ell_j)$, etc., such that the labels are constant in each stretch and change from one stretch to the next. Any stretch, say $(\ell_p,\ldots,\ell_q)$, has a first point $D_p = (a,x)$ and a last point $D_q = (b,y)$. Let m denote the number of stretches and let (a_i,x_i) and (b_i,y_i), with $a_i \leqslant b_i$, be the first and last points of the i-th stretch.

We now rewrite (5.1) by summing over $m \in \mathbb{N}$, $(a_1,b_1,\ldots,a_m,b_m)$, and $(x_1,y_1,\ldots,x_m,y_m)$. The sum over the labels of $\boldsymbol{\ell}$ leads to a combinatorial factor $3\cdot 2^{m-1}$, because there are 3 choices for the label of the first stretch and two choices for the label of the following stretches. Once we fix (a_1,x_1) and (b_1,y_1), summing over all possible configurations inside the first stretch then gives the factor

$$\sum_{r=1}^{\infty} \sigma_N^{2(r+1)} \sum_{\substack{a_1=t_0<t_1<\ldots<t_r=b_1 \\ z_0=x_1,\,z_1,z_2,\ldots,z_{r-1}\in\mathbb{Z}^2,\,z_r=y_1}} \prod_{i=1}^{r} q_{t_{i-1},t_i}(z_{i-1},z_i)^2 = \sigma_N^2\, U_N(b_1-a_1,\, y_1-x_1)\,,$$

where we recall that U_N is defined in (2.15)-(2.16). A similar factor arises from each stretch, which leads to the following crucial identity (see Figure 1):

$$\begin{aligned} M_{s,t}^{N,\mathrm{NT}}(\phi,\psi) &= \sum_{m=2}^{\infty} 3\cdot 2^{m-1} I_{s,t}^{(N,m)}(\phi,\psi), \qquad \text{where} \\ I_{s,t}^{(N,m)}(\phi,\psi) &:= \frac{\sigma_N^{2m}}{N^3} \sum_{\substack{s<a_1\leqslant b_1<a_2\leqslant b_2<\ldots<a_m\leqslant b_m<t \\ x_1,y_1,x_2,y_2,\ldots,x_m,y_m\in\mathbb{Z}^2}} q_{s,a_1}^N(\phi,x_1)^2\, q_{s,a_2}^N(\phi,x_2)\,\cdot \\ &\cdot U_N(b_1-a_1,y_1-x_1)\, q_{b_1,a_2}(y_1,x_2)\, U_N(b_2-a_2,y_2-x_2)\,\cdot \\ &\cdot \prod_{i=3}^{m} \Big\{ q_{b_{i-2},a_i}(y_{i-2},x_i)\, q_{b_{i-1},a_i}(y_{i-1},x_i)\, U_N(b_i-a_i,y_i-x_i) \Big\}\,\cdot \\ &\cdot q^N_{b_{m-1},t}(y_{m-1},\psi)\, q^N_{b_m,t}(y_m,\psi)^2\,, \end{aligned} \tag{5.2}$$

with the convention that $\prod_{i=3}^m\{\ldots\} = 1$ for $m=2$. Note that the sum starts with $m=2$ because in (5.1), we have $|\boldsymbol{A}|,|\boldsymbol{B}|,|\boldsymbol{C}| \geqslant 1$.

If we compare (5.2) with (4.5) and (4.2), we see that Proposition 4.2 follows from the following result and dominated convergence. ☐

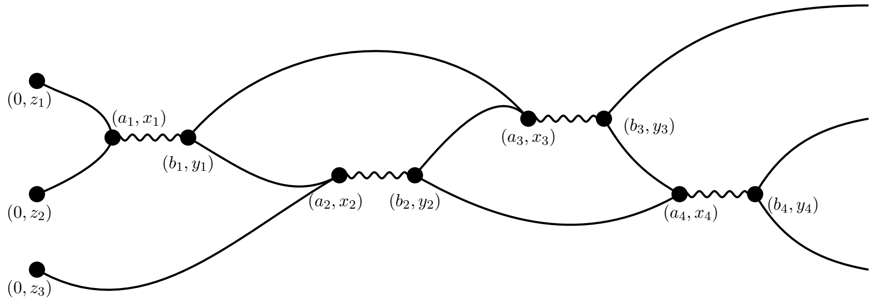

FIGURE 1. Diagramatic representation of the expansion (5.2) of the third moment. Curly lines between nodes (a_i, x_i) and (b_i, y_i) have weight $U_N(b_i - x_i, y_i - x_i)$, coming for pairwise matchings between a single pair of copies AB, BC or CA, while solid, curved lines between nodes (a_i, x_i) and (b_{i-1}, y_{i-1}) or between (a_i, x_i) and (b_{i-2}, y_{i-2}) indicate a weight $q_{b_{i-1},a_i}(y_{i-1}, x_i)$ and $q_{b_{i-2},a_i}(y_{i-2}, x_i)$, respectively.

Lemma 5.1. *For $m \geqslant 2$, let $I^{(N,m)}_{Nt}(\phi,\psi) := I^{(N,m)}_{0,Nt}(\phi,\psi)$ be defined as in* (5.2)*, and let $\mathcal{I}^{(m)}_t(\phi,\psi)$ be defined as in* (4.2)*. Then*

$$\lim_{N\to\infty} I^{(N,m)}_{Nt}(\phi,\psi) = \pi^m \, \mathcal{I}^{(m)}_t(\phi,\psi) \qquad \forall\, m \geqslant 2\,. \tag{5.3}$$

Furthermore, for any $C > 0$ we have

$$|I^{(N,m)}_{Nt}(\phi,\psi)| \leqslant e^{-Cm} \qquad \textit{for all } m, N \textit{ sufficiently large}\,. \tag{5.4}$$

The proof of Lemma 5.1 is given later, see Subsection 5.3. We first prove the next result on $\mathcal{I}^{(m)}_t(\phi,\psi)$, which will reveal a structure that will be used in the proof of Lemma 5.1.

Lemma 5.2. *For $\phi \in C_c(\mathbb{R}^2)$, $\psi \in C_b(\mathbb{R}^2)$, and $\mathcal{I}^{(m)}_t(\phi,\psi)$ defined as in* (4.2)*, we have:*

$$\forall\, a \in (0,\infty)\,, \qquad \sum_{m=2}^{\infty} a^m \, |\mathcal{I}^{(m)}_t(\phi,\psi)| < \infty. \tag{5.5}$$

5.2. PROOF OF LEMMA 5.2. In light of Remark 1.3, we may assume $t = 1$. Recall that

$$\begin{aligned}
\mathcal{I}^{(m)}(\phi,\psi) := \mathcal{I}^{(m)}_1(\phi,\psi) := & \idotsint\limits_{\substack{0<a_1<b_1<a_2<b_2<\ldots<a_m<b_m<1 \\ x_1,y_1,x_2,y_2,\ldots,x_m,y_m\in\mathbb{R}^2}} \Phi^2_{a_1}(x_1)\, \Phi_{a_2}(x_2)\,\cdot \\
& \cdot\; G_\vartheta(b_1-a_1, y_1-x_1)\, g_{\frac{a_2-b_1}{2}}(x_2-y_1)\, G_\vartheta(b_2-a_2, y_2-x_2)\,\cdot \\
& \cdot\; \prod_{i=3}^{m} g_{\frac{a_i-b_{i-2}}{2}}(x_i-y_{i-2})\, g_{\frac{a_i-b_{i-1}}{2}}(x_i-y_{i-1})\, G_\vartheta(b_i-a_i, y_i-x_i)\,\cdot \\
& \cdot\; \Psi_{1-b_{m-1}}(y_{m-1})\Psi^2_{1-b_m}(y_m)\; \mathrm{d}\vec{a}\,\mathrm{d}\vec{b}\,\mathrm{d}\vec{x}\,\mathrm{d}\vec{y},
\end{aligned} \tag{5.6}$$

where $G_\vartheta(t,x) := G_\vartheta(t) g_{t/4}(x)$, with $g_{t/4}(x)$ being the heat kernel, see (1.9), and G_ϑ defined in (1.18). We also recall that $\Phi_a(x) := (\phi * g_{a/2})(x)$, $\Psi_{1-b}(y) = (\Psi * g_{(1-b)/2})(y)$.

Note that we obtain an upper bound if we replace ϕ by $|\phi|$, so we may assume that $\phi \geqslant 0$. Similarly, we may replace ψ by the constant $|\psi|_\infty$, and we take $|\psi|_\infty \leqslant 1$ for simplicity. We thus bound $\mathcal{I}^{(m)}(\phi,\psi) \leqslant \mathcal{I}^{(m)}(\phi,1)$, with $\phi \geqslant 0$, and we focus on $\mathcal{I}^{(m)}(\phi,1)$.

We first show that, by integrating out the space variables, we can bound

$$\begin{aligned}
&\mathcal{I}^{(m)}(\phi,1) \leqslant C_\phi\, J^{(m)}, \qquad \text{where} \quad C_\phi := |\phi|_\infty^2 \int_{\mathbb{R}^2} \phi(z)\,\mathrm{d}z\,, \qquad \text{and} \\
&J^{(m)} := \int\!\!\cdots\!\!\int_{0<a_1<b_1<\ldots<a_m<b_m<1} G_\vartheta(b_1-a_1)G_\vartheta(b_2-a_2)\prod_{i=3}^m \frac{G_\vartheta(b_i-a_i)}{\sqrt{(a_i-b_{i-1})(a_i-b_{i-2})}}\,\mathrm{d}\vec{a}\,\mathrm{d}\vec{b}\,.
\end{aligned} \tag{5.7}$$

Note that in (5.6) we have $\Psi \equiv 1$ (by $\psi \equiv 1$) and y_m appears only in $G_\vartheta(b_m-a_m, y_m-x_m)$. Then we can integrate out $y_m \in \mathbb{R}^2$ to obtain

$$\int_{\mathbb{R}^2} G_\vartheta(b_m-a_m, y_m-x_m)\,\mathrm{d}y_m = G_\vartheta(b_m-a_m).$$

We are then left with two factors containing x_m, and the corresponding integral is

$$\begin{aligned}
&\int_{\mathbb{R}^2} g_{\frac{a_m-b_{m-1}}{2}}(x_m-y_{m-1})\, g_{\frac{a_m-b_{m-2}}{2}}(x_m-y_{m-2})\mathrm{d}x_m = g_{\frac{2a_m-b_{m-1}-b_{m-2}}{2}}(y_{m-1}-y_{m-2}) \\
&\quad \leqslant g_{\frac{2a_m-b_{m-1}-b_{m-2}}{2}}(0) = \frac{1}{\pi(2a_m-b_{m-1}-b_{m-2})} \leqslant \frac{1}{2\pi}\,\frac{1}{\sqrt{(a_m-b_{m-1})(a_m-b_{m-2})}}\,,
\end{aligned}$$

having used $\alpha\beta \leqslant \frac{1}{2}(\alpha^2+\beta^2)$ in the last inequality.

We now iterate. Integrating out each y_i, for $i \geqslant 2$, replaces $G_\vartheta(b_i-a_i, y_i-x_i)$ by $G_\vartheta(b_i-a_i)$, while integrating out each x_i, for $i \geqslant 3$, replaces $g_{\frac{a_i-b_{i-2}}{2}}(x_i-y_{i-2})\, g_{\frac{a_i-b_{i-1}}{2}}(x_i-y_{i-1})$ by $(2\pi\sqrt{(a_i-b_{i-1})(a_i-b_{i-2})})^{-1}$. This leads to

$$\begin{aligned}
\mathcal{I}^{(m)}(\phi,1) \leqslant \frac{1}{(2\pi)^{m-2}} & \int\!\!\cdots\!\!\int_{\substack{0<a_1<b_1<a_2<b_2<\ldots<a_m<b_m<1 \\ x_1,y_1,x_2\in\mathbb{R}^2}} \mathrm{d}\vec{a}\;\mathrm{d}\vec{b}\;\mathrm{d}x_1\,\mathrm{d}y_1\,\mathrm{d}x_2 \\
& \Phi_{a_1}^2(x_1)\,\Phi_{a_2}(x_2)\, g_{\frac{b_1-a_1}{4}}(y_1-x_1)\, g_{\frac{a_2-b_1}{2}}(x_2-y_1)\,\cdot \\
& \cdot\; G_\vartheta(b_1-a_1)\,G_\vartheta(b_2-a_2)\prod_{i=3}^m \frac{G_\vartheta(b_i-a_i)}{\sqrt{(a_i-b_{i-1})(a_i-b_{i-2})}}\;.
\end{aligned} \tag{5.8}$$

We finally bound $\Phi_{a_2}(x_2) \leqslant |\phi|_\infty$, see (3.1), then perform the integrals over x_2 and y_1, which both give 1, and note that $\int_{\mathbb{R}^2} \Phi_{a_1}(x_1)^2\,\mathrm{d}x_1 \leqslant |\phi|_\infty \int_{\mathbb{R}^2}\phi(z)\,\mathrm{d}z$, which yields (5.7).

We can now bound the quantity in Lemma 5.2 using (5.7), to get

$$\sum_{m=2}^\infty a^m |\mathcal{I}_t^{(m)}(\phi,\psi)| \leqslant (2\pi)^2\, C_\phi \sum_{m=2}^\infty \left(\frac{a}{2\pi}\right)^m J^{(m)}\,. \tag{5.9}$$

It remains to show that $J^{(m)}$ decay super-exponentially fast. For any $\lambda > 0$, we have

$$J^{(m)} \leqslant e^\lambda \int\!\!\cdots\!\!\int_{0<a_1<b_1<\ldots<a_m<b_m<1} e^{-\lambda b_1} G_\vartheta(b_1-a_1) e^{-\lambda(b_2-b_1)} G_\vartheta(b_2-a_2) \prod_{i=3}^m \frac{e^{-\lambda(b_i-b_{i-1})}G_\vartheta(b_i-a_i)}{\sqrt{(a_i-b_{i-1})(a_i-b_{i-2})}}\,\mathrm{d}\vec{a}\,\mathrm{d}\vec{b}.$$

Denote $u_i := a_i - b_{i-1}$ and $v_i := b_i - a_i$ for $1 \leqslant i \leqslant m$, where $b_0 := 0$. Then observe that $a_i - b_{i-2} = u_{i-1} + v_{i-1} + u_i \geqslant u_{i-1} + u_i$. Since $b_i - b_{i-1} \geqslant v_i$, we can bound $J^{(m)}$ by

$$\begin{aligned} J^{(m)} &\leqslant e^{\lambda} \int\!\!\cdots\!\!\int_{\substack{u_i, v_i \in (0,1) \\ \sum_{i=1}^m (u_i+v_i)<1}} \left\{ \prod_{i=1}^m e^{-\lambda v_i} G_\vartheta(v_i) \right\} \left\{ \prod_{i=3}^m \frac{1}{\sqrt{(u_{i-1}+u_i)u_i}} \right\} \mathrm{d}\vec{u}\, \mathrm{d}\vec{v} \\ &\leqslant e^{\lambda} \left(\int_0^1 e^{-\lambda v}\, G_\vartheta(v)\, \mathrm{d}v \right)^m \int\!\!\cdots\!\!\int_{u_2,\ldots,u_m \in (0,1)} \prod_{i=3}^m \frac{1}{\sqrt{u_i(u_i+u_{i-1})}}\, \mathrm{d}\vec{u} \\ &\leqslant e^{\lambda} \left(\int_0^1 e^{-\lambda v}\, \hat{G}_\vartheta(v)\, \mathrm{d}v \right)^m \int_0^1 \phi^{(m-2)}(u_2)\, \mathrm{d}u_2, \end{aligned} \tag{5.10}$$

where in the last inequality we have bounded $G_\vartheta(\cdot) \leqslant \hat{G}_\vartheta(\cdot)$, see (2.25), and we define

$$\phi^{(0)}(u) := 1\,, \qquad \text{and} \qquad \phi^{(k)}(u) := \int_0^1 \frac{1}{\sqrt{s(s+u)}}\, \phi^{(k-1)}(s)\, \mathrm{d}s\,, \qquad \forall k \in \mathbb{N}\,. \tag{5.11}$$

We will show the following results.

Lemma 5.3. *There is a constant $\mathsf{c}_\vartheta < \infty$ such that for every $\lambda \geqslant 1$*

$$\int_0^1 e^{-\lambda v}\, \hat{G}_\vartheta(v)\, \mathrm{d}v \leqslant C_\lambda := \frac{\mathsf{c}_\vartheta}{2 + \log \lambda}\,. \tag{5.12}$$

Lemma 5.4. *For all $k \in \mathbb{N}$, the function $\phi^{(k)}(\cdot)$ is decreasing on $(0,1)$ and satisfies*

$$\phi^{(k)}(v) \leqslant 32^k \sum_{i=0}^k \frac{1}{i!} \left(\frac{1}{2} \log \frac{e^2}{v} \right)^i \leqslant 32^k \frac{e}{\sqrt{v}}\,, \qquad \forall v \in (0,1)\,. \tag{5.13}$$

With Lemmas 5.3 and 5.4, it follows from (5.10) that

$$J^{(m)} \leqslant e^{\lambda}\, C_\lambda^m\, 32^{m-2} \int_0^1 \frac{e}{\sqrt{v}}\, \mathrm{d}v \leqslant e^{\lambda}\, C_\lambda^m\, 32^{m-2}\, 2e \leqslant e^{\lambda}\, (32\, C_\lambda)^m\,. \tag{5.14}$$

If we choose $\lambda = m$, then by (5.9) and the definition (5.12) of C_λ we get

$$\sum_{m=2}^\infty a^m\, |\mathcal{I}_t^{(m)}(\phi,\psi)| \leqslant C_\phi \sum_{m=2}^\infty a^m J^{(m)} \leqslant C_\phi \sum_{m=2}^\infty \left(\frac{32\, a\, e\, \mathsf{c}_\vartheta}{2 + \log m} \right)^m < \infty\,, \tag{5.15}$$

which concludes the proof of Lemma 5.2. □

It remains to prove Lemmas 5.3 and 5.4.

Proof of Lemma 5.3. Recall that $\hat{G}_\vartheta(\cdot)$ is defined in (2.25) and it is decreasing. Then

$$\int_{\frac{1}{\lambda}}^1 e^{-\lambda v}\, \hat{G}_\vartheta(v)\, \mathrm{d}v \leqslant \hat{G}_\vartheta(\tfrac{1}{\lambda}) \int_{\frac{1}{\lambda}}^\infty e^{-\lambda v}\, \mathrm{d}v = \hat{G}_\vartheta(\tfrac{1}{\lambda})\, \frac{e^{-1}}{\lambda} \leqslant e^{-1} \int_0^{\frac{1}{\lambda}} \hat{G}_\vartheta(v)\, \mathrm{d}v\,,$$

hence

$$\int_0^1 e^{-\lambda v}\, \hat{G}_\vartheta(v)\, \mathrm{d}v \leqslant (1 + e^{-1}) \int_0^{\frac{1}{\lambda}} \hat{G}_\vartheta(v)\, \mathrm{d}v = \frac{(1+e^{-1})\, c_\vartheta}{2 + \log \lambda} \leqslant \frac{2\, c_\vartheta}{2 + \log \lambda}\,. \tag{5.16}$$

We have proved that (5.12) holds, provided we chose $\mathsf{c}_\vartheta := 2\, c_\vartheta$. □

Proof of Lemma 5.4. The second inequality in (5.13) follows from $\sum_{i=0}^{k} \frac{x^i}{i!} \leqslant e^x$.

Let us prove the first inequality in (5.13). Recall the definition (5.11) of $\phi^{(k)}$. Then

$$\phi^{(1)}(v) = \int_0^1 \frac{\mathrm{d}s}{\sqrt{s(s+v)}} = \int_0^{\frac{1}{v}} \frac{\mathrm{d}z}{\sqrt{z(z+1)}} \leqslant \int_0^1 \frac{\mathrm{d}z}{\sqrt{z}} + \int_1^{\frac{1}{v}} \frac{\mathrm{d}z}{z} = 2 + \log\frac{1}{v}\,. \tag{5.17}$$

To iterate this argument and bound $\phi^{(k)}$, we claim that

$$\phi^{(k)}(v) \leqslant \sum_{i=0}^{k} \frac{c_{k,i}}{2^i\, i!} \left(\log\frac{e^2}{v}\right)^i \qquad \forall v \in (0,1)\,, \tag{5.18}$$

for suitable choices of the coefficients $c_{k,i}$. For $k=1$, we see from (5.17) that

$$c_{1,0} = 0\,, \qquad c_{1,1} = 2\,. \tag{5.19}$$

Inductively, we assume that (5.18) holds for $k-1$ and we will deduce it for k. Note that plugging (5.18) for $k-1$ into (5.11) gives

$$\phi^{(k)}(v) \leqslant \sum_{j=0}^{k-1} \frac{c_{k-1,j}}{2^j\, j!} \int_0^1 \frac{\left(\log\frac{e^2}{s}\right)^j}{\sqrt{s(s+v)}}\,\mathrm{d}s\,. \tag{5.20}$$

To identify $c_{k,i}$ for $0 \leqslant i \leqslant k$, we need the following Lemma, proved later.

Lemma 5.5. *For all $k \in \mathbb{N}_0$, we have*

$$\int_0^1 \frac{\left(\log\frac{e^2}{s}\right)^k}{\sqrt{s(s+v)}}\,\mathrm{d}s \leqslant 2^{k+1} k! \sum_{i=0}^{k+1} \frac{\left(\log\frac{e^2}{v}\right)^i}{2^i\, i!} \qquad \forall v \in (0,1)\,. \tag{5.21}$$

If we plug (5.21) with $k=j$ into (5.20) we get that, for all $v \in (0,1)$,

$$\phi^{(k)}(v) \leqslant \sum_{j=0}^{k-1} \frac{c_{k-1,j}}{2^j\, j!} \left\{ 2^{j+1}\, j! \sum_{i=0}^{j+1} \frac{\left(\log\frac{e^2}{v}\right)^i}{2^i\, i!} \right\} = \sum_{i=0}^{k} \frac{1}{2^i\, i!} \left(2 \sum_{j=(i-1)^+}^{k-1} c_{k-1,j} \right) \left(\log\frac{e^2}{v}\right)^i .$$

This shows that (5.18) indeed holds, with

$$c_{k,i} = 2 \sum_{j=(i-1)^+}^{k-1} c_{k-1,j}\,. \tag{5.22}$$

We have the following combinatorial bound on the coefficients $c_{k,i}$, which we prove later by comparing with the number of paths for a suitable random walk.

Lemma 5.6. *For every $k \in \mathbb{N}$ and $i \in \{0,\ldots,k\}$ we have $c_{k,i} \leqslant 32^k$.*

Plugging this bound into (5.18) we obtain, for all $k \in \mathbb{N}$ and $v \in (0,1)$,

$$\phi^{(k)}(v) \leqslant 32^k \sum_{i=0}^{k} \frac{1}{i!} \left(\frac{1}{2}\log\frac{e^2}{v}\right)^i, \tag{5.23}$$

which is the first inequality in (5.13). This concludes the proof of Lemma 5.4. □

It remains to prove Lemmas 5.5 and 5.6.

Proof of Lemma 5.5. By a change of variable $s = vz$,

$$\int_0^1 \frac{\left(\log \frac{e^2}{s}\right)^k}{\sqrt{s(s+v)}}\,\mathrm{d}s = \int_0^{\frac{1}{v}} \frac{\left(\log \frac{e^2}{vz}\right)^k}{\sqrt{z(z+1)}}\,\mathrm{d}z \leqslant \underbrace{\int_0^1 \frac{\left(\log \frac{e^2}{vz}\right)^k}{\sqrt{z}}\,\mathrm{d}z}_{A} + \underbrace{\int_1^{\frac{1}{v}} \frac{\left(\log \frac{e^2}{vz}\right)^k}{z}\,\mathrm{d}z}_{B}. \tag{5.24}$$

Let us look at B: the change of variable $z = v^{\alpha-1}$, with $\alpha \in (0,1)$, gives

$$B = \int_0^1 \frac{\left(\log \frac{e^2}{v^\alpha}\right)^k}{v^{\alpha-1}}\, v^{\alpha-1} \log \tfrac{1}{v}\,\mathrm{d}\alpha = \log \tfrac{1}{v} \int_0^1 \left(2 + \alpha \log \tfrac{1}{v}\right)^k \mathrm{d}\alpha \leqslant \tfrac{1}{k+1}\left(\log \tfrac{e^2}{v}\right)^{k+1}.$$

We now look at A: the change of variable $z = x\, e^2/v$, with $x \in (0, \frac{v}{e^2})$, followed by $x = e^{-2y}$, with $y \in (\frac{1}{2}\log\frac{e^2}{v}, \infty)$, yields

$$A = \frac{e}{\sqrt{v}} \int_0^{\frac{v}{e^2}} \frac{\left(\log \frac{1}{x}\right)^k}{\sqrt{x}}\,\mathrm{d}x = \frac{e}{\sqrt{v}} \int_{\frac{1}{2}\log\frac{e^2}{v}}^\infty \frac{(2y)^k}{e^{-y}}\, 2\, e^{-2y}\,\mathrm{d}y = \frac{e\, 2^{k+1}}{\sqrt{v}} \int_{\frac{1}{2}\log\frac{e^2}{v}}^\infty y^k\, e^{-y}\,\mathrm{d}y\,.$$

Let $(\boldsymbol{N}_t)_{t\geqslant 0}$ be a Poisson process with intensity one, and let $(X_i)_{i\geqslant 1}$ denote its jump sizes, which are i.i.d. exponential variables with parameter one. For all $t \geqslant 0$ we can write

$$\int_t^\infty y^k\, e^{-y}\,\mathrm{d}y = \Gamma(k+1)\,\mathrm{P}\left(\sum_{i=1}^k X_i \geqslant t\right) = \Gamma(k+1)\ \mathrm{P}\big(\boldsymbol{N}_t \leqslant k\big) = e^{-t} \sum_{i=0}^k \frac{k!}{i!}\, t^i\,. \tag{5.25}$$

Choosing $t = \frac{1}{2}\log\frac{e^2}{v}$, it follows that

$$A = \frac{e\, 2^{k+1}}{\sqrt{v}} e^{-\frac{1}{2}\log\frac{e^2}{v}} \sum_{i=0}^k \frac{k!}{i!} \left(\tfrac{1}{2}\log\tfrac{e^2}{v}\right)^i = \sum_{i=0}^k \frac{k!}{i!}\, 2^{k+1-i} \left(\log \tfrac{e^2}{v}\right)^i. \tag{5.26}$$

We have thus shown that

$$A + B = \sum_{i=0}^{k+1} \frac{k!}{i!}\, 2^{k+1-i} \left(\log \tfrac{e^2}{v}\right)^i,$$

which coincides with the RHS of (5.21). □

Proof of Lemma 5.6. We iterate the recursion relation (5.22), to get

$$\begin{aligned} c_{k,i} &= 2 \sum_{j_{k-1}=(i-1)^+}^{k-1} c_{k-1,j_{k-1}} = 4 \sum_{j_{k-1}=(i-1)^+}^{k-1} \ \sum_{j_{k-2}=(j_{k-1}-1)^+}^{k-2} c_{k-2,j_{k-2}} = \cdots \\ &= 2^{k-1} \sum_{j_{k-1}=(i-1)^+}^{k-1} \ \sum_{j_{k-2}=(j_{k-1}-1)^+}^{k-2} \cdots \sum_{j_2=(j_3-1)^+}^{2} \ \sum_{j_1=(j_2-1)^+}^{1} c_{1,j_1}\,. \end{aligned} \tag{5.27}$$

Since $c_{1,1} = 2$ and $c_{1,0} = 0$, see (5.19), we can restrict to $j_1 = 1$. Also observe that

$$j_i \geqslant (j_{i+1}-1)^+ \qquad \text{if and only if} \qquad j_i \geqslant 0 \ \text{ and } \ j_{i+1} \leqslant j_i + 1\,,$$

hence we can reverse the order of the sums in (5.27) and write

$$c_{k,i} = 2^k \left|\mathcal{S}_k(i)\right|, \tag{5.28}$$

where $|\mathcal{S}_k(i)|$ denotes the cardinality of the set

$$\mathcal{S}_k(i) := \left\{(j_1,\ldots,j_k) \in \mathbb{N}_0^k :\ j_1 = 1,\ j_k = i,\ j_{n+1} \leqslant j_n + 1\ \forall\, n = 1,\ldots,k-1\right\}. \tag{5.29}$$

In words, $\mathcal{S}_k(i)$ is the set of non-negative integer-valued paths $(j_1, \ldots, j_k)$ that start from $j_1 = 1$, arrive at $j_k = i$, and can make upward jumps of size at most 1, while the downward jumps can be of arbitrary size (with the constraint that the path is non-negative).

To complete the proof, it remains to show that

$$|\mathcal{S}_k(i)| \leqslant 16^k .$$

We define a correspondence which associates to any path $\boldsymbol{j} = (j_1, \ldots, j_k) \in \mathcal{S}_k(i)$ a *nearest neighbor path* $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_n)$, with length $n = n(\boldsymbol{j}) \in \{k, \ldots, 2k\}$, with increments in $\{-1, 0, 0^*, +1\}$, where by 0^* we mean an increment of size 0 with an extra label "$*$" (that will be useful to get an injective map). The correspondence is simple: whenever the path $\boldsymbol{j}$ has a downward jump (which can be of arbitrary size), we transform it into a sequence of downward jumps of size 1, followed by a jump of size 0^*.

Note that if $m = m(\boldsymbol{j})$ denotes the number of downward jumps in the path $\boldsymbol{j}$, then the new path $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_n)$ has length

$$n = n(\boldsymbol{j}) = k + (\sigma_1 + \ldots + \sigma_m) ,$$

where σ_i is the size of the i-th downward jump of $\boldsymbol{j}$. The total size of downward jumps is

$$(\sigma_1 + \ldots + \sigma_m) = \Delta^-(\boldsymbol{j}) := \sum_{i=1}^{k-1} (j_{i+1} - j_i)^- .$$

Defining $\Delta^+(\boldsymbol{j}) := \sum_{i=1}^{k-1} (j_{i+1} - j_i)^+$, we have

$$\Delta^+(\boldsymbol{j}) - \Delta^-(\boldsymbol{j}) = j_k - j_1 = i - 1 .$$

However $\Delta^+(\boldsymbol{j}) \leqslant k - 1$, because the upward jumps are of size at most 1, hence

$$\Delta^-(\boldsymbol{j}) \leqslant (k-1) - (i-1) \leqslant k ,$$

which shows that $n = n(\boldsymbol{j}) \leqslant 2k$, as we claimed.

Note that the correspondence $\boldsymbol{j} \mapsto \boldsymbol{\ell}$ is injective: the original path $\boldsymbol{j}$ can be reconstructed from $\boldsymbol{\ell}$, thanks to the labeled increments 0^*, which distinguishes consecutive downward jumps from a single downward jump with the same total length. Since the path $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_n)$ has $n - 1$ increments, each of which takes four possible values, we get the desired estimate:

$$\big|\mathcal{S}_k(i)\big| \leqslant \sum_{n=k}^{2k} 4^{n-1} \leqslant \sum_{n=1}^{2k} 4^{n-1} = \frac{16^k - 1}{3} \leqslant 16^k . \qquad \square$$

5.3. Proof of Lemma 5.1. We follow the same strategy as in the proof of Lemma 5.2.

We first prove the exponential bound (5.4). We recall that $I_{Nt}^{(N,m)}(\phi, \psi) := I_{0,Nt}^{(N,m)}(\phi, \psi)$, see (5.2). We may take $t = 1$, $\phi \geqslant 0$, and $\psi \equiv 1$, so that the last terms in (5.2) are $q^N_{b_{m-1},t}(y_{m-1}, \psi) \equiv 1$, $q^N_{b_m,t}(y_m, \psi) \equiv 1$. We can thus rewrite (5.2) as follows:

$$\begin{aligned} I_N^{(N,m)}(\phi, 1) := \frac{\sigma_N^{2m}}{N^3} & \sum_{\substack{0 < a_1 \leqslant b_1 < a_2 \leqslant b_2 < \ldots < a_m \leqslant b_m < N \\ x_1, y_1, x_2, y_2, \ldots, x_m, y_m \in \mathbb{Z}^2}} q^N_{0,a_1}(\phi, x_1)^2 \, q^N_{0,a_2}(\phi, x_2) \, \cdot \\ & \cdot U_N(b_1 - a_1, y_1 - x_1) \, q_{b_1,a_2}(y_1, x_2) \, U_N(b_2 - a_2, y_2 - x_2) \, \cdot \\ & \cdot \prod_{i=3}^{m} \Big\{ q_{b_{i-2},a_i}(y_{i-2}, x_i) \, q_{b_{i-1},a_i}(y_{i-1}, x_i) \, U_N(b_i - a_i, y_i - x_i) \Big\} . \end{aligned} \tag{5.30}$$

Similar to (5.7), we first prove the following bound:

$$I_N^{(N,m)}(\phi,1) \leqslant C_\phi\, J^{(N,m)}\,, \qquad \text{where} \tag{5.31}$$
$$J^{(N,m)} := c^m \frac{\sigma_N^{2m}}{N^2} \sum_{\substack{0<a_1\leqslant b_1<\ldots \\ \ldots<a_m\leqslant b_m<N}} U_N(b_1-a_1)\, U_N(b_2-a_2) \prod_{i=3}^m \frac{U_N(b_i-a_i)}{\sqrt{(a_i-b_{i-1})(a_i-b_{i-2})}}\,,$$

for suitable constants $C_\phi, c<\infty$. We first note that y_m appears in (5.30) only in the term $U_N(b_m-a_m, y_m-x_m)$ and hence we can sum it out as

$$\sum_{y_m\in\mathbb{Z}^2} U_N(b_m-a_m, y_m-x_m) =: U_N(b_m-a_m). \tag{5.32}$$

We next sum over x_m: since $q_{s,t}(x,y) \leqslant \sup_z q_{t-s}(z) \leqslant \frac{c}{t-s}$, see (2.3) and (1.8), we have

$$\begin{aligned} \sum_{x_m\in\mathbb{Z}^2} q_{b_{m-1},a_m}(y_{m-1},x_m)\, q_{b_{m-2},a_m}(y_{m-2},x_m) &\leqslant \sup_{x,y\in\mathbb{Z}^2} q_{b_{m-2},a_m}(y,x) \\ &\leqslant \frac{c}{(a_m-b_{m-2})} \leqslant \frac{c}{\sqrt{(a_m-b_{m-1})(a_m-b_{m-2})}}\,. \end{aligned} \tag{5.33}$$

We can now iterate, integrating out y_i for $i\geqslant 2$ and x_i for $i\geqslant 3$, to obtain

$$\begin{aligned} I_N^{(N,m)}(\phi,1) \leqslant \frac{c^{m-2}\sigma_N^{2m}}{N^3} \sum_{0<a_1\leqslant b_1<a_2\cdots<a_m\leqslant b_m<N} \; \sum_{x_1,x_2,y_1\in\mathbb{Z}^2} q_{0,a_1}^N(\phi,x_1)^2\, q_{0,a_2}^N(\phi,x_2) \\ \cdot\, U_N(b_1-a_1,y_1-x_1)\, q_{b_1,a_2}(y_1,x_2)\, U_N(b_2-a_2) \prod_{i=3}^m \frac{U_N(b_i-a_i)}{\sqrt{(a_i-b_{i-1})(a_i-b_{i-2})}}\,. \end{aligned} \tag{5.34}$$

After bounding $q_{0,a_2}^N(\phi,x_2)\leqslant|\phi|_\infty$, see (2.8), the sum over x_2 gives 1, because $q_{b_1,a_2}(y_1,\cdot)$ is a probability kernel. Then the sum over y_1 gives $U_N(b_1-a_1)$. Finally, the sum over x_1 gives

$$\sum_{x_1\in\mathbb{Z}^2} q_{0,a_1}^N(\phi,x_1)^2 \leqslant |\phi|_\infty \sum_{x_1\in\mathbb{Z}^2} q_{0,a_1}^N(\phi,x_1) = |\phi|_\infty \sum_{z\in\mathbb{Z}^2} \phi\big(\tfrac{z}{\sqrt{N}}\big) \leqslant c_\phi\, N \tag{5.35}$$

for a suitable $c_\phi<\infty$, because ϕ has compact support. This completes the proof of (5.31).

Next we bound $J^{(N,m)}$ in (5.31), similarly to the continuum analogue (5.10). Namely, we denote $u_i = a_i - b_{i-1}$ and $v_i := b_i - a_i$ for $1\leqslant i\leqslant m$, with $b_0 := 0$, we insert the factor $e^\lambda \prod_{i=1}^m e^{-\lambda(\frac{v_i}{N})} > 1$, and then we use $a_i - b_{i-2} \geqslant u_{i-1}+u_i$ to obtain the bound

$$J^{(N,m)} \leqslant e^\lambda\, c^m \left(\sigma_N^2 \sum_{v=0}^N e^{-\lambda\frac{v}{N}} U_N(v)\right)^m \left\{\frac{1}{N^2} \sum_{1\leqslant u_1,\ldots,u_m\leqslant N} \prod_{i=3}^m \frac{1}{\sqrt{u_i(u_{i-1}+u_i)}}\right\}. \tag{5.36}$$

Note that $\sigma_N^2 \leqslant \frac{c_1}{\log N}$, see (1.14), and $U_N(u)\leqslant c_2\,(\mathbb{1}_{\{u=0\}} + \frac{\log N}{N}\hat{G}_\vartheta(\frac{u}{N}))$, see (2.21) and (2.25). Since $\hat{G}_\vartheta(\cdot)$ is decreasing, we can bound the Riemann sum by the integral and get

$$\begin{aligned} \sigma_N^2 \sum_{v=0}^N e^{-\lambda\frac{v}{N}} U_N(v) &\leqslant \frac{c_1c_2}{\log N}\left(1 + \frac{\log N}{N}\sum_{v=1}^N e^{-\lambda\frac{v}{N}}\,\hat{G}_\vartheta\big(\tfrac{v}{N}\big)\right) \\ &\leqslant \frac{c_1c_2}{\log N}\left(1+\log N\int_0^1 e^{-\lambda v}\,\hat{G}_\vartheta(v)\,\mathrm{d}v\right) \leqslant c_1c_2\left(\frac{1}{\log N}+C_\lambda\right), \end{aligned} \tag{5.37}$$

where in the last inequality we have applied (5.12).

The multiple sum over the u_i's in (5.36) is bounded by the iterated integral in (5.10), by monotonicity (note that if we replace u_i by Nu_i, with $u_i \in \frac{1}{N}\mathbb{Z} \cap (0,1)$, then we get the correct prefactor $1/N^m$, thanks to the term $1/N^2$ in (5.36)). Then

$$J^{(N,m)} \leqslant e^{\lambda}\, c^m \left(\tfrac{1}{\log N} + C_\lambda\right)^m \int_0^1 \phi^{(m-2)}(u)\, \mathrm{d}u \leqslant e^{\lambda}\, (32\, c)^m \left(\tfrac{1}{\log N} + C_\lambda\right)^m , \tag{5.38}$$

because the integral is at most 32^m, by (5.13) (see also (5.14)). Since $C_\lambda = \frac{\mathsf{c}_\vartheta}{2+\log\lambda}$, see (5.12), if we choose λ and N large enough, then it is clear by (5.38) that $J^{(N,m)}$ decays faster than any exponential in m. This proves (5.4).

We next prove (5.3), for simplicity with $t = 1$. This is easily guessed because $I_1^{(N,m)}(\phi, \psi)$ (see (5.2)) is close to a Riemann sum for $\pi^m \mathcal{I}_1^{(m)}(\phi, \psi)$ (see (4.2)), by the asymptotic relations

$$\sigma_N^2 \sim \frac{\pi}{\log N}\,, \qquad q_{0,a}^N(\phi, x) \sim \Phi_{\frac{a}{N}}\left(\tfrac{x}{\sqrt{N}}\right), \qquad q_{b,1}^N(y, \psi) \sim \Psi_{1-\frac{b}{N}}\left(\tfrac{y}{\sqrt{N}}\right), \tag{5.39}$$

$$q_{b,a}(y, x) \sim \frac{1}{N}\, g_{\frac{a-b}{N}}\left(\tfrac{x-y}{\sqrt{N}}\right), \qquad U_N(b-a, y-x) \sim \frac{\log N}{N^2}\, G_\vartheta\left(\tfrac{b-a}{N}, \tfrac{y-x}{\sqrt{N}}\right), \tag{5.40}$$

see (1.10), (1.14), (3.4) and (1.8), (2.22).† We stress that plugging (5.39)-(5.40) into (5.2) we obtain the correct prefactor $1/N^{2m}$, thanks to the extra term $1/N^3$ in (5.2).

To justify the replacements (5.39)-(5.40), we proceed by approximations. Henceforth $m \geqslant 2$ is fixed. We define $\mathcal{I}_1^{(m),(\varepsilon)}(\phi, \psi)$ by restricting the integral in (4.2) to the set

$$\left\{a_i - b_{i-1} \geqslant \varepsilon \quad \forall 1 \leqslant i \leqslant m+1\,, \qquad b_i - a_i \geqslant \varepsilon \quad \forall 1 \leqslant i \leqslant m\right\}, \tag{5.41}$$

where $b_0 := 0$ and $a_{m+1} := 1$. Note that $\mathcal{I}_1^{(m)}(\phi, \psi) - \mathcal{I}_1^{(m),(\varepsilon)}(\phi, \psi)$ is small, if we choose $\varepsilon > 0$ small, simply because the integrated integral $\mathcal{I}_1^{(m)}(\phi, \psi)$ is finite.

We similarly define $I_N^{(N,m),(\varepsilon)}(\phi, \psi)$ by restricting the sum in (5.2) to the set

$$\left\{a_i - b_{i-1} \geqslant \varepsilon N \quad \forall 1 \leqslant i \leqslant m+1\,, \qquad b_i - a_i \geqslant \varepsilon N \quad \forall 1 \leqslant i \leqslant m\right\}, \tag{5.42}$$

where $b_0 := 0$ and $a_{m+1} := N$. The difference $I_N^{(N,m)}(\phi, \psi) - I_N^{(N,m),(\varepsilon)}(\phi, \psi)$ is bounded by the sum in (5.31) restricted to the complementary set of (5.42). By the uniform bound (2.21), this sum is bounded by the integral in (5.7) restricted to the complementary set of (5.41). Then $I_N^{(N,m)}(\phi, \psi) - I_N^{(N,m),(\varepsilon)}(\phi, \psi)$ is small, uniformly in large N, for $\varepsilon > 0$ small.

As a consequence, to prove (5.3) it suffices to show that

$$\lim_{N\to\infty} I_N^{(N,m),(\varepsilon)}(\phi, \psi) = \pi^m\, \mathcal{I}_1^{(m),(\varepsilon)}(\phi, \psi) \qquad \text{for each } \varepsilon > 0.$$

We next make a second approximation. For large $M > 0$, we define $\mathcal{I}_1^{(m),(\varepsilon,M)}(\phi, \psi)$ by further restricting the integral in (4.2) to the bounded set

$$\begin{aligned} \Big\{|x_1| \leqslant M, \quad & |y_i - x_i| \leqslant M\sqrt{b_i - a_i} \quad \forall 1 \leqslant i \leqslant m\,, \\ & |x_i - y_{i-1}| \leqslant M\sqrt{a_i - b_{i-1}} \quad \forall\, 2 \leqslant i \leqslant m\Big\}. \end{aligned} \tag{5.43}$$

We similarly define $I_N^{(N,m),(\varepsilon,M)}(\phi, \psi)$, by further restricting the sum in (5.2) to the set

$$\begin{aligned} \Big\{|x_1| \leqslant M\sqrt{N}, \quad & |y_i - x_i| \leqslant M\sqrt{b_i - a_i} \quad \forall 1 \leqslant i \leqslant m\,, \\ & |x_i - y_{i-1}| \leqslant M\sqrt{a_i - b_{i-1}} \quad \forall\, 2 \leqslant i \leqslant m\Big\}. \end{aligned} \tag{5.44}$$

†For simplicity, in relations (5.40) we have omitted the "periodicity correction" $2\,\mathbb{1}_{\{(n,x)\in\mathbb{Z}^3_{\text{even}}\}}$, see (1.8) and (2.22), because this disappears upon summation.

Clearly, $\lim_{M\to\infty} \mathcal{I}_1^{(m),(\varepsilon,M)}(\phi,\psi) = \mathcal{I}_1^{(m),(\varepsilon)}(\phi,\psi)$. We claim that, analogously,

$$\lim_{M\to\infty} \limsup_{N\to\infty} \big| I_N^{(N,m),(\varepsilon,M)}(\phi,\psi) - I_N^{(N,m),(\varepsilon)}(\phi,\psi) \big| = 0 \,. \tag{5.45}$$

Then we can complete the proof of Lemma 5.1: the asymptotic relations (5.39) and (5.40) hold uniformly on the restricted sets (5.42) and (5.44), so by dominated convergence

$$\text{for every } \varepsilon > 0\,,\ M < \infty : \qquad \lim_{N\to\infty} I_N^{(N,m),(\varepsilon,M)}(\phi,\psi) = \pi^m \, \mathcal{I}_1^{(m),(\varepsilon,M)}(\phi,\psi)\,.$$

It remains to prove (5.45). We can upper bound the difference in (5.45) as in (5.31)-(5.34): we sum out the spatial variables recursively, starting from y_m, then x_m, then y_{m-1}, etc.

- When we sum out y_m, if $|y_m - x_m| > M\sqrt{b_m - a_m}$, then by (5.32) and (2.23) we pick up at most a fraction $\delta(M) \leqslant C/M^2$ of the upper bound in (5.34). The same applies when we sum out y_i for $2 \leqslant i \leqslant m-1$, if $|y_i - x_i| > M\sqrt{b_i - a_i}$.
- When we sum out x_m, if $|x_m - y_{m-1}| > M\sqrt{a_m - b_{m-1}}$, then we restrict the sum in (5.33) accordingly, and we pick up again at most a fraction $\delta(M) \leqslant 1/M^2$ of the upper bound in (5.34), simply because $\sum_{|x|>M\sqrt{n}} q_n(x) = \mathrm{P}(|S_n| > M\sqrt{n}) \leqslant 1/M^2$. The same applies when we sum out x_i for $3 \leqslant i \leqslant m-1$.
- The same argument applies to the sums over x_2 and y_1, see the lines following (5.34).
- For the last sum over x_1, if $|x_1| > M\sqrt{N}$, by (2.8) and the fact that ϕ has compact support, we pick up at most a fraction $\delta(M) = O(1/M^2)$ of the sum (5.35).

Since for fixed m, there are only finitely many cases that violate (5.44), while $\delta(M) \to 0$ as $M \to \infty$, then (5.45) follows readily. □

6. Further bounds without triple intersections

We recall that the centered third moment $\mathbb{E}\big[\big(Z_{s,t}^{N,\beta_N}(\phi,\psi) - \mathbb{E}[Z_{s,t}^{N,\beta_N}(\phi,\psi)]\big)^3\big]$ of the partition function averaged over both endpoints admits the expansion (4.3). We then denoted by $M_{s,t}^{N,\mathrm{NT}}(\phi,\psi)$ the contribution to (4.3) coming from *no triple intersecitons*, see (4.4).

We now consider the partition functions $Z_{s,t}^{N,\beta_N}(w,\psi)$, $Z_{s,t}^{N,\beta_N}(\phi,z)$ averaged over *one* endpoint, see (2.4), (2.5), and also the point-to-point partition function $Z_{s,t}^{\beta_N}(w,z)$, see (2.1) (we sometimes write $Z_{s,t}^{N,\beta_N}(w,z)$, even though it carries no explicit dependence on N).

The centered third moment $\mathbb{E}\big[\big(Z_{s,t}^{N,\beta_N}(*,\dagger) - \mathbb{E}[Z_{s,t}^{N,\beta_N}(*,\dagger)]\big)^3\big]$ for $* \in \{\phi, w\}$, $\dagger \in \{\psi, z\}$ can be written as in (4.3), starting from the polynomial chaos expansions (2.11)-(2.12). In analogy with (4.4), we decompose

$$\mathbb{E}\big[\big(Z_{s,t}^{N,\beta_N}(*,\dagger) - \mathbb{E}[Z_{s,t}^{N,\beta_N}(*,\dagger)]\big)^3\big] = M_{s,t}^{N,\mathrm{NT}}(*,\dagger) + M_{s,t}^{N,\mathrm{T}}(*,\dagger)\,, \tag{6.1}$$

where $M_{s,t}^{N,\mathrm{T}}(*,\dagger)$ and $M_{s,t}^{N,\mathrm{NT}}(*,\dagger)$ are the contributions with and without triple intersections. In this section we prove the following bounds, which will be used to prove Proposition 4.3.

Lemma 6.1 (Bounds without triple intersections). *Let $\phi \in C_c(\mathbb{R}^2)$, $\psi \in C_b(\mathbb{R}^2)$ and $w, z \in \mathbb{Z}^2$. For any $\varepsilon > 0$, as $N \to \infty$, we have*

$$M_{0,N}^{N,\mathrm{NT}}(w,\psi) = O(N^{\varepsilon}), \tag{6.2}$$

$$\sum_{1 \leqslant a \leqslant N} \sum_{z \in \mathbb{Z}^2} M_{0,a}^{N,\mathrm{NT}}(w,z) = O(1), \tag{6.3}$$

$$\sum_{1 \leqslant a \leqslant N} \sum_{z \in \mathbb{Z}^2} M_{0,a}^{N,\mathrm{NT}}(\phi,z) = O(N^{\frac{5}{2}+\varepsilon}). \tag{6.4}$$

We prove relations (6.2)–(6.4) separately below. For the quantity $M_{s,t}^{N,\mathrm{NT}}(*,\dagger)$, when both arguments $*, \dagger$ are functions, we derived the representation (5.2). Analogous representations hold when one of the arguments $*, \dagger$ is a point. For instance, in the point-to-point case:

$$\begin{aligned}
M_{s,t}^{N,\mathrm{NT}}(w,z) &= \sum_{m=2}^{\infty} 3 \cdot 2^{m-1}\, I_{s,t}^{(N,m)}(w,z), \qquad \text{where} \\
I_{s,t}^{(N,m)}(w,z) &:= \sigma_N^{2m} \sum_{\substack{s<a_1 \leqslant b_1 < a_2 \leqslant b_2 < \ldots < a_m \leqslant b_m < t \\ x_1,y_1,x_2,y_2,\ldots,x_m,y_m \in \mathbb{Z}^2}} q_{s,a_1}(w,x_1)^2\, q_{s,a_2}(w,x_2) \,\cdot \\
&\quad \cdot U_N(b_1-a_1, y_1-x_1)\, q_{b_1,a_2}(y_1,x_2)\, U_N(b_2-a_2, y_2-x_2) \,\cdot \\
&\quad \cdot \prod_{i=3}^{m} \Big\{ q_{b_{i-2},a_i}(y_{i-2},x_i)\, q_{b_{i-1},a_i}(y_{i-1},x_i)\, U_N(b_i-a_i, y_i-x_i) \Big\} \,\cdot \\
&\quad \cdot q_{b_{m-1},t}(y_{m-1},z)\, q_{b_m,t}(y_m,z)^2 .
\end{aligned} \tag{6.5}$$

Note that in contrast to (5.2) there is no factor N^{-3}, because the definition of $Z_{s,t}^{\beta_N}(w,z)$, unlike $Z_{s,t}^{N,\beta_N}(\phi,\psi)$, contains no such factor, cf. (2.1) and (2.6).

The identity (6.5) holds also for $M_{s,t}^{N,\mathrm{NT}}(\phi,z)$ (replace $q_{s,a_i}(w,x_i)$ by $q_{s,a_i}^N(\phi,x_i)$, $i=1,2$) and for $M_{s,t}^{N,\mathrm{NT}}(w,\psi)$ (replace $q_{b_i,t}(y_i,z)$ by $q_{b_i,t}^N(y_i,\psi)$, $i=m-1,m$).

Proof of (6.2)**.** To estimate $I_{0,N}^{(N,m)}(w,\psi)$, we replace ψ by the constant $|\psi|_\infty$, and we take $|\psi|_\infty \leqslant 1$. We then focus on $I_{0,N}^{(N,m)}(w,1)$, and we can set $w=0$, by translation invariance. By the analogue of (6.5) (note that $q_{b_i,t}^N(y_i,\psi) \equiv 1$ for $\psi \equiv 1$), we get

$$\begin{aligned}
I_{0,N}^{(N,m)}(w,\psi) &\leqslant \sigma_N^{2m} \sum_{\substack{0<a_1 \leqslant b_1 < a_2 \leqslant b_2 < \ldots < a_m \leqslant b_m < N \\ x_1,y_1,x_2,y_2,\ldots,x_m,y_m \in \mathbb{Z}^2}} q_{a_1}(x_1)^2 U_N(b_1-a_1, y_1-x_1) \,\cdot \\
&\quad \cdot \prod_{i=2}^{m} \Big\{ q_{b_{i-2},a_i}(y_{i-2},x_i)\, q_{b_{i-1},a_i}(y_{i-1},x_i)\, U_N(b_i-a_i, y_i-x_i) \Big\},
\end{aligned} \tag{6.6}$$

where we stress that the product starts from $i=2$ and we set $b_0 := 0$ and $y_0 := 0$. By the definition of U_N in (2.15)-(2.16), we have the following identity, for fixed $b_1 \in \mathbb{N}$, $y_1 \in \mathbb{Z}^2$:

$$\sigma_N^2 \sum_{0<a_1 \leqslant b_1,\ x_1 \in \mathbb{Z}^2} q_{a_1}(x_1)^2\, U_N(b_1-a_1, y_1-x_1) = U_N(b_1,y_1). \tag{6.7}$$

Therefore we can rewrite (6.6) as

$$\begin{aligned} I_{0,N}^{(N,m)}(w,\psi) \leqslant \sigma_N^{2(m-1)} & \sum_{\substack{0<b_1<a_2\leqslant b_2<\ldots<a_m\leqslant b_m<N \\ y_1,x_2,y_2,\ldots,x_m,y_m\in\mathbb{Z}^2}} U_N(b_1,y_1)\,\cdot \\ & \cdot \prod_{i=2}^m \Big\{ q_{b_{i-2},a_i}(y_{i-2},x_i)\, q_{b_{i-1},a_i}(y_{i-1},x_i)\, U_N(b_i-a_i,y_i-x_i)\Big\}. \end{aligned} \tag{6.8}$$

We now sum out the spatial variables y_m, x_m, ..., y_2, x_2, y_1, arguing as in (5.32)-(5.33), to get the following upper bound, analogous to (5.31), for a suitable $c<\infty$:

$$I_{0,N}^{(N,m)}(w,\psi) \leqslant c^m\,\sigma_N^{2(m-1)} \sum_{\substack{0<b_1<a_2\leqslant b_2<\ldots \\ \ldots<a_m\leqslant b_m<N}} U_N(b_1) \prod_{i=2}^m \frac{U_N(b_i-a_i)}{\sqrt{(a_i-b_{i-1})(a_i-b_{i-2})}}\,. \tag{6.9}$$

Then we set $u_i := a_i - b_{i-1}$, $v_i := b_i - a_i$ for $2\leqslant i\leqslant m$, and we rename $u_1 := b_1$. This allows to bound $a_i - b_{i-2} \geqslant u_i + u_{i-1}$ for all $i\geqslant 2$ (including $i=2$, since $a_i - b_{i-2} = a_2 \geqslant u_2 + b_1$). Then, for $\lambda>0$, we insert the factor $e^{\lambda}\prod_{i=2}^m e^{-\lambda(\frac{v_i}{N})} > 1$ and we estimate, as in (5.36),

$$\begin{aligned} I_{0,N}^{(N,m)}(w,\psi) \leqslant e^{\lambda}\, c^m & \left(\sigma_N^2 \sum_{v=0}^N e^{-\lambda\frac{v}{N}} U_N(v)\right)^{m-1} \cdot \\ & \cdot \left\{ \sum_{u_1=1}^N U_N(u_1) \sum_{0<u_2,\ldots,u_m<N} \prod_{i=2}^m \frac{1}{\sqrt{u_i(u_i+u_{i-1})}} \right\}. \end{aligned} \tag{6.10}$$

The first parenthesis is $\leqslant c\big(\frac{1}{\log N} + C_\lambda\big)$, see (5.37). Then we replace u_i by Nu_i, with $u_i \in \frac{1}{N}\mathbb{Z}$, and bound Riemann sums by integrals, by monotonicity. This yields (for a possibly larger c)

$$\begin{aligned} I_{0,N}^{(N,m)}(w,\psi) \leqslant e^{\lambda}\, c^m & \big(\tfrac{1}{\log N} + C_\lambda\big)^{m-1} \cdot \\ & \cdot \left\{ \sum_{\substack{u_1\in\frac{1}{N}\mathbb{Z} \\ \frac{1}{N}\leqslant u_1\leqslant 1}} U_N(Nu_1) \int_{0<u_2,\ldots,u_m<1} \prod_{i=2}^m \frac{1}{\sqrt{u_i(u_i+u_{i-1})}}\, \mathrm{d}\vec{u} \right\}. \end{aligned} \tag{6.11}$$

The integral equals $\phi^{(m-1)}(u_1)$, see (5.11). We bound $U_N(Nu_1) \leqslant c_2 \frac{\log N}{N}\, \hat{G}_\vartheta(u_1)$ by (2.21) and (2.25), since $u_1>0$. Recalling that $\phi^{(m-1)}(\cdot)$ is decreasing, we get

$$\begin{aligned} I_{0,N}^{(N,m)}(w,\psi) & \leqslant e^{\lambda}\, c^m \big(\tfrac{1}{\log N} + C_\lambda\big)^{m-1} \left\{ \frac{\log N}{N} \sum_{\substack{u_1\in\frac{1}{N}\mathbb{Z} \\ 0<u_1<1}} \hat{G}_\vartheta(u_1) \right\} \phi^{(m-1)}(\tfrac{1}{N}) \\ & \leqslant e^{\lambda}\, c^m \big(\tfrac{1}{\log N} + C_\lambda\big)^{m-1} (\log N)\, C_\lambda\, \phi^{(m-1)}(\tfrac{1}{N})\,, \end{aligned} \tag{6.12}$$

where for the last inequality, recalling that $\hat{G}_\vartheta(\cdot)$ is decreasing, we bounded the Riemann sum in brackets by the integral $\int_0^1 \hat{G}_\vartheta(u_1)\,\mathrm{d}u_1 \leqslant C_\lambda$, see (5.12).

Putting together (6.5) and (6.12), we can finally estimate

$$M_{0,N}^{N,\mathrm{NT}}(w,\psi) \leqslant 3 \sum_{m\geqslant 2} 2^m\, I_{0,N}^{(N,m)}(w,\psi) \leqslant 3e^{\lambda}\,(\log N) \sum_{m\geqslant 2} \big[2c\,\big(\tfrac{1}{\log N} + C_\lambda\big)\big]^m\, \phi^{(m-1)}(\tfrac{1}{N})\,,$$

and using the first inequality in (5.13) we obtain

$$\begin{aligned} M_{0,N}^{N,\mathrm{NT}}(w,\psi) &\leqslant 3e^{\lambda}\,(\log N) \sum_{m\geqslant 2} \big[64c\,\big(\tfrac{1}{\log N}+C_\lambda\big)\big]^m \sum_{i=0}^{m} \frac{1}{i!}\big(\tfrac{1}{2}\log(e^2N)\big)^i \\ &\leqslant 3e^{\lambda}\,(\log N) \sum_{i\geqslant 0} \frac{1}{i!}\big(\tfrac{1}{2}\log(e^2N)\big)^i \sum_{m\geqslant i} \big[64c\,\big(\tfrac{1}{\log N}+C_\lambda\big)\big]^m \\ &= \tfrac{3e^\lambda}{1-[64c\,(\frac{1}{\log N}+C_\lambda)]}\,(\log N) \sum_{i\geqslant 0} \frac{1}{i!}\Big(32c\,\big(\tfrac{1}{\log N}+C_\lambda\big)\,\log(e^2N)\Big)^i \\ &= \tfrac{3e^\lambda}{1-[64c\,(\frac{1}{\log N}+C_\lambda)]}\,(\log N)\,\big(e^2N\big)^{32c\,(\frac{1}{\log N}+C_\lambda)}\,. \end{aligned}$$

Since $\lim_{\lambda\to\infty} C_\lambda = 0$, see (5.12), given $\varepsilon > 0$ we can fix λ large so that $32c\,C_\lambda < \frac{\varepsilon}{2}$. Then for large N the exponent of (e^2N) in the last term is $< \varepsilon$, which proves (6.2). □

Proof of (6.3)**.** From the first line of (6.5) we can write

$$\sum_{1\leqslant a\leqslant N}\sum_{z\in\mathbb{Z}^2} M_{0,a}^{N,\mathrm{NT}}(w,z) = \sum_{m\geqslant 2} 3\cdot 2^{m-1} \sum_{1\leqslant a\leqslant N}\sum_{z\in\mathbb{Z}^2} I_{0,a}^{(N,m)}(w,z) \tag{6.13}$$

To estimate $\sum_{1\leqslant a\leqslant N}\sum_{z\in\mathbb{Z}^2} I_{0,a}^{(N,m)}(w,z)$, we use the representation (6.5) with $s=0$ and $t=a$. We may also set $w=0$ (by translation invariance). We first perform the sum over a_1 and b_m, using (6.7) and the symmetric relation

$$\sigma_N^2 \sum_{a_m\leqslant b_m<a,\ y_m\in\mathbb{Z}^2} U_N(b_m-a_m, y_m-x_m)\, q_{b_m,a}(y_m,z)^2 = U_N(a-a_m, z-x_m)\,. \tag{6.14}$$

We then obtain

$$\begin{aligned} \sum_{\substack{1\leqslant a\leqslant N\\ z\in\mathbb{Z}^2}} I_{0,a}^{(N,m)}(w,z) = \sigma_N^{2(m-2)} &\sum_{\substack{0<b_1<a_2\leqslant b_2<\ldots<a_m<a\leqslant N\\ y_1,x_2,\ldots,y_{m-1},x_m,z\in\mathbb{Z}^2}} U_N(b_1,y_1)\,\cdot \\ &\cdot \prod_{i=2}^{m-1} \Big\{ q_{b_{i-2},a_i}(y_{i-2},x_i)\, q_{b_{i-1},a_i}(y_{i-1},x_i)\, U_N(b_i-a_i, y_i-x_i)\Big\} \\ \cdot \Big\{ q_{b_{m-2},a_m}(y_{m-2},x_m)\, q_{b_{m-1},a_m}&(y_{m-1},x_m)\, U_N(a-a_m, z-x_m)\Big\}\, q_{b_{m-1},a}(y_{m-1},z). \end{aligned} \tag{6.15}$$

If we rename $y_m := z$ and $b_m := a$, then we see that (6.15) differs from (6.8) only for the factor $\sigma_N^{2(m-2)}$ (instead of $\sigma_N^{2(m-1)}$) and for the presence of the last kernel $q_{b_{m-1},a}(y_{m-1},z) = q_{b_{m-1},b_m}(y_{m-1},y_m)$. The latter can be estimated using (1.8):

$$q_{b_{m-1},a}(y_{m-1},z) \leqslant \frac{c}{b_m-b_{m-1}} \leqslant \frac{c}{\sqrt{b_m-b_{m-1}}} \tag{6.16}$$

for some suitable constant c. As in (6.9), we first sum out the spatial variables, getting

$$\sum_{\substack{1\leqslant a\leqslant N\\ z\in\mathbb{Z}^2}} I_{0,a}^{(N,m)}(w,z) \leqslant c^m\,\sigma_N^{2(m-2)} \sum_{\substack{0<b_1<a_2\leqslant b_2<\ldots\\ \ldots<a_m\leqslant b_m<N}} U_N(b_1) \prod_{i=2}^{m} \frac{U_N(b_i-a_i)}{\sqrt{(a_i-b_{i-1})(a_i-b_{i-2})}}\, \frac{1}{\sqrt{b_m-b_{m-1}}}\,.$$

Then we set $u_1 := b_1$ and $u_i := a_i - b_{i-1}$, $v_i := b_i - a_i$ for $2\leqslant i\leqslant m$, which allows to bound $a_i - b_{i-2} \geqslant u_i + u_{i-1}$ for $i\geqslant 2$, as well as $b_m - b_{m-1} \geqslant u_m$. Then, for $\lambda > 0$, we insert the factor $e^{\lambda}\prod_{i=2}^{m} e^{-\lambda(\frac{v_i}{N})} > 1$ and, by (5.37), we obtain the following analogue of (6.10):

$$\sum_{\substack{1\leqslant a\leqslant N\\ z\in\mathbb{Z}^2}} I_{0,a}^{(N,m)}(w,z) \leqslant c^m\, e^{\lambda}\, (\log N)\big(\tfrac{1}{\log N}+C_\lambda\big)^{m-1}\cdot \\ \cdot\left\{\sum_{u_1=1}^{N} U_N(u_1) \sum_{0<u_2,\dots,u_m<N} \prod_{i=2}^{m} \frac{1}{\sqrt{u_i(u_i+u_{i-1})}}\,\frac{1}{\sqrt{u_m}}\right\}, \tag{6.17}$$

where the extra $\log N$ comes from having $\sigma_N^{2(m-2)}$ instead of $\sigma_N^{2(m-1)}$ (by (1.14) and (1.10)).

We now switch to macroscopic variables, replacing u_i by Nu_i, with $u_i \in \frac{1}{N}\mathbb{Z}\cap(0,1)$, and bound $U_N(Nu_1) \leqslant c_1\,\frac{\log N}{N}\,\hat{G}_\vartheta(u_1)$ since $u_1>0$, by (2.21) and (2.25). We then replace the Riemann sum in brackets by the corresponding integrals, similar to (6.11), with an important difference (for later purposes): since $u_i\in\frac{1}{N}\mathbb{Z}$ and $u_i>0$, we can restrict the integration on $u_i \geqslant \frac{1}{N}$ (possibly enlarging the value of c). This leads to

$$\sum_{\substack{1\leqslant a\leqslant N\\ z\in\mathbb{Z}^2}} I_{0,a}^{(N,m)}(w,z) \leqslant (\log N)\, e^{\lambda}\, c^m\, \big(\tfrac{1}{\log N}+C_\lambda\big)^{m-1}\cdot \\ \cdot\,\frac{\log N}{\sqrt{N}}\left\{\int_{\frac{1}{N}\leqslant u_1,u_2,\dots,u_m<1} \hat{G}_\vartheta(u_1)\left(\prod_{i=2}^{m}\frac{1}{\sqrt{u_i(u_i+u_{i-1})}}\right)\frac{1}{\sqrt{u_m}}\,\mathrm{d}\vec{u}\right\}, \tag{6.18}$$

where the factor $\frac{\log N}{\sqrt{N}}$ comes from the estimate on $U_N(Nu_1)$ and from the last kernel $1/\sqrt{u_m}$.

If we define $\widehat{\phi}^{(k)}(\cdot)$ as the following modification of (5.11):

$$\widehat{\phi}^{(0)}(u) := \frac{1}{\sqrt{u}}\,, \qquad \text{and for } k\geqslant 1\text{:}\quad \widehat{\phi}^{(k)}(u) := \int_{\frac{1}{N}}^{1} \frac{1}{\sqrt{s(s+u)}}\,\widehat{\phi}^{(k-1)}(s)\,\mathrm{d}s\,, \tag{6.19}$$

then, recalling (5.12), we can rewrite (6.18) as follows:

$$\sum_{\substack{1\leqslant a\leqslant N\\ z\in\mathbb{Z}^2}} I_{0,a}^{(N,m)}(w,z) \leqslant e^{\lambda}\, c^m\, \frac{(\log N)^2}{\sqrt{N}}\,(C_{\lambda,N})^{m-1}\int_{\frac{1}{N}}^{1}\hat{G}_\vartheta(u)\,\widehat{\phi}^{(m-1)}(u)\,\mathrm{d}u\,, \\ \text{where we set } \; C_{\lambda,N} := \tfrac{1}{\log N}+C_\lambda = \tfrac{1}{\log N}+\tfrac{\mathsf{c}_\vartheta}{2+\log\lambda}\,. \tag{6.20}$$

Similar to Lemma 5.4, we have the following bound on $\widehat{\phi}^{(k)}$, that we prove later.

Lemma 6.2. *For all $k\in\mathbb{N}$, the function $\widehat{\phi}^{(k)}(v)$ is decreasing on $(0,1)$, and satisfies*

$$\widehat{\phi}^{(k)}(u) \leqslant 32^k \sum_{i=0}^{k}\frac{1}{2^i\, i!}\frac{\big(\log(e^2Nu)\big)^i}{\sqrt{u}} \leqslant 32^k\, e\,\sqrt{N}\,, \qquad \forall u\in\big(\tfrac{1}{N},1\big). \tag{6.21}$$

We need to estimate the integral in (6.20), when we plug in the bound (6.21). We first consider the contribution from $u<\frac{1}{\sqrt{N}}$. In this case $\hat{G}_\vartheta(u)\leqslant\frac{4c_\vartheta}{(\log N)^2}\frac{1}{u}$, see (2.25), hence

$$\int_{\frac{1}{N}}^{\frac{1}{\sqrt{N}}} \hat{G}_\vartheta(u)\frac{\big(\log(e^2Nu)\big)^i}{\sqrt{u}}\mathrm{d}u \leqslant \frac{4c_\vartheta}{(\log N)^2}\int_{\frac{1}{N}}^{\frac{1}{\sqrt{N}}}\frac{\big(\log(e^2Nu)\big)^i}{u^{\frac{3}{2}}}\mathrm{d}u \\ \leqslant \frac{4ec_\vartheta\sqrt{N}}{(\log N)^2}\int_1^\infty\frac{\big(\log w\big)^i}{w^{\frac{3}{2}}}\mathrm{d}w = \frac{8ec_\vartheta\sqrt{N}}{(\log N)^2}\int_0^\infty 2^i s^i e^{-s}\mathrm{d}s = C\,\frac{\sqrt{N}}{(\log N)^2}\,2^i\,i!\,, \tag{6.22}$$

where we first made the change of variables $e^2Nu = w$, and then $w = e^{2s}$, and denote $C = 8ec_\vartheta$ for short. Then it follows by (6.21) that

$$\int_{\frac{1}{N}}^{\frac{1}{\sqrt{N}}} \hat{G}_\vartheta(u)\, \widehat{\phi}^{(m-1)}(u)\, \mathrm{d}u \leqslant C\, 32^m \underbrace{m\, \frac{\sqrt{N}}{(\log N)^2}}_{\mathsf{A}_{m,N}}.$$

We then consider the contribution from $u \geqslant \frac{1}{\sqrt{N}}$. Since $\hat{G}_\vartheta(u) \leqslant \frac{c_\vartheta}{u}$, we have

$$\int_{\frac{1}{\sqrt{N}}}^{1} \hat{G}_\vartheta(u) \frac{\big(\log(e^2Nu)\big)^i}{\sqrt{u}} \mathrm{d}u \leqslant c_\vartheta\, \big(\log(e^2N)\big)^i \int_{\frac{1}{\sqrt{N}}}^{1} \frac{1}{u^{3/2}}\, \mathrm{d}u = 2c_\vartheta\, N^{\frac{1}{4}}\, \big(\log(e^2N)\big)^i,$$

hence by (6.21)

$$\int_{\frac{1}{\sqrt{N}}}^{1} \hat{G}_\vartheta(u)\, \widehat{\phi}^{(m-1)}(u)\, \mathrm{d}u \leqslant C\, 32^m \underbrace{N^{\frac{1}{4}} \sum_{i=0}^{m-2} \frac{1}{2^i\, i!} \big(\log(e^2N)\big)^i}_{\mathsf{B}_{m,N}}.$$

By (6.13) and (6.20), we finally see that

$$\begin{aligned}\sum_{1\leqslant a\leqslant N} \sum_{z\in\mathbb{Z}^2} M_{0,a}^{N,\mathrm{NT}}(w,z) &\leqslant 3 \sum_{m\geqslant 2} 2^{m-1}\, e^\lambda\, c^m\, \frac{(\log N)^2}{\sqrt{N}}\, (C_{\lambda,N})^{m-1}\, C\, 32^m \big\{\mathsf{A}_{m,N} + \mathsf{B}_{m,N}\big\} \\ &\leqslant C'\, e^\lambda \Bigg\{ \sum_{m\geqslant 2} (64\, c\, C_{\lambda,N})^{m-1}\, m \\ &\qquad + \sum_{m\geqslant 2} (64\, c\, C_{\lambda,N})^{m-1} \Bigg(\frac{(\log N)^2}{N^{\frac{1}{4}}} \sum_{i=0}^{m-2} \frac{1}{2^i\, i!} \big(\log(e^2N)\big)^i \Bigg) \Bigg\},\end{aligned}$$

with $C' := 3\cdot 32c$. If we fix λ large enough, then for large N we have $64\, c\, C_{N,\lambda} < 1$ (recall (6.20)), then the first sum in the RHS is finite, in agreement with our goal (6.3). Concerning the second sum, we can estimate it by

$$\begin{aligned}&\leqslant \frac{(\log N)^2}{N^{\frac{1}{4}}} \sum_{i\geqslant 0} \frac{1}{2^i\, i!} \big(\log(e^2N)\big)^i \sum_{m\geqslant i+2} (64\, c\, C_{\lambda,N})^{m-1} \\ &\leqslant \frac{(\log N)^2}{N^{\frac{1}{4}}} \sum_{i\geqslant 0} \frac{1}{2^i\, i!} \big(\log(e^2N)\big)^i\, \frac{(64\, c\, C_{\lambda,N})^i}{1-64\, c\, C_{\lambda,N}} = \frac{(\log N)^2}{N^{\frac{1}{4}}}\, \frac{(e^2N)^{32\, c\, C_{\lambda,N}}}{1-64\, c\, C_{\lambda,N}}.\end{aligned}$$

If we fix λ large enough, then for large N we have that the exponent is $32\, c\, C_{\lambda,N} < \frac{1}{4}$, hence the last term is $o(1)$ as $N\to\infty$. This completes the proof of (6.3). □

In order to prove Lemma 6.2, we need the following analogue of Lemma 5.5.

Lemma 6.3. *For all $i \in \mathbb{N}_0$ and $v \in \big(\frac{1}{N}, 1\big)$,*

$$\int_{\frac{1}{N}}^{1} \frac{\big(\log(e^2Ns))\big)^i}{s\sqrt{s+v}}\, \mathrm{d}s \leqslant \frac{2^{i+1}}{\sqrt{v}}\, i! \sum_{j=0}^{i+1} \frac{\big(\log(e^2Nv)\big)^j}{2^j j!}. \tag{6.23}$$

Proof. We can bound

$$\int_{\frac{1}{N}}^{1} \frac{\big(\log(e^2Ns))\big)^i}{s\sqrt{s+v}}\,\mathrm{d}s \leqslant \underbrace{\frac{1}{\sqrt{v}}\int_{\frac{1}{N}}^{v} \frac{\big(\log(e^2Ns))\big)^i}{s}\,\mathrm{d}s}_{B} + \underbrace{\int_{v}^{1} \frac{\big(\log(e^2Ns))\big)^i}{s\sqrt{s}}\,\mathrm{d}s}_{A}\,.$$

For B, we make the change of variable $u = \log(e^2Ns)$ to obtain

$$B = \frac{1}{\sqrt{v}}\int_{2}^{\log e^2Nv} u^i \mathrm{d}u \leqslant \frac{(\log e^2Nv)^{i+1}}{\sqrt{v}(i+1)}. \tag{6.24}$$

For A, we make the change of variable $y = \frac{1}{2}\log e^2Ns$ and apply (5.25) to obtain

$$A \leqslant 2^{i+1}e\sqrt{N}\int_{\frac{1}{2}\log e^2Nv}^{\infty} y^i e^{-y}\mathrm{d}y \leqslant 2^{i+1}e\sqrt{N}e^{-\frac{1}{2}\log e^2Nv}\sum_{j=0}^{i}\frac{i!}{j!}\Big(\frac{1}{2}\log e^2Nv\Big)^j. \tag{6.25}$$

Combined with the bound for B, this gives precisely (6.23). □

Proof of Lemma 6.2. We follow the proof of Lemma 5.4. We first show that for all $k \in \mathbb{N}$

$$\widehat{\phi}^{(k)}(v) \leqslant \frac{2^k}{\sqrt{v}}\sum_{i=0}^{k}\frac{\hat{c}_{k,i}}{2^i\, i!}\big(\log(e^2Nv)\big)^i \qquad \forall v \in \Big(\frac{1}{N},1\Big), \tag{6.26}$$

for suitable coefficients $\hat{c}_{k,i}$. For $k = 1$, note that by (6.19)

$$\widehat{\phi}^{(1)}(v) = \int_{\frac{1}{N}}^{1}\frac{1}{\sqrt{s(s+v)}}\frac{1}{\sqrt{s}}\mathrm{d}s \leqslant \int_{\frac{1}{N}}^{v}\frac{1}{\sqrt{v}}\frac{1}{s}\,\mathrm{d}s + \int_{v}^{1}\frac{1}{s\sqrt{s}}\,\mathrm{d}s \leqslant \frac{\log(e^2Nv)}{\sqrt{v}}.$$

Therefore (6.26) holds for $k = 1$ with $\hat{c}_{1,0} = 0$ and $\hat{c}_{1,1} = 1$.

Assume that we have established (6.26) up to $k-1$, then

$$\widehat{\phi}^{(k)}(v) = \int_{\frac{1}{N}}^{1}\frac{1}{\sqrt{s(s+v)}}\,\widehat{\phi}^{(k-1)}(s)\,\mathrm{d}s \leqslant 2^{k-1}\sum_{i=0}^{k-1}\frac{\hat{c}_{k-1,i}}{2^i\, i!}\int_{\frac{1}{N}}^{1}\frac{\big(\log(e^2Ns)\big)^i}{s\sqrt{s+v}}\,\mathrm{d}s. \tag{6.27}$$

Applying Lemma 6.3, we obtain

$$\widehat{\phi}^{(k)}(v) \leqslant 2^{k-1}\sum_{i=0}^{k-1}\frac{\hat{c}_{k-1,i}}{2^i\, i!}\frac{2^{i+1}}{\sqrt{v}}i!\sum_{j=0}^{i+1}\frac{\big(\log(e^2Nv)\big)^j}{2^j j!} = \frac{2^k}{\sqrt{v}}\sum_{j=0}^{k}\Big(\sum_{i=(j-1)^+}^{k-1}\hat{c}_{k-1,i}\Big)\frac{(\log e^2Nv)^j}{2^j j!}\,.$$

This shows that (6.26) holds, provided the coefficients $\hat{c}_{k,i}$ satisfy the recursion

$$\hat{c}_{k,j} = \sum_{i=(j-1)^+}^{k-1}\hat{c}_{k-1,i}, \tag{6.28}$$

which differs from the recursion (5.22) for $c_{k,i}$ by a missing factor of 2. Note that $\hat{c}_{1,1}$ here is also only half of $c_{1,1}$ in (5.19). Therefore we have the identity $\hat{c}_{k,i} = 2^{-k}c_{k,i}$, and Lemma 5.6 gives the bound $\hat{c}_{k,i} \leqslant 16^k$. Substituting this bound into (6.26) then proves Lemma 6.2. □

Proof of (6.4)**.** We start from the analogue of (6.5), with $q_{s,a_i}(w,x_1), q_{s,a_i}(w,x_1)$ replaced by $q^N_{s,a_i}(\phi,x_1), q^N_{s,a_i}(\phi,x_2)$. Applying relation (6.14), we can write

$$\begin{aligned}\sum_{\substack{1\leqslant a\leqslant N\\ z\in\mathbb{Z}^2}} I^{(N,m)}_{0,a}(\phi,z) = \sigma_N^{2(m-1)} & \sum_{\substack{0<a_1\leqslant b_1<a_2<\ldots<a_m<a\leqslant N\\ x_1,y_1,x_2,y_2,\ldots,x_m,z\in\mathbb{Z}^2}} q^N_{0,a_1}(\phi,x_1)^2 q^N_{0,a_2}(\phi,x_2)\,\cdot\\ & \cdot U_N(b_1-a_1,y_1-x_1) q_{b_1,a_2}(y_1,x_2) U_N(b_2-a_2,y_2-x_2)\,\cdot \qquad (6.29)\\ & \cdot \prod_{i=3}^m \Big\{ q_{b_{i-2},a_i}(y_{i-2},x_i)\, q_{b_{i-1},a_i}(y_{i-1},x_i)\, U_N(b_i-a_i,y_i-x_i)\Big\} q_{b_{m-1},a}(y_{m-1},z).\end{aligned}$$

We rename $y_m := z$, $b_m := a$ and bound $q_{b_{m-1},a}(y_{m-1},z) \leqslant (c\sqrt{b_m-b_{m-1}})^{-1}$, as in (6.16). Next we sum over the space variables $y_m, x_m, \ldots$ until y_3, x_3, y_2, as in (5.32)-(5.33), which has the effect of replacing $U_N(b_i-a_i, y_i-x_i)$ by $U_N(b_i-a_i)$ and $q_{b_{i-2},a_i}(y_{i-2},x_i)\, q_{b_{i-1},a_i}(y_{i-1},x_i)$ by $c\,(\sqrt{(a_i-b_{i-1})(a_i-b_{i-2})})^{-1}$. Then we bound $q^N_{0,a_2}(\phi,x_2) \leqslant |\phi|_\infty$, see (2.8), after which the sum over x_2 gives 1, the sum over y_1 gives $U_N(b_1-a_1)$, and the sum over x_1 is bounded by $c\,N$, as in (5.35). This leads to estimate the RHS of (6.29) by

$$c^m\, N\, \sigma_N^{2(m-1)} \sum_{\substack{0<a_1\leqslant b_1<\ldots\\ \ldots<a_m\leqslant b_m<N}} U_N(b_1-a_1)\, U_N(b_2-a_2) \prod_{i=3}^m \frac{U_N(b_i-a_i)}{\sqrt{(a_i-b_{i-1})(a_i-b_{i-2})}}\, \frac{1}{\sqrt{b_m-b_{m-1}}}\,.$$

We now set $u_i := a_i - b_{i-1}$ and $v_i := b_i - a_i$ for $1 \leqslant i \leqslant m$, with $b_0 := 0$, and bound $a_i - b_{i-2} \geqslant u_i + u_{i-1}$, while $b_m - b_{m-1} \geqslant u_m$. Then we insert the factor $e^\lambda \prod_{i=1}^m e^{-\lambda(\frac{v_i}{N})} > 1$, for $\lambda > 0$, and by (5.37) we bound the last display by

$$c^m\, e^\lambda\, N\,(\log N)\,\big(\tfrac{1}{\log N} + C_\lambda\big)^m \left\{ \sum_{0<u_1,\ldots,u_m<N} \prod_{i=3}^m \frac{1}{\sqrt{u_i(u_i+u_{i-1})}}\, \frac{1}{\sqrt{u_m}} \right\}, \qquad (6.30)$$

which is an analogue of (6.17). The exponent of $(\frac{1}{\log N} + C_\lambda)$ equals m, because we have m factors $U_N(b_i - a_i)$, and the extra $\log N$ comes from having $m-1$ powers of σ_N^2.

We now switch to macroscopic variables, replacing u_i by Nu_i, with $u_i \in \frac{1}{N}\mathbb{Z} \cap (0,1)$, and replace the Riemann sum in brackets by the corresponding integrals, where as in (6.18) we restrict the integration on $u_i \geqslant \frac{1}{N}$ (possibly enlarging the value of c). This leads to

$$\begin{aligned}\sum_{\substack{1\leqslant a\leqslant N\\ z\in\mathbb{Z}^2}} I^{(N,m)}_{0,a}(\phi,z) &\leqslant c^m\, e^\lambda\, N\,(\log N)\,\big(\tfrac{1}{\log N} + C_\lambda\big)^m\,\cdot\\ &\quad \cdot N^{\frac32} \left\{ \int\limits_{\frac1N \leqslant u_1,u_2,\ldots,u_m<1} \left(\prod_{i=3}^m \frac{1}{\sqrt{u_i(u_i+u_{i-1})}} \right) \frac{1}{\sqrt{u_m}}\, \mathrm{d}\vec{u} \right\},\end{aligned} \qquad (6.31)$$

where the factor $N^{\frac32}$ arises by matching the normalization factor N^{-m} of the Riemann sum and the term $N^{-(m-2)-\frac12}$ generated by the square roots, when we set $u_i \rightsquigarrow Nu_i$.

Note that the variable u_1 does not appear in the function to be integrated in (6.31), so the integral over u_1 is at most 1. Recalling the definition (6.19) of $\widehat{\phi}^{(k)}$, we have

$$\sum_{\substack{1\leqslant a\leqslant N\\ z\in\mathbb{Z}^2}} I^{(N,m)}_{0,a}(\phi,z) \leqslant c^m\, e^\lambda\, N^{\frac52}\,(\log N)\,\big(\tfrac{1}{\log N} + C_\lambda\big)^m \int_{\frac1N}^1 \phi^{(m-2)}(u_2)\,\mathrm{d}u_2\,.$$

By Lemma 6.2, we have

$$\int_{\frac{1}{N}}^{1} \widehat{\phi}^{(m-2)}(u)\mathrm{d}u \leqslant 32^{m-2} \sum_{i=0}^{m-2} \frac{1}{2^i\, i!} \int_0^1 \frac{(\log(e^2 N u))^i}{\sqrt{u}}\, \mathrm{d}u \leqslant 32^{m-2} \sum_{i=0}^{m-2} \frac{(\log(e^2 N))^i}{2^i i!}\, 2\,.$$

Therefore, if we set $C_{\lambda,N} := C_{\lambda,N} := \frac{1}{\log N} + C_\lambda$ as in (6.20), recalling (6.5) we get

$$\begin{aligned}
\sum_{\substack{1\leqslant a\leqslant N\\ z\in\mathbb{Z}^2}} M_{0,a}^{N,\mathrm{NT}}(\phi,z) &\leqslant 3 \sum_{m\geqslant 2} 2^m \sum_{\substack{1\leqslant a\leqslant N\\ z\in\mathbb{Z}^2}} I_{0,a}^{N,m}(\phi,z)\\
&\leqslant 3\, e^\lambda\, N^{\frac{5}{2}}\, (\log N) \sum_{m\geqslant 2} (64c\, C_{\lambda,N})^m \sum_{i=0}^{m} \frac{(\log(e^2N))^i}{2^i i!}\\
&\leqslant 3\, e^\lambda\, N^{\frac{5}{2}}\, (\log N) \sum_{i\geqslant 0} \frac{(\log(e^2N))^i}{2^i i!} \sum_{m\geqslant i} (64c\, C_{\lambda,N})^m\\
&\leqslant \frac{3\, e^\lambda}{1-64c\, C_{\lambda,N}}\, N^{\frac{5}{2}}\, (\log N) \sum_{i\geqslant 0} \frac{\big(32c\, C_{\lambda,N}\, \log(e^2N)\big)^i}{i!}\\
&= \frac{3\, e^\lambda}{1-64c\, C_{\lambda,N}}\, N^{\frac{5}{2}}\, (\log N)\, (e^2N)^{32c\, C_{\lambda,N}}\,.
\end{aligned}$$

Given $\varepsilon > 0$ we can fix λ large so that $32c\, C_\lambda < \frac{\varepsilon}{2}$. Then we have $C_{\lambda,N} = \frac{1}{\log N} + C_\lambda < \frac{2}{3}\varepsilon$ for large N. This concludes the proof of (6.4). ☐

7. Bounds on triple intersections

In this section, we prove Proposition 4.3. First we derive a representation for $M_{s,t}^{N,\mathrm{T}}(\phi,\psi)$, which denotes the sum in (4.3) restricted to $\boldsymbol{A}\cap\boldsymbol{B}\cap\boldsymbol{C}\neq\varnothing$ (recall (4.4)).

We denote by $\boldsymbol{D} = (D_1,\ldots,D_{|\boldsymbol{D}|}) := \boldsymbol{A}\cap\boldsymbol{B}\cap\boldsymbol{C}$, with $D_i = (d_i, w_i)$, the locations of the triple intersections. If we fix two consecutive triple intersections, say $D_{i-1} = (a,w)$ and $D_i = (b,z)$, the contribution to (4.3) is given by

$$\mathbb{E}\big[\big(Z_{a,b}^{N,\beta_N}(w,z)\big)^3\big] - M_{a,b}^{N,\mathrm{T}}(w,z)\,,$$

where $M_{a,b}^{N,\mathrm{T}}(w,z)$ is defined in (6.1), together with $M_{a,b}^{N,\mathrm{T}}(\phi,z)$ and $M_{a,b}^{N,\mathrm{T}}(w,\psi)$. Then we obtain from (4.3) the following representation for $M_{s,t}^{N,\mathrm{T}}(\phi,\psi)$ (where $\mathbb{E}[\xi^3] := \mathbb{E}[\xi_{n,z}^3]$):

$$\begin{aligned}
M_{s,t}^{N,\mathrm{T}}(\phi,\psi) := \frac{1}{N^3} \sum_{\substack{\boldsymbol{D}\subseteq\{s+1,\ldots,t-1\}\times\mathbb{Z}^2\\ |\boldsymbol{D}|\geqslant 1}} & \mathbb{E}[\xi^3]^{|\boldsymbol{D}|}\, \Big(\mathbb{E}\big[\big(Z_{s,d_1}^{N,\beta_N}(\phi,w_1)\big)^3\big] - M_{s,d_1}^{N,\mathrm{T}}(\phi,w_1)\Big)\,\cdot\\
&\cdot \prod_{i=2}^{|\boldsymbol{D}|} \Big(\mathbb{E}\big[\big(Z_{d_{i-1},d_i}^{N,\beta_N}(w_{i-1},w_i)\big)^3\big] - M_{d_{i-1},d_i}^{N,\mathrm{T}}(w_{i-1},w_i)\Big)\,\cdot\\
&\cdot \Big(\mathbb{E}\big[\big(Z_{d_{|\boldsymbol{D}|},t}^{N,\beta_N}(w_{|\boldsymbol{D}|},\psi)\big)^3\big] - M_{d_{|\boldsymbol{D}|},t}^{N,\mathrm{T}}(w_{|\boldsymbol{D}|},\psi)\Big)\,.
\end{aligned} \tag{7.1}$$

To prove Proposition 4.3 we may assume $t = 1$, by Remark 1.3, and also $\phi \geqslant 0$, $\psi \geqslant 0$ (otherwise just replace ϕ by $|\phi|$ and ψ by $|\psi|$ to obtain upper bounds). If we rename

$(d_1, w_1) = (a, x)$ and $(d_{|\boldsymbol{D}|}, w_{|\boldsymbol{D}|}) = (b, y)$ in (7.1), we get the upper bound

$$\begin{aligned} |M_{0,N}^{N,\mathrm{T}}(\phi,\psi)| \leqslant |\mathbb{E}[\xi^3]| \cdot \underbrace{\frac{1}{N^3} \sum_{\substack{1\leqslant a\leqslant N \\ x\in\mathbb{Z}^2}} \Big(\mathbb{E}\big[\big(Z_{0,a}^{N,\beta_N}(\phi,x)\big)^3\big] - M_{0,a}^{N,\mathrm{T}}(\phi,x)\Big)}_{A_N} \\ \cdot \Big(\sum_{n=0}^{\infty} \varrho_N^n \Big) \cdot \underbrace{\sup_{\substack{1\leqslant b\leqslant N \\ y\in\mathbb{Z}^2}} \Big(\mathbb{E}\big[\big(Z_{b,N}^{N,\beta_N}(y,\psi)\big)^3\big] - M_{b,N}^{N,\mathrm{T}}(y,\psi)\Big)}_{B_N}, \end{aligned} \tag{7.2}$$

where we set

$$\varrho_N := |\mathbb{E}[\xi^3]| \sum_{\substack{1\leqslant a\leqslant N \\ z\in\mathbb{Z}^2}} \Big(\mathbb{E}\big[\big(Z_{0,a}^{N,\beta_N}(0,z)\big)^3\big] - M_{0,a}^{N,\mathrm{T}}(0,z)\Big). \tag{7.3}$$

Note that $\mathbb{E}[\xi^3]$ actually depends on N, and vanishes as $N \to \infty$. Indeed, recalling that $\xi_{n,z} = e^{\beta_N \omega_{n,z} - \lambda(\beta_N)} - 1$ and $\lambda(\beta) = \frac{1}{2}\beta^2 + O(\beta^3)$ as $\beta \to 0$, see (2.10) and (1.1), we have

$$\mathbb{E}[\xi^3] = e^{\lambda(3\beta_N) - 3\lambda(\beta_N)} - 3\, e^{\lambda(2\beta_N) - 2\lambda(\beta_N)} + 2 = O(\beta_N^3) = O\big((\log N)^{-\frac{3}{2}}\big), \tag{7.4}$$

where the last equality holds by (1.14) and (1.10).

Then, to prove Proposition 4.3, by the bound (7.2) it would suffice to show that

$$\limsup_{N\to\infty} A_N \cdot B_N < \infty \qquad \text{and} \qquad \limsup_{N\to\infty} \varrho_N < 1,$$

so that the series $\sum_{n=0}^{\infty} \varrho_N^n = (1-\varrho_N)^{-1}$ is bounded. We are going to prove the following stronger result, which implies the bound $|M_{0,N}^{N,\mathrm{T}}(\phi,\psi)| = o(N^{-1/2+\eta})$, for any fixed $\eta > 0$.

Lemma 7.1. *The following relations hold as $N \to \infty$, for any fixed $\varepsilon > 0$:*

(a) $A_N = o(N^{\varepsilon - 1/2})$;

(b) $B_N = o(N^{\varepsilon})$;

(c) $\varrho_N = O((\log N)^{-1/2})$.

Before the proof, we recall that $\mathbb{E}\big[\big(Z_{a,b}^{N,\beta_N}(*,\dagger) - q_{a,b}^N(*,\dagger)\big)^3\big] = M_{a,b}^{N,\mathrm{T}}(*,\dagger) + M_{a,b}^{N,\mathrm{NT}}(*,\dagger)$, for any $* \in \{w,\phi\}$, $\dagger \in \{z,\psi\}$, hence

$$\begin{aligned} &\mathbb{E}\big[\big(Z_{a,b}^{N,\beta_N}(*,\dagger)\big)^3\big] - M_{a,b}^{N,\mathrm{T}}(*,\dagger) \\ &\quad = q_{a,b}^N(*,\dagger)^3 + 3\, q_{a,b}^N(*,\dagger)\, \mathbb{V}\mathrm{ar}\,\big(Z_{a,b}^{N,\beta_N}(*,\dagger)\big) + M_{a,b}^{N,\mathrm{NT}}(*,\dagger). \end{aligned} \tag{7.5}$$

Also note that $M_{a,b}^{N,\mathrm{NT}}(*,\dagger) \geqslant 0$, see (4.3) and (6.5).

Proof of Lemma 7.1. We first prove point (b). By definition, see (2.7),

$$q_{b,N}^N(y,\psi) = \sum_{z\in\mathbb{Z}^2} q_{N-b}(z-y)\psi\big(\tfrac{z}{\sqrt{N}}\big) \leqslant |\psi|_\infty.$$

If we replace ψ by the constant 1 in the averaged partition function $Z_{b,N}^{N,\beta_N}(y,\psi)$ we obtain the point-to-plane partition function $Z_{N-b}^{\beta_N}(y)$, see (2.4) and (1.4). Then, by (1.32),

$$\mathbb{V}\mathrm{ar}\,\big(Z_{b,N}^{N,\beta_N}(y,\psi)\big) \leqslant \mathbb{E}\big[Z_{b,N}^{N,\beta_N}(y,\psi)^2\big] \leqslant |\psi|_\infty^2\, \mathbb{E}\big[Z_{N-b}^{\beta_N}(y)^2\big] = O(\log N). \tag{7.6}$$

Lastly, by (6.2), we have

$$M_{b,N}^{N,\mathrm{NT}}(y,\psi) = O(N^{\varepsilon}) .$$

It suffices to plug these estimates into (7.5) with $* = y$, $\dagger = \psi$ and point (b) follows.

Next we prove point (a). First note that

$$\begin{aligned}
\frac{1}{N^3} \sum_{\substack{1\leqslant a\leqslant N\\ x\in\mathbb{Z}^2}} q_{0,a}^N(\phi,x)^3 &= \frac{1}{N^3} \sum_{\substack{1\leqslant a\leqslant N\\ x\in\mathbb{Z}^2}} \Big(\sum_{y\in\mathbb{Z}^2} \phi\big(\tfrac{y}{\sqrt{N}}\big) q_a(x-y)\Big)^3 \\
&\leqslant \frac{|\phi|_\infty^2}{N^3} \sum_{\substack{1\leqslant a\leqslant N\\ x\in\mathbb{Z}^2}} \sum_{y\in\mathbb{Z}^2} q_a(x-y)\, \phi\big(\tfrac{y}{\sqrt{N}}\big) = \frac{|\phi|_\infty^2}{N^3} \sum_{1\leqslant a\leqslant N} \sum_{y\in\mathbb{Z}^2} \phi\big(\tfrac{y}{\sqrt{N}}\big) \\
&= \frac{|\phi|_\infty^2}{N} \sum_{y\in\mathbb{Z}^2} \frac{1}{N} \phi\big(\tfrac{y}{\sqrt{N}}\big) = O\Big(\frac{1}{N}\Big),
\end{aligned}$$

where the last sum converges to $\int \phi(x)\mathrm{d}x$ by Riemann sum approximation. Next note that we can bound $\mathbb{V}\mathrm{ar}\big(Z_{0,a}^{N,\beta_N}(\phi,x)\big) \leqslant |\phi|_\infty^2\, \mathbb{E}[Z_a^{\beta_N}(x)^2] = O(\log N)$, arguing as in (7.6), hence

$$\begin{aligned}
\frac{1}{N^3} \sum_{\substack{1\leqslant a\leqslant N\\ x\in\mathbb{Z}^2}} q_{0,a}^N(\phi,x)\, \mathbb{V}\mathrm{ar}\big(Z_{0,a}^{N,\beta_N}(\phi,x)\big) &\leqslant \frac{1}{N^3} \sum_{\substack{1\leqslant a\leqslant N\\ x,y\in\mathbb{Z}^2}} \phi\big(\tfrac{y}{\sqrt{N}}\big) q_a(x-y)\, O(\log N) \\
&= \frac{1}{N}\, O(\log N) \sum_{y\in\mathbb{Z}^2} \frac{1}{N} \phi\big(\tfrac{y}{\sqrt{N}}\big) = O\Big(\frac{\log N}{N}\Big) .
\end{aligned}$$

Lastly, by (6.4), we have

$$\frac{1}{N^3} \sum_{\substack{1\leqslant a\leqslant N\\ x\in\mathbb{Z}^2}} M_{0,a}^{N,\mathrm{NT}}(\phi,x) = O(N^{\varepsilon-\frac{1}{2}}).$$

Plugging these estimates into (7.5) with $* = \phi$ and $\dagger = x$, point (a) follows.

We finally prove point (c). By the local limit theorem (1.8) we have $q_a(x) \leqslant \frac{c}{a}$ for some $c < \infty$, uniformly in $a \in \mathbb{N}$ and $x \in \mathbb{Z}^2$. Therefore, recalling (7.4), we have

$$\mathbb{E}[\xi^3] \sum_{\substack{1\leqslant a\leqslant N\\ z\in\mathbb{Z}^2}} q_{0,a}^N(0,z)^3 \leqslant \mathbb{E}[\xi^3] \sum_{\substack{1\leqslant a\leqslant N\\ z\in\mathbb{Z}^2}} \frac{c^2}{a^2} q_a(z) = \mathbb{E}[\xi^3] \sum_{1\leqslant a\leqslant N} \frac{c^2}{a^2} = O((\log N)^{-3/2}) .$$

Next we bound $\mathbb{V}\mathrm{ar}\big(Z_{0,a}^{\beta_N}(0,z)\big) \leqslant \sigma_N^{-2}\, U_N(a,z)$, see (2.15), and note that

$$\sum_{z\in\mathbb{Z}^2} U_N(a,z) = U_N(a) \leqslant C\, c_\vartheta\, \frac{\log N}{a\, \log(e^2 N/a)} ,$$

by (2.18), (2.20) and (2.25). Bounding $q_a(x) \leqslant \frac{c}{a}$ and $\sigma_N^{-2} = O(\log N)$, see (1.14) and (1.10), we obtain

$$\mathbb{E}[\xi^3] \sum_{\substack{1\leqslant a\leqslant N\\ z\in\mathbb{Z}^2}} q_{0,a}^N(0,z)\, \mathbb{V}\mathrm{ar}\big(Z_{0,a}^{N,\beta_N}(0,z)\big) \leqslant c'\, \mathbb{E}[\xi^3] \sum_{1\leqslant a\leqslant N} \frac{1}{a^2}\, \frac{(\log N)^2}{\log(e^2 N/a)} .$$

For $a \leqslant \sqrt{N}$ we can bound $\log(e^2 N/a) \geqslant \log(e^2\sqrt{N}) \geqslant \frac{1}{2}\log N$, while for $\sqrt{N} < a \leqslant N$ we can simply bound $\log(e^2 N/a) \geqslant \log e^2 = 2$. This shows that the last sum is uniformly bounded, since $\sum_{a\geqslant 1} \frac{2\log N}{a^2} + \sum_{a>\sqrt{N}} \frac{(\log N)^2}{2\,a^2} = O(\log N) + O(\frac{(\log N)^2}{\sqrt{N}})$. We thus obtain

$$\mathbb{E}[\xi^3] \sum_{\substack{1\leqslant a\leqslant N \\ z\in\mathbb{Z}^2}} q_{0,a}^N(0,z)\,\mathbb{V}\mathrm{ar}\left(Z_{0,a}^{N,\beta_N}(0,z)\right) = O\big(\mathbb{E}[\xi^3]\,\log N\big) = O\big((\log N)^{-1/2}\big)\,.$$

Lastly, by (6.3), we also have

$$\mathbb{E}[\xi^3] \sum_{\substack{1\leqslant a\leqslant N \\ z\in\mathbb{Z}^2}} M_{0,a}^{N,\mathrm{NT}}(0,z) = \mathbb{E}[\xi^3]\,O(1) = O\big((\log N)^{-3/2}\big)\,.$$

If we plug the previous bounds into (7.5) with $* = 0$ and $\dagger = z$, point (c) is proved. □

8. Proof for the stochastic heat equation

In this section we prove Theorems 1.7 and 1.9 on the variance and third moment of the solution to the stochastic heat equation.

We first give a useful representation of $u^\varepsilon(t,\phi) := \int_{\mathbb{R}^2} \phi(x) u^\varepsilon(t,x)\,\mathrm{d}x$. By a Feynman-Kac representation and the definition of the Wick exponential (see [CSZ17b] for details), it follows that $u^\varepsilon(t,\phi)$ is equal in distribution to the Wiener chaos expansion

$$\begin{aligned} u^\varepsilon(t,\phi) \overset{d}{=} \int_{\mathbb{R}^2} \phi(x)\,\mathrm{d}x + \sum_{r\geqslant 1} \beta_\varepsilon^r \int_{0<t_1<\dots<t_r<\varepsilon^{-2}t} \int_{(\mathbb{R}^2)^r} \prod_{i=1}^r W(\mathrm{d}t_i\,\mathrm{d}x_i)\,\cdot \\ \cdot \left\{ \int_{\mathbb{R}^2} \mathrm{d}x\ \varepsilon^2 \phi(\varepsilon x) \int_{(\mathbb{R}^2)^r} \prod_{i=1}^r g_{t_i - t_{i-1}}(\hat{x}_{i-1},\hat{x}_i)\, j(\hat{x}_i - x_i)\,\mathrm{d}\hat{x}_i \right\} \end{aligned} \tag{8.1}$$

with the convention that $t_0 := 0$ and $\hat{x}_0 = x$.

Expression (8.1) is the starting point to prove both Theorems 1.7 and 1.9. To analyze this expression, we first need to extend the renewal theory framework, described in Subsections 1.3 and 2.2, to continuum distributions. The key results, described in the next subsection, are analogous to those obtained in the discrete setting, see [CSZ18, Remark 1.7].

8.1. Renewal framework.

Fix a continuous function $r : [0,\infty) \to (0,\infty)$ such that†

$$r(t) = \frac{1}{4\pi t}\big(1 + o(1)\big) \qquad \text{as } t \to \infty\,. \tag{8.2}$$

For $\varepsilon > 0$, we consider i.i.d. random variables $\big(\mathcal{T}_i^\varepsilon\big)_{i\geqslant 1}$ with density

$$\mathrm{P}(\mathcal{T}_i^\varepsilon \in \mathrm{d}t) = \frac{r(t)}{\mathcal{R}_\varepsilon}\,\mathbb{1}_{[0,\varepsilon^{-2}]}(t)\,\mathrm{d}t\,, \tag{8.3}$$

where $\mathcal{R}_\varepsilon := \int_0^{\varepsilon^{-2}} r(t)\,\mathrm{d}t$ is the normalization constant. Note that $\mathcal{T}_1^\varepsilon + \ldots + \mathcal{T}_k^\varepsilon$ is a continuum analogue of $\tau_k^{(N)}$ in (1.43), see (1.41)-(1.42), with the identification $N = \varepsilon^{-2}$.

Let us quote some relevant results from [CSZ18] that will be needed in the sequel.

- By [CSZ18, Proposition 1.3], we have the convergence in distribution

$$\Big(\varepsilon^2\big(\mathcal{T}_1^\varepsilon + \ldots + \mathcal{T}_{\lfloor s\log\varepsilon^{-2}\rfloor}^\varepsilon\big)\Big)_{s\geqslant 0} \xrightarrow[\varepsilon\to 0]{d} (Y_s)_{s\geqslant 0}\,, \tag{8.4}$$

  where $(Y_s)_{s\geqslant 0}$ is the Dickman subordinator, whose marginal density is given by (1.45).

†The precise constant 4π in (8.2) is the one relevant for us, but any other positive constant would do.

- By [CSZ18, Lemma 6.1], the following large deviations bound holds, with $c \in (0,1)$:

$$\mathrm{P}\big(\mathcal{T}_1^\varepsilon + \ldots + \mathcal{T}_{\lfloor s \log \varepsilon^{-2} \rfloor}^\varepsilon \leqslant \varepsilon^{-2}\big) \leqslant e^{s - cs\log s}\,, \qquad \forall \varepsilon \in (0,1),\ \forall s \in [0,\infty)\,. \tag{8.5}$$

Let us now take λ_ε such that

$$\lambda_\varepsilon := 1 + \frac{\vartheta}{\log \varepsilon^{-2}}\big(1 + o(1)\big)\,, \qquad \text{for some } \vartheta \in \mathbb{R}\,. \tag{8.6}$$

Then it follows by Riemann sum approximation (set $r = s\,\log\varepsilon^{-2}$) that for all $T \in [0,1]$

$$\frac{1}{\log \varepsilon^{-2}} \sum_{r \geqslant 1} \lambda_\varepsilon^r \, \mathrm{P}\big(\mathcal{T}_1^\varepsilon + \ldots + \mathcal{T}_r^\varepsilon \leqslant \varepsilon^{-2} T\big) \xrightarrow[\varepsilon \to 0]{} \int_0^\infty e^{\vartheta u}\, \mathrm{P}(Y_u \leqslant T)\, \mathrm{d}u\,. \tag{8.7}$$

This relation will play a crucial role. We now list some approximations that we can make in the left hand side of (8.7), without affecting the convergence.

(1) *We can restrict the sum to $r \leqslant K \log \varepsilon^{-2}$, for large $K > 0$.* Indeed, it is easily seen by (8.5) and (8.6) that the contribution of $r > K \log \varepsilon^{-2}$ to the sum in (8.7) is small, uniformly in ε, for K large.

(2) *We can restrict the probability to the event "there are no consecutive short increments"*, where we say that an increment $\mathcal{T}_i^\varepsilon$ is *short* if and only if $\mathcal{T}_i^\varepsilon \leqslant (\log \varepsilon^{-2})^2$. Indeed, the probability that an increment is short is, by (8.2)-(8.3),

$$p_\varepsilon := \mathrm{P}\big(\mathcal{T}_i^\varepsilon \leqslant (\log \varepsilon^{-2})^2\big) = \frac{\int_0^{(\log \varepsilon^{-2})^2} r(t)\,\mathrm{d}t}{\int_0^{\varepsilon^{-2}} r(t)\,\mathrm{d}t} = O\bigg(\frac{\log(\log \varepsilon^{-2})}{\log \varepsilon^{-2}}\bigg)\,, \tag{8.8}$$

hence the probability of having two consecutive short increments among $\mathcal{T}_1^\varepsilon, \ldots, \mathcal{T}_r^\varepsilon$ is

$$\mathrm{P}\Bigg(\bigcup_{i=1}^{r-1} \big\{\mathcal{T}_i^\varepsilon \leqslant (\log \varepsilon^{-2})^2, \mathcal{T}_{i+1}^\varepsilon \leqslant (\log \varepsilon^{-2})^2\big\}\Bigg) \leqslant r\, p_\varepsilon^2 \leqslant O\bigg(\frac{r\,\big(\log(\log \varepsilon^{-2})\big)^2}{(\log \varepsilon^{-2})^2}\bigg)\,,$$

which vanishes as $\varepsilon \to 0$, when we restrict to $r \leqslant K \log \varepsilon^{-2}$.

(3) *We can further restrict the probability to the event "the first increment $\mathcal{T}_1^\varepsilon$ is long, i.e. not short"*, simply because $p_\varepsilon \to 0$ as $\varepsilon \to 0$, see (8.8).

8.2. Proof of Theorem 1.7. It follows from the expansion (8.1) that $\mathbb{E}[u^\varepsilon(t,\phi)] = \int_{\mathbb{R}^2} \phi(x)\,\mathrm{d}x$ and that the variance of $u^\varepsilon(t,\phi)$ is given by

$$\mathbb{V}\mathrm{ar}\,\big(u^\varepsilon(t,\phi)\big) = \varepsilon^4 \int_{\mathbb{R}^2 \times \mathbb{R}^2} \phi(\varepsilon\hat{x})\,\phi(\varepsilon\tilde{x})\; K_t^\varepsilon(\hat{x},\tilde{x})\;\mathrm{d}\hat{x}\,\mathrm{d}\tilde{x} \tag{8.9}$$

where, using the conventions $\hat{x}_0 = \hat{x}$, $\tilde{x}_0 = \tilde{x}$ and $\vec{t} = (t_1, \ldots, t_r)$, we define

$$\begin{aligned} K_t^\varepsilon(\hat{x},\tilde{x}) := \sum_{r \geqslant 1} \beta_\varepsilon^{2r} \int_{0 < t_1 < \cdots < t_r < \varepsilon^{-2} t} \mathrm{d}\vec{t} \int_{(\mathbb{R}^2)^r} \prod_{i=1}^r \mathrm{d}x_i \int_{(\mathbb{R}^2)^{2r}} \prod_{i=1}^r \mathrm{d}\hat{x}_i\,\mathrm{d}\tilde{x}_i \\ \cdot \prod_{i=1}^r g_{t_i - t_{i-1}}(\hat{x}_{i-1},\hat{x}_i)\, g_{t_i - t_{i-1}}(\tilde{x}_{i-1},\tilde{x}_i)\, j(\hat{x}_i - x_i)\, j(\tilde{x}_i - x_i) \\ = \sum_{r \geqslant 1} \beta_\varepsilon^{2r} \int_{0 < t_1 < \cdots < t_r < \varepsilon^{-2} t} \mathrm{d}\vec{t} \int_{(\mathbb{R}^2)^{2r}} \prod_{i=1}^r \mathrm{d}\hat{x}_i\,\mathrm{d}\tilde{x}_i \\ \cdot \prod_{i=1}^r g_{t_i - t_{i-1}}(\hat{x}_{i-1},\hat{x}_i)\, g_{t_i - t_{i-1}}(\tilde{x}_{i-1},\tilde{x}_i)\, J(\hat{x}_i - \tilde{x}_i)\,, \end{aligned} \tag{8.10}$$

where the second equality holds because $j(-x) = j(x)$ and we recall that $J = j * j$.

We now exploit the identity

$$g_t(x)\, g_t(y) = 4\, g_{2t}(x-y)\, g_{2t}(x+y)\,. \tag{8.11}$$

If we set $\hat{x}_i - \tilde{x}_i =: z_i$ and $\hat{x}_i + \tilde{x}_i = w_i$ and take into account that the Jacobian of the transformation $(x,y) \mapsto (x-y, x+y)$ on $(\mathbb{R}^2)^2$ equals 1/4, we obtain, with $z_0 = \hat{x} - \tilde{x}$,

$$\begin{aligned} K_t^\varepsilon(\hat{x}, \tilde{x}) &= \sum_{r\geqslant 1} \beta_\varepsilon^{2r} \int_{0<t_1<\cdots<t_r<\varepsilon^{-2}t} \mathrm{d}\vec{t} \int_{(\mathbb{R}^2)^{2r}} \mathrm{d}\vec{z}\, \mathrm{d}\vec{w} \\ &\qquad \prod_{i=1}^r g_{2(t_i-t_{i-1})}(w_i - w_{i-1})\, g_{2(t_i-t_{i-1})}(z_i - z_{i-1})\, J(z_i) \\ &= \sum_{r\geqslant 1} \beta_\varepsilon^{2r} \int_{0<t_1<\cdots<t_r<\varepsilon^{-2}t} \mathrm{d}\vec{t} \int_{(\mathbb{R}^2)^r} \mathrm{d}\vec{z} \prod_{i=1}^r g_{2(t_i-t_{i-1})}(z_i - z_{i-1})\, J(z_i)\,. \end{aligned} \tag{8.12}$$

Note that variables z_i with $i \geqslant 1$ lie in $\mathrm{supp}(J)$, which is a compact subset of $\mathbb{R}^2$, while $z_0 = \hat{x} - \tilde{x}$ is of order ε^{-1}, in view of (8.9). For this reason, it is convenient to isolate the integrals over t_1, z_1 and change variable $t_1 \to \varepsilon^{-2} t_1$. Observing that $g_{\varepsilon^{-2}t}(x) = \varepsilon^2 g_t(\varepsilon x)$, and renaming (t_1, z_1) as (s, z), we obtain

$$K_t^\varepsilon(\hat{x}, \tilde{x}) = \int_0^t \mathrm{d}s \int_{\mathbb{R}^2} \mathrm{d}z\, g_{2s}\big(\varepsilon(z - (\hat{x} - \tilde{x}))\big)\, J(z)\, \boldsymbol{K}_{t-s}^\varepsilon(z)\,, \tag{8.13}$$

where we define the new kernel $\boldsymbol{K}_T^\varepsilon(z)$ as follows:

$$\boldsymbol{K}_T^\varepsilon(z) := \sum_{r\geqslant 0} \beta_\varepsilon^{2(r+1)} \int_{0<t_1<\ldots<t_r<\varepsilon^{-2}T} \mathrm{d}\vec{t} \int_{(\mathbb{R}^2)^r} \mathrm{d}\vec{z} \prod_{i=1}^r g_{2(t_i-t_{i-1})}(z_i - z_{i-1})\, J(z_i)\,, \tag{8.14}$$

where $z_0 := z$ and we agree that for $r = 0$ the integrals equal 1.

This key expression will be analyzed using renewal theory. Note that by (1.9)

$$g_{2t}(y-x) = \frac{1}{4\pi t} + O\bigg(\frac{1}{t^2}\bigg) \qquad \text{as } t \to \infty\,, \qquad \text{uniformly in } x, y \in \mathrm{supp}(J)\,, \tag{8.15}$$

so the dependence on the space variables z_i in (8.14) should decouple. We can make this precise using the approximations described in Subsection 8.1. We proceed in three steps.

Step 1: First approximation. Note that β_ε, see (1.35), may be rewritten as follows:

$$\beta_\varepsilon^2 = \frac{4\pi}{\log \varepsilon^{-2}} + \frac{4\varrho + o(1)}{(\log \varepsilon^{-2})^2}\,. \tag{8.16}$$

We first obtain a domination of $\boldsymbol{K}_T^\varepsilon(z)$ by a renewal quantity. Let us define

$$\bar{r}(t) := \sup_{z' \in \mathrm{supp}(J)} \int_{\mathbb{R}^2} g_{2t}(z - z')\, J(z)\, \mathrm{d}z\,. \tag{8.17}$$

Note that $\bar{r}(t) = \frac{1}{4\pi t} + O(\frac{1}{t^2})$ as $t \to \infty$, thanks to (8.15), hence

$$\bar{\mathcal{R}}_\varepsilon = \int_0^{\varepsilon^{-2}} \bar{r}(t)\, \mathrm{d}t = \frac{1}{4\pi} \log \varepsilon^{-2} + O(1) \qquad \text{as } \varepsilon \to 0\,. \tag{8.18}$$

If we denote by $(\bar{\mathcal{T}}_i^\varepsilon)_{i\in\mathbb{N}}$ i.i.d. random variables defined as in (8.3), more precisely

$$\mathrm{P}(\bar{\mathcal{T}}_i^\varepsilon \in \mathrm{d}t) = \frac{\bar{r}(t)}{\bar{\mathcal{R}}_\varepsilon}\, \mathbb{1}_{[0,\varepsilon^{-2}]}(t)\, \mathrm{d}t\,, \tag{8.19}$$

we can bound $\boldsymbol{K}_T^\varepsilon(z)$ from above for $T \leqslant 1$, uniformly in $z \in \operatorname{supp}(J)$, as follows:

$$\begin{aligned}\sup_{z\in\operatorname{supp}(J)} \boldsymbol{K}_T^\varepsilon(z) &\leqslant \beta_\varepsilon^2 \sum_{r\geqslant 0} \beta_\varepsilon^{2r} \int_{0<t_1<\ldots<t_r<\varepsilon^{-2}T} \prod_{i=1}^r \bar{r}(t_i - t_{i-1})\,\mathrm{d}\vec{t} \\ &= \beta_\varepsilon^2 \sum_{r\geqslant 0} (\beta_\varepsilon^2\,\bar{\mathcal{R}}_\varepsilon)^r\,\mathrm{P}(\bar{\mathcal{T}}_1^\varepsilon + \ldots + \bar{\mathcal{T}}_r^\varepsilon < \varepsilon^{-2}T) \\ &\leqslant \tfrac{c_1}{\log\varepsilon^{-2}} \sum_{r\geqslant 0} \big(1 + \tfrac{c_2}{\log\varepsilon^{-2}}\big)^r\,\mathrm{P}(\bar{\mathcal{T}}_1^\varepsilon + \ldots + \bar{\mathcal{T}}_r^\varepsilon < \varepsilon^{-2}T)\,,\end{aligned} \tag{8.20}$$

where the last inequality holds by (8.16) and (8.18), for suitable $c_1, c_2 \in (0,\infty)$.

The last line of (8.20) is comparable to the left hand side of (8.7), so we can apply the approximations (1)-(3) described in Subsection 8.1. In terms of $\boldsymbol{K}_T^\varepsilon(z)$, see (8.14), these approximations correspond to restricting the sum to $r \leqslant K\log\varepsilon^{-2}$ for a large constant $K>0$, by (1), and to restricting the integral over $\vec{t}$ to the following set, by (2)-(3):

$$\begin{aligned}\mathfrak{I}_T^\varepsilon := \big\{0 < t_1 < \cdots < t_r < \varepsilon^{-2}\,T : \quad & t_1 > (\log\varepsilon^{-1})^2 \ \text{ and, for every } 1 \leqslant i \leqslant r-1, \\ & \text{either } \ t_i - t_{i-1} > (\log\varepsilon^{-2})^2 \ \text{ or } \ t_{i+1} - t_i > (\log\varepsilon^{-2})^2 \,\big\}\,.\end{aligned} \tag{8.21}$$

Summarizing, when we send $\varepsilon \to 0$ followed by $K \to \infty$, we can write

$$\boldsymbol{K}_T^\varepsilon(z) = \tilde{\boldsymbol{K}}_{T,K}^\varepsilon(z) + o(1) \qquad \text{uniformly for } z \in \operatorname{supp}(J)\,, \tag{8.22}$$

where we define, with $t_0 := 0$ and $z_0 := z$,

$$\tilde{\boldsymbol{K}}_{T,K}^\varepsilon(z) := \sum_{r=0}^{K\log\varepsilon^{-2}} \beta_\varepsilon^{2(r+1)} \int_{\mathfrak{I}_T^\varepsilon} \mathrm{d}\vec{t} \int_{(\mathbb{R}^2)^r} \mathrm{d}\vec{z} \prod_{i=1}^r g_{2(t_i-t_{i-1})}(z_i - z_{i-1})\,J(z_i)\,. \tag{8.23}$$

Step 2: Second approximation. Given $r \in \mathbb{N}$, let us denote by S_ε and L_ε the subsets of indexes $i \in \{1,\ldots,r\}$ corresponding to short and long increments:

$$\begin{aligned}\mathsf{S}_\varepsilon &:= \{i \in \{1,\ldots,r\} :\ t_i - t_{i-1} \leqslant (\log\varepsilon^{-2})^2\}\,, \\ \mathsf{L}_\varepsilon &:= \{i \in \{1,\ldots,r\} :\ t_i - t_{i-1} > (\log\varepsilon^{-2})^2\}\,.\end{aligned}$$

We can then decompose the last product in (8.23) as follows:

$$\prod_{i=1}^r g_{2(t_i-t_{i-1})}(z_i - z_{i-1})\,J(z_i) = \prod_{i\in\mathsf{S}_\varepsilon} g_{2(t_i-t_{i-1})}(z_i - z_{i-1}) \prod_{i\in\mathsf{L}_\varepsilon} g_{2(t_i-t_{i-1})}(z_i - z_{i-1}) \prod_{i=1}^r J(z_i)\,.$$

We now make replacements and integrations, in order to simplify this expression.

For each $i \in \mathsf{L}_\varepsilon$ we replace $g_{2(t_i-t_{i-1})}(z_i - z_{i-1})$ by $r(t_i - t_{i-1})$, where we set

$$r(t) := \langle J, g_{2t}\,J\rangle := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} J(x) g_{2t}(x-y) J(y)\,\mathrm{d}x\,\mathrm{d}y\,. \tag{8.24}$$

The error from each such replacement is $\exp\big\{O\big((\log\varepsilon^{-1})^{-2}\big)\big\}$, since one easily sees that $g_{2(t_i-t_{i-1})}(z_i - z_{i-1}) = g_{2(t_i-t_{i-1})}(x-y)\exp\big\{O\big((t_i - t_{i-1})^{-1}\big)\big\}$ and we have $t_i - t_{i-1} > (\log\varepsilon^{-2})^2$ (recall that $x-y$ and $z_{i+1} - z_i$ are in the support of J, which is compact). Since we are restricted to $r \leqslant K\log\varepsilon^{-2}$, see (8.23), we have $|\mathsf{L}_\varepsilon| \leqslant K\log\varepsilon^{-2}$, hence the total error from all these replacements is $\exp\big\{O\big((\log\varepsilon^{-1})^{-1}\big)\big\} = (1+o(1))$. We have shown that

$$\prod_{i=1}^r g_{2(t_i-t_{i-1})}(z_i - z_{i-1})\,J(z_i) = \big(1+o(1)\big) \prod_{i\in\mathsf{S}_\varepsilon} g_{2(t_i-t_{i-1})}(z_i - z_{i-1}) \prod_{i\in\mathsf{L}_\varepsilon} r(t_i - t_{i-1}) \prod_{i=1}^r J(z_i)\,.$$

We now proceed by integrating successively $\mathrm{d}z_i$ for $i = r, r-1, \ldots, 1$ as follows:

- for $i \in \mathsf{L}_\varepsilon$ the integral over $\mathrm{d}z_i$ amounts to $\int_{\mathbb{R}^2} J(z_i)\,\mathrm{d}z_i = 1$;
- for $i \in \mathsf{S}_\varepsilon$ we integrate both $\mathrm{d}z_{i-1}$ and $\mathrm{d}z_i$ which gives, see (8.24),

$$\int_{(\mathbb{R}^2)^2} J(z_i)\, g_{2(t_i-t_{i-1})}(z_i - z_{i-1})\, J(z_i)\,\mathrm{d}z_{i-1}\,\mathrm{d}z_i = \langle J, g_{2(t_i-t_{i-1})} J\rangle = r(t_i - t_{i-1})\,,$$

This sequence of integrations is consistent, i.e. it does not result to integrating a variable $\mathrm{d}z_i$ twice, because on the set $\mathfrak{I}_T^\varepsilon$, see (8.23) and (8.21), there are not two consecutive indices i in S_ε. Therefore, uniformly for $r \leqslant K \log \varepsilon^{-2}$, we have shown that

$$\int_{(\mathbb{R}^2)^r} \mathrm{d}\vec{z} \prod_{i=1}^{r} g_{2(t_i-t_{i-1})}(z_i - z_{i-1})\, J(z_i) = \big(1 + o(1)\big) \prod_{i=1}^{r} r(t_i - t_{i-1})\,. \tag{8.25}$$

Note that $r(t) = \frac{1}{4\pi t} + O(\frac{1}{t^2})$, by (8.24) and (8.15), so we can consider i.i.d. random variables $\mathcal{T}_i^\varepsilon$ with law (8.3). When we plug (8.25) into (8.23), the approximations (1)-(3) described in Subsection 8.1 show that we can remove the restrictions $r \leqslant K \log \varepsilon^{-2}$ on the sum and $\vec{t} \in \mathfrak{I}_T^\varepsilon$ on the integral. Recalling (8.22), we have finally shown that as $\varepsilon \to 0$

$$\boldsymbol{K}_T^\varepsilon(z) = \big(1 + o(1)\big)\, \hat{\boldsymbol{K}}_T^\varepsilon + o(1) \qquad \text{uniformly for } z \in \operatorname{supp}(J)\,, \tag{8.26}$$

where, recalling (8.16), we define

$$\begin{aligned} \hat{\boldsymbol{K}}_T^\varepsilon &:= \beta_\varepsilon^2 \sum_{r \geqslant 0} \beta_\varepsilon^{2r} \int_{0<t_1<\ldots<t_r<\varepsilon^{-2}T} \prod_{i=1}^{r} r(t_i - t_{i-1})\,\mathrm{d}\vec{t} \\ &= \big(4\pi + o(1)\big) \frac{1}{\log \varepsilon^{-2}} \sum_{r \geqslant 0} (\beta_\varepsilon^2\, \mathcal{R}_\varepsilon)^r\, \mathrm{P}(\mathcal{T}_1^\varepsilon + \ldots + \mathcal{T}_r^\varepsilon < \varepsilon^{-2} T)\,. \end{aligned} \tag{8.27}$$

Step 3: Variance computation. We can finally complete the proof of Theorem 1.7, by proving relation (1.37). Assume that we have shown that, for some $\vartheta \in \mathbb{R}$,

$$\beta_\varepsilon^2\, \mathcal{R}_\varepsilon = 1 + \frac{\vartheta}{\log \varepsilon^{-2}} \big(1 + o(1)\big)\,. \tag{8.28}$$

Then, by (8.7) and (8.27), we can write

$$\lim_{\varepsilon \to 0} \hat{\boldsymbol{K}}_T^\varepsilon = 4\pi \int_0^\infty e^{\vartheta u}\, \mathrm{P}(Y_u \leqslant T)\,\mathrm{d}u\,, \tag{8.29}$$

and the convergence is uniform in $T \in [0,1]$ (because both sides are increasing and the right hand side is continuous in T). Looking back at (8.9), (8.13) and (8.26), after the change of variables $\hat{x}, \tilde{x} \to \varepsilon^{-1}\hat{x}, \varepsilon^{-1}\tilde{x}$, we obtain

$$\begin{aligned} \mathbb{V}\mathrm{ar}\,\big(u^\varepsilon(t,\phi)\big) = \big(1 + o(1)\big) &\int_{\mathbb{R}^2 \times \mathbb{R}^2} \mathrm{d}\hat{x}\,\mathrm{d}\tilde{x}\; \phi(\hat{x})\,\phi(\tilde{x}) \\ &\cdot \int_0^t \mathrm{d}s \int_{\mathbb{R}^2} \mathrm{d}z\, g_{2s}\big(\varepsilon z - (\hat{x} - \tilde{x})\big)\, J(z)\, \hat{\boldsymbol{K}}_{t-s}^\varepsilon + o(1)\,. \end{aligned}$$

Recalling (8.29), since $\int_{\mathbb{R}^2} J(z)\,\mathrm{d}z = 1$, we have shown that

$$\lim_{\varepsilon \to 0} \mathbb{V}\mathrm{ar}\,\big(u^\varepsilon(t,\phi)\big) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} \mathrm{d}\hat{x}\,\mathrm{d}\tilde{x}\; \phi(\hat{x})\,\phi(\tilde{x})\, Q(\hat{x} - \tilde{x})\,,$$

where

$$Q(x) := 4\pi \int_0^t \mathrm{d}s\, g_{2s}(x) \int_0^\infty e^{\vartheta u}\, \mathrm{P}(Y_u \leqslant t - s)\,\mathrm{d}u\,.$$

Recalling that $f_s(\cdot)$ denotes the density of Y_s, see (1.45), and using the definition (1.18) of $G_\vartheta(\cdot)$, we can rewrite $Q(x)$ as follows:

$$\begin{aligned} Q(x) &= 4\pi \int_0^t \mathrm{d}s\, g_{2s}(x) \int_0^\infty e^{\vartheta u} \left(\int_0^{t-s} f_u(r)\,\mathrm{d}r \right) \mathrm{d}u \\ &= 4\pi \int_0^t \mathrm{d}s\, g_{2s}(x) \int_0^{t-s} G_\vartheta(r)\,\mathrm{d}r = 4\pi \int_{0<s<v<t} g_{2s}(x)\, G_\vartheta(v-s)\,\mathrm{d}s\,\mathrm{d}v \\ &= 2\pi \int_{0<s<v<t} g_s(x/\sqrt{2})\, G_\vartheta(v-s)\,\mathrm{d}s\,\mathrm{d}v\,. \end{aligned}$$

A look at (1.20) shows that $Q(x) = 2K_{t,\vartheta}(x/\sqrt{2})$, hence relation (1.37) is proved.

It only remains to prove (8.28) and to identify ϑ. Note that by (1.9)

$$\int_0^{\varepsilon^{-2}} g_{2t}(x-y)\,\mathrm{d}t = \frac{1}{4\pi} \int_0^{\varepsilon^{-2}} \frac{e^{-\frac{|x-y|^2}{4t}}}{t}\,\mathrm{d}t = \frac{1}{4\pi} \int_{\frac{\varepsilon^2|x-y|^2}{4}}^\infty \frac{e^{-u}}{u}\,\mathrm{d}u$$

Using the following representation of the Euler-Mascheroni constant:

$$\int_0^\infty \left(\frac{1}{t(t+1)} - \frac{e^{-t}}{t} \right) \mathrm{d}t = \gamma\,,$$

see [GR07, Entry 8.367 (9), page 906], and since

$$\int_a^\infty \frac{1}{t(t+1)}\,\mathrm{d}t = \int_a^\infty \left(\frac{1}{t} - \frac{1}{t+1} \right) \mathrm{d}t = \log(1+a^{-1})\,,$$

we see that as $\varepsilon \to 0$,

$$\int_0^{\varepsilon^{-2}} g_{2t}(x-y)\,\mathrm{d}t = \frac{1}{4\pi} \left\{ \log\left(1 + \frac{4}{\varepsilon^2|x-y|^2}\right) - \gamma + o(1) \right\}.$$

Recalling the definition (8.24) of $r(t)$, we have

$$\begin{aligned} \mathcal{R}_\varepsilon &:= \int_0^{\varepsilon^{-2}} r(t)\,\mathrm{d}t = \int_{(\mathbb{R}^2)^2} J(x)J(y) \int_0^{\varepsilon^{-2}} g_{2t}(x-y)\,\mathrm{d}t\ \mathrm{d}x\mathrm{d}y \\ &= \frac{1}{4\pi} \left\{ \log\varepsilon^{-2} + \log 4 + 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} J(x)\log\frac{1}{|x-y|}\,J(y)\,\mathrm{d}x\,\mathrm{d}y - \gamma + o(1) \right\}. \end{aligned}$$

Finally, recalling (8.16), we obtain

$$\beta_\varepsilon^2\,\mathcal{R}_\varepsilon = 1 + \frac{\log 4 + 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} J(x)\log\frac{1}{|x-y|}\,J(y)\,\mathrm{d}x\,\mathrm{d}y - \gamma + \varrho/\pi}{\log\varepsilon^{-2}}\big(1+o(1)\big)\,.$$

This shows that (8.28) holds, with the expression in (1.38) for ϑ. □

8.3. Proof of Theorem 1.9. We use the expansion (8.1) to evaluate

$$\mathrm{E}\left[\left(u^\varepsilon(t,\phi) - \int_{\mathbb{R}^2} \phi(x)\,\mathrm{d}x \right)^3 \right]. \tag{8.30}$$

We are going to expand the third power and compute the expectation, which amounts to "pairwise matchings" of the instances of the noise $W(\mathrm{d}t_i\,\mathrm{d}x_i)$ (note that "triple matchings" are automatically ruled out, because Gaussians have vanishing third moment). This will lead to an expression, see (8.39) below, which is similar to the one we found for the directed polymer, cf. (5.2), with some additional complications due to the continuous setting.

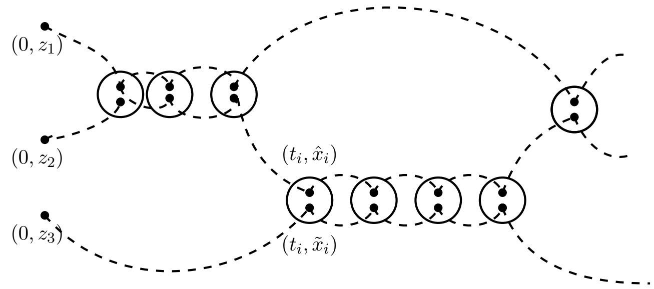

FIGURE 2. Diagramatic representation of the expansion (8.39) of the third moment of the solution of SHE. Due to the space-mollification of the noise, we have non trivial correlations between space-time points $(t_i, \hat{x}_i)$ and $(t_i, \tilde{x}_i)$ — which intuitively belong to two copies of the continuum polymer path, i.e. Brownian motion — only when $\hat{x}_i - \tilde{x}_i$ is in the support of $J(\cdot)$. This is slightly different from the lattice case, cf. the corresponding expansion (5.2) for the directed polymer, where non trivial correlations occur only if $\hat{x}_i = \tilde{x}_i$, i.e. two copies of the polymer exactly meet. The disks represent the support of $J(\cdot)$ and should be understood as disks in space $\mathbb{R}^2$ (we drew them in space-time for graphical clarity). An array of consecutive disks represents the quantity $\mathcal{U}_\varepsilon(s, t; \boldsymbol{x}, \boldsymbol{y})$ in (8.31), with $(s, \boldsymbol{x})$ and $(t, \boldsymbol{y})$ corresponding to space time location of the points inside the first and the last disk in a sequence. They are the analogues of the wiggled lines in Figure 1.

Before entering the technical details, let us give the heuristic picture, which is represented in Figure 2. When taking the third power of the expansion (8.1), we have three sets of coordinates, that we may label a, b, c, that have to match in pairs. Each matching can be of three types ab, bc, ac, and we say that consecutive matching of the same type form a *stretch*. The contribution of each stretch is encoded by a quantity $\mathcal{U}_\varepsilon(s, t; \boldsymbol{x}, \boldsymbol{y})$.

The rest of the proof is divided in two steps.

- In the first step, we define the single-stretch quantity $\mathcal{U}_\varepsilon(s, t; \boldsymbol{x}, \boldsymbol{y})$ and we provide some key estimates on it, based on local renewal theorems obtained in [CSZ18].
- In the second step, we express the centered third moment (8.30) as a sum over the contributions of stretches, see (8.39). We then derive the asymptotic behavior of this expression and show that it is bounded, completing the proof of Theorem 1.9.

Step 1: Single stretch. We introduce a quantity $\mathcal{U}_\varepsilon(s, t; \boldsymbol{x}, \boldsymbol{y})$ which is an analogue of $U_N(t-s, y-x)$ in the discrete setting, see (2.15), linked to the point-to-point variance. Due to the presence of the mollifier, the space variables are couples $\boldsymbol{x} = (\hat{x}, \tilde{x}), \boldsymbol{y} = (\hat{y}, \tilde{y}) \in (\mathbb{R}^2)^2$.

Here is the definition:

$$\begin{aligned}\mathcal{U}_\varepsilon(s,t;\boldsymbol{x},\boldsymbol{y}) := {}& \beta_\varepsilon^2\, g_{t-s}(\hat{x},\hat{y})\, g_{t-s}(\tilde{x},\tilde{y})\, J(\hat{y}-\tilde{y}) \\ & + \beta_\varepsilon^2 \sum_{r\geqslant 1} \beta_\varepsilon^{2r} \int\limits_{s<t_1<\cdots<t_r<t} \prod_{i=1}^r \mathrm{d}t_i \int\limits_{(\mathbb{R}^2)^{2r}} \prod_{i=1}^r \mathrm{d}\hat{z}_i\, \mathrm{d}\tilde{z}_i \\ & \qquad \cdot g_{t_1-s}(\hat{x},\hat{z}_1)\, g_{t_1-s}(\tilde{x},\tilde{z}_1)\, J(\hat{z}_1-\tilde{z}_1)\,\cdot \\ & \qquad \cdot \prod_{i=2}^r g_{t_i-t_{i-1}}(\hat{z}_{i-1},\hat{z}_i)\, g_{t_i-t_{i-1}}(\tilde{z}_{i-1},\tilde{z}_i)\, J(\hat{z}_i-\tilde{z}_i)\,\cdot \\ & \qquad \cdot\, g_{t-t_r}(\hat{z}_r,\hat{y})\, g_{t-t_r}(\tilde{z}_r,\tilde{y})\, J(\hat{y}-\tilde{y})\,,\end{aligned} \tag{8.31}$$

where we recall that $J = j * j$ and we agree that the product equals 1 for $r = 1$.

Let us now evaluate $\int_{(\mathbb{R}^2)^2} \mathcal{U}_\varepsilon(s,t;\boldsymbol{x},\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}$. We use the identity (8.11) and make the change of variables $w_i := \hat{z}_i + \tilde{z}_i$, $z_i := \hat{z}_i - \tilde{z}_i$ for $i = 1, \ldots, r$, as well as $w_{r+1} := \hat{y} + \tilde{y}$, $z_{r+1} := \hat{y} - \tilde{y}$. Integrating out all the w_i's for $i = r+1, r, \ldots, 1$, as we did in (8.10)-(8.12), we obtain

$$\begin{aligned}\int_{(\mathbb{R}^2)^2} \mathcal{U}_\varepsilon(s,t;\boldsymbol{x},\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} = {}& \beta_\varepsilon^2 \sum_{r\geqslant 0} \beta_\varepsilon^{2r} \int\limits_{s<t_1<\cdots<t_r<t} \prod_{i=1}^r \mathrm{d}t_i \int\limits_{(\mathbb{R}^2)^{r+1}} \prod_{i=1}^{r+1} \mathrm{d}z_i \\ & \qquad \cdot g_{2(t_1-s)}\big(z_1 - (\hat{x}-\tilde{x})\big)\, J(z_1)\,\cdot \\ & \qquad \cdot \prod_{i=2}^{r+1} g_{2(t_i-t_{i-1})}(z_i - z_{i-1})\, J(z_i)\,,\end{aligned} \tag{8.32}$$

where we set $t_{r+1} := t$. We can rewrite this relation more compactly as follows:

$$\int_{(\mathbb{R}^2)^2} \mathcal{U}_\varepsilon(s,t;\boldsymbol{x},\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} = \boldsymbol{U}_\varepsilon(t-s;\hat{x}-\tilde{x})\,, \tag{8.33}$$

where we set, with $t_0 := 0$ and $z_0 := z$,

$$\boldsymbol{U}_\varepsilon(t;z) := \sum_{r\geqslant 0} \beta_\varepsilon^{2(r+1)} \int\limits_{0<t_1<\cdots<t_r<t} \mathrm{d}\vec{t} \int\limits_{(\mathbb{R}^2)^{r+1}} \mathrm{d}\vec{z} \prod_{i=1}^{r+1} g_{2(t_i-t_{i-1})}(z_i - z_{i-1})\, J(z_i)\,. \tag{8.34}$$

Note that $\boldsymbol{U}_\varepsilon(t;z)$ looks similar to $\boldsymbol{K}_t^\varepsilon(z)$, see (8.14), with an important difference: the product in (8.34) includes one more term $i = r+1$. This extra term makes $\boldsymbol{U}_\varepsilon(t;z)$ close to a *local renewal function*, as we now explain.

Since we content ourselves with an upper bound, recalling the definition (8.17) of $\bar{r}(t)$, we can estimate

$$\sup_{z\in\mathrm{supp}(J)} \boldsymbol{U}_\varepsilon(t;z) \leqslant \sum_{r\geqslant 0} \beta_\varepsilon^{2(r+1)} \int\limits_{0<t_1<\cdots<t_r<t} \mathrm{d}\vec{t} \prod_{i=1}^{r+1} \bar{r}(t_i - t_{i-1})\,. \tag{8.35}$$

Let us introduce i.i.d. random variables $(\bar{\mathcal{T}}_i^\varepsilon)_{i\in\mathbb{N}}$ as in (8.19), and denote by $\bar{f}_k^\varepsilon(t)$ the density of the associated random walk:

$$\bar{f}_k^\varepsilon(t) := \frac{\mathrm{P}(\bar{\mathcal{T}}_1^\varepsilon + \ldots + \bar{\mathcal{T}}_k^\varepsilon \in \mathrm{d}t)}{\mathrm{d}t}\,.$$

We can then rewrite (8.35) as follows:

$$\sup_{z\in\mathrm{supp}(J)} \boldsymbol{U}_\varepsilon(t;z) \leqslant \sum_{r\geqslant 0} \lambda_\varepsilon^{r+1}\, \bar{f}_{r+1}^\varepsilon(t)\,, \qquad \text{where} \qquad \lambda_\varepsilon := \beta_\varepsilon^2\, \bar{\mathcal{R}}_\varepsilon\,. \tag{8.36}$$

The right hand side can be viewed as a (weighted) *renewal function*: it is the continuum version of the quantity $U_N(n)$ in (2.18) (with the usual identification $N = \varepsilon^{-2}$). We already remarked that $\lambda_\varepsilon = 1 + O(\frac{1}{\log \varepsilon^{-2}})$, by (8.16) and (8.18). Proposition 2.1 holds in this continuum setting [CSZ18, Remark 1.7], hence by the analogue of relation (2.21) we get

$$\sup_{z \in \mathrm{supp}(J)} \boldsymbol{U}_\varepsilon(t; z) \leqslant C \, \frac{\log \varepsilon^{-2}}{\varepsilon^{-2}} \, G_\vartheta(\varepsilon^2 t) \,. \tag{8.37}$$

In conclusion, by (8.33), we have proved the crucial upper bound

$$\sup_{\boldsymbol{x} \in (\mathbb{R}^2)^2 \colon \hat{x} - \tilde{x} \in \mathrm{supp}(J)} \int_{(\mathbb{R}^2)^2} \mathcal{U}_\varepsilon(s, t; \boldsymbol{x}, \boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \;\leqslant\; C \, \varepsilon^2 \, \log \varepsilon^{-2} \, G_\vartheta(\varepsilon^2 (t - s)) \,. \tag{8.38}$$

Step 2: Third moment computation. We expand the third power in (8.30) using the Wiener chaos representation (8.1). We then compute the expectation, which forces pairwise matchings of the instances of the noise $W(\mathrm{d}t_i \, \mathrm{d}x_i)$. Since

$$\int_{\mathbb{R}^2} j(\hat{x}_i - x_i) \, j(\tilde{x}_i - x_i) \, \mathrm{d}x_i = J(\hat{x}_i - \tilde{x}_i) \,,$$

we obtain the following expression (analogous to the directed polymer case, see (5.2)), where $\mathcal{U}_\varepsilon(a_i, b_i; \boldsymbol{x}_i, \boldsymbol{y}_i)$ are the contributions of stretches of consecutive pairwise matchings:

$$\mathrm{E}\Big[\Big(u^\varepsilon(t, \phi) - \textstyle\int_{\mathbb{R}^2} \phi(x) \, \mathrm{d}x\Big)^3\Big] = \displaystyle\sum_{m \geqslant 2} 3 \, \mathcal{I}_t^{(\varepsilon, m)} \qquad \text{with}$$

$$\begin{aligned} \mathcal{I}_t^{(\varepsilon,m)} := \beta_\varepsilon^{2m} & \idotsint\limits_{\substack{0 < a_1 < b_1 < a_2 < b_2 < \ldots < a_m < b_m < \varepsilon^{-2} t \\ \boldsymbol{x}_1, \boldsymbol{y}_1, \ldots, \boldsymbol{x}_m, \boldsymbol{y}_m \in (\mathbb{R}^2)^2}} \mathrm{d}\vec{a} \, \mathrm{d}\vec{b} \, \mathrm{d}\vec{\boldsymbol{x}} \, \mathrm{d}\vec{\boldsymbol{y}} \int_{(\mathbb{R}^2)^3} \mathrm{d}z_1 \, \mathrm{d}z_2 \, \mathrm{d}z_3 \; \varepsilon^6 \, \phi(\varepsilon z_1) \phi(\varepsilon z_2) \phi(\varepsilon z_3) \, \cdot \\ & \cdot \, g_{a_1}(z_1, \tilde{x}_1) \, g_{a_1}(z_2, \hat{x}_1) \, J(\hat{x}_1 - \tilde{x}_1) \, \mathcal{U}_\varepsilon(a_1, b_1; \boldsymbol{x}_1, \boldsymbol{y}_1) \, \cdot \\ & \cdot \sum_{\hat{Y}_1 \in \{\hat{y}_1, \tilde{y}_1\}} g_{a_2}(z_3, \tilde{x}_2) \, g_{a_2 - b_1}(\hat{Y}_1, \hat{x}_2) \, J(\hat{x}_2 - \tilde{x}_2) \, \mathcal{U}_\varepsilon(a_2, b_2; \boldsymbol{x}_2, \boldsymbol{y}_2) \, \cdot \\ & \cdot \prod_{i=3}^m \sum_{\substack{\hat{Y}_{i-1} \in \{\hat{y}_{i-1}, \tilde{y}_{i-1}\} \\ \tilde{Y}_{i-2} \in \{\hat{y}_{i-2}, \tilde{y}_{i-2}\} \setminus \{\hat{Y}_{i-2}\}}} g_{a_i - b_{i-1}}(\hat{Y}_{i-1}, \hat{x}_i) \, g_{a_i - b_{i-2}}(\tilde{Y}_{i-2}, \tilde{x}_i) \, \cdot \\ & \qquad \cdot \, J(\hat{x}_i - \tilde{x}_i) \, \mathcal{U}_\varepsilon(a_i, b_i; \boldsymbol{x}_i, \boldsymbol{y}_i) \,. \end{aligned} \tag{8.39}$$

Remark 8.1. *This formula looks actually slightly different than the corresponding expansion for the directed polymer* (5.2)*, for the presence of the sums over $\hat{Y}_{i-1}$ and $\tilde{Y}_{i-2}$. The reason is that, each time that two copies of the continuum polymers "spilt" (i.e. at the end of each stretch) we have to decide which one will meet the unmatched copy and which one will wait until the next split. But since the two continuum polymers do not match exactly but rather lie inside the support of $J(\cdot)$, the symmetry that was present in the discrete case is broken. This gives rise to the sum over $\hat{Y}_{i-1} \in \{\hat{y}_{i-1}, \tilde{y}_{i-1}\}$ and $\tilde{Y}_{i-2} \in \{\hat{y}_{i-2}, \tilde{y}_{i-2}\} \setminus \{\hat{Y}_{i-2}\}$.*

We estimate $\mathcal{I}_t^{(\varepsilon,m)}$ as follows. We start by integrating $\boldsymbol{y}_m$ using (8.38), to get

$$\int_{(\mathbb{R}^2)^2} \mathcal{U}_\varepsilon(a_m, b_m; \boldsymbol{x}_m, \boldsymbol{y}_m) \, \mathrm{d}\boldsymbol{y}_m \leqslant C \, \varepsilon^2 \, \log \varepsilon^{-2} \, G_\vartheta(\varepsilon^2 (b_m - a_m)) \,,$$

uniformly over the allowed $\boldsymbol{x}_m$. Next we integrate over $\hat{x}_m$ and $\tilde{x}_m$, to get:

$$\begin{aligned}
&\int_{(\mathbb{R}^2)^2} \mathrm{d}\hat{x}_m \, \mathrm{d}\tilde{x}_m \, g_{a_m-b_{m-1}}(\hat{Y}_{m-1}, \hat{x}_m) \, g_{a_m-b_{m-2}}(\tilde{Y}_{m-2}, \tilde{x}_m) J(\hat{x}_m - \tilde{x}_m) \\
&\quad = \big(g_{a_m-b_{m-1}} * J * g_{a_m-b_{m-2}}\big)(\hat{Y}_{m-1} - \tilde{Y}_{m-2}) \\
&\quad = \big(g_{2a_m-b_{m-1}-b_{m-2}} * J\big)(\hat{Y}_{m-1} - \hat{Y}_{m-2}) \\
&\quad \leqslant \|g_{2a_m-b_{m-1}-b_{m-2}} * J\|_\infty \leqslant \|g_{2a_m-b_{m-1}-b_{m-2}}\|_\infty \\
&\quad = \frac{1}{2\pi(2a_m - b_{m-1} - b_{m-2})} \leqslant \frac{1}{4\pi\sqrt{(a_m - b_{m-1})\,(a_m - b_{m-2})}} \,,
\end{aligned}$$

having used $2xy \leqslant x^2 + y^2$ in the last equality.

We iterate this procedure for $i = m-1, m-2, \ldots$ until $i = 3$: we can first integrate out $\boldsymbol{y}_i$ and then $\hat{x}_i$ and $\tilde{x}_i$. This replaces $\mathcal{U}_\varepsilon(a_i, b_i; \boldsymbol{x}_i, \boldsymbol{y}_i)$ by $C\,\varepsilon^2 \log \varepsilon^{-2}\, G_\vartheta(\varepsilon^2(b_i - a_i))$ and $g_{a_i-b_{i-1}}(\hat{Y}_{i-1}, \hat{x}_i)\, g_{a_i-b_{i-2}}(\tilde{Y}_{i-2}, \tilde{x}_i)$ by $(4\pi\sqrt{(a_i - b_{i-1})\,(a_i - b_{i-2})})^{-1}$. We also recall that $\beta_\varepsilon^2 \leqslant C(\log \varepsilon^{-2})^{-1}$, see (8.16). Looking back at (8.39), we obtain for some $C < \infty$

$$\begin{aligned}
\mathcal{I}_t^{(\varepsilon,m)} \leqslant (\beta_\varepsilon^2)^2\,(C\,\varepsilon^2)^{m-2} & \idotsint\limits_{\substack{0<a_1<b_1<a_2<b_2<\ldots<a_m<b_m<\varepsilon^{-2}t \\ \boldsymbol{x}_1,\boldsymbol{y}_1,\boldsymbol{x}_2,\boldsymbol{y}_2 \in (\mathbb{R}^2)^2}} \mathrm{d}\vec{a}\,\mathrm{d}\vec{b}\,\mathrm{d}\vec{\boldsymbol{x}}\,\mathrm{d}\vec{\boldsymbol{y}} \int\limits_{(\mathbb{R}^2)^3} \mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\;\varepsilon^6\,\phi(\varepsilon z_1)\phi(\varepsilon z_2)\phi(\varepsilon z_3)\,\cdot \\
&\cdot\, g_{a_1}(z_1, \tilde{x}_1)\, g_{a_1}(z_2, \hat{x}_1)\, J(\hat{x}_1 - \tilde{x}_1)\, \mathcal{U}_\varepsilon(a_1, b_1; \boldsymbol{x}_1, \boldsymbol{y}_1)\,\cdot \\
&\cdot \sum_{\hat{Y}_1 \in \{\hat{y}_1, \tilde{y}_1\}} g_{a_2}(z_3, \tilde{x}_2)\, g_{a_2-b_1}(\hat{Y}_1, \hat{x}_2)\, J(\hat{x}_2 - \tilde{x}_2)\, \mathcal{U}_\varepsilon(a_2, b_2; \boldsymbol{x}_2, \boldsymbol{y}_2)\,\cdot \\
&\cdot \prod_{i=3}^m \frac{G_\vartheta(\varepsilon^2(b_i - a_i))}{\sqrt{(a_m - b_{m-1})\,(a_m - b_{m-2})}} \,.
\end{aligned}$$

We can now conclude with the last bounds.

- We integrate out $\boldsymbol{y}_2$, replacing $\mathcal{U}_\varepsilon(a_2, b_2; \boldsymbol{x}_2, \boldsymbol{y}_2)$ by $C\,\varepsilon^2 \log \varepsilon^{-2}\, G_\vartheta(\varepsilon^2(b_2 - a_2))$, see (8.38). Then we bound $\phi(\varepsilon z_3) \leqslant \|\phi\|_\infty$ and we integrate out z_3 (which makes $g_{a_2}(z_3, \tilde{x}_2)$ disappear) followed by $\tilde{x}_2$ and $\hat{x}_2$ (which make $g_{a_2-b_1}(\hat{Y}_1, \hat{x}_2)\, J(\hat{x}_2 - \tilde{x}_2)$ disappear).
- We integrate out $\boldsymbol{y}_1$, replacing $\mathcal{U}_\varepsilon(a_1, b_1; \boldsymbol{x}_1, \boldsymbol{y}_1)$ by $C\,\varepsilon^2 \log \varepsilon^{-2}\, G_\vartheta(\varepsilon^2(b_1 - a_1))$, see (8.38). Then we bound $\phi(\varepsilon z_1) \leqslant \|\phi\|_\infty$ and we integrate out z_1 (which makes $g_{a_1}(z_1, \tilde{x}_1)$ disappear) followed by $\tilde{x}_1$ and $\hat{x}_1$ (which make $g_{a_1}(z_2, \hat{x}_1)\, J(\hat{x}_1 - \tilde{x}_1)$ disappear). Lastly, we integrate out z_1, which turns the factor ε^6 into ε^4.

This leads to

$$\begin{aligned}
\mathcal{I}_t^{(\varepsilon,m)} \leqslant (C\,\varepsilon^2)^m\,\varepsilon^4 & \idotsint\limits_{0<a_1<b_1<a_2<b_2<\ldots<a_m<b_m<\varepsilon^{-2}t} \mathrm{d}\vec{a}\,\mathrm{d}\vec{b}\; G_\vartheta(\varepsilon^2(b_1 - a_1))\, G_\vartheta(\varepsilon^2(b_2 - a_2))\,\cdot \\
&\qquad\qquad \cdot \prod_{i=3}^m \frac{G_\vartheta(\varepsilon^2(b_i - a_i))}{\sqrt{(a_m - b_{m-1})\,(a_m - b_{m-2})}} \,.
\end{aligned}$$

Finally, the change of variables $a_i \to \varepsilon^{-2} a_i$, $b_i \to \varepsilon^{-2} b_i$ gives

$$\mathcal{I}_t^{(\varepsilon,m)} \leqslant C^m \int\cdots\int_{0<a_1<b_1<a_2<b_2<\ldots<a_m<b_m<t} \mathrm{d}\vec{a}\,\mathrm{d}\vec{b}\; G_\vartheta(b_1-a_1)\, G_\vartheta(b_2-a_2)\,\cdot$$
$$\cdot \prod_{i=3}^m \frac{G_\vartheta(b_i-a_i)}{\sqrt{(a_m-b_{m-1})\,(a_m-b_{m-2})}}\,.$$

Note that the right hand side, *which carries no dependence on* ε, coincides for $t=1$ with $J^{(m)}$ defined in (5.7). We already showed that $J^{(m)}$ decays super-exponentially fast as $m\to\infty$, see (5.14)-(5.15). Looking at the first line of (8.39), we see that the proof is completed. □

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Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, via Cozzi 55, 20125 Milano, Italy

E-mail address: `francesco.caravenna@unimib.it`

Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, 119076 Singapore

E-mail address: `matsr@nus.edu.sg`

Department of Statistics, University of Warwick, Coventry CV4 7AL, UK

E-mail address: `N.Zygouras@warwick.ac.uk`