ON THE MAXIMUM OF CONDITIONED RANDOM WALKS AND TIGHTNESS FOR PINNING MODELS

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Abstract. We consider real random walks with finite variance. We prove an optimal integrability result for the diffusively rescaled maximum, when the walk or its bridge is conditioned to stay positive, or to avoid zero. As an application, we prove tightness under diffusive rescaling for general pinning and wetting models based on random walks.

1. Introduction

In this paper we deal with random walks on $\mathbb{R}$, with zero mean and finite variance.

In Section 2 we consider the random walks, or their bridges, conditioned to stay positive on a finite time interval. We prove that the maximum of the walk, diffusively rescaled, has a *uniformly integrable square*. The same result is proved under the conditioning that the walk avoids zero.

In Section 3 we present an application to *pinning and wetting models* built over random walks. More generally, we consider probabilities which admit suitable regeneration epochs, which cut the path into independent "excursions". We prove that these models, under diffusive rescaling, are tight in the space of continuous functions. This fills a gap in the proof of [DGZ05, Lemma 4].

Sections 4, 5, 6 contain the proofs.

This paper generalizes and supersedes the unpublished manuscript [CGZ07b].

2. Random walks conditioned to stay positive, or to avoid zero

We use the conventions $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $(X_i)_{i \in \mathbb{N}}$ be i.i.d. real random variables. Let $(S_n)_{n \in \mathbb{N}_0}$ be the associated random walk:

$$S_0 := 0\,, \qquad S_n := X_1 + \ldots + X_n \quad \text{for } n \in \mathbb{N}\,.$$

Assumption 2.1. $\mathrm{E}[X_1] = 0$, $\mathrm{E}[X_1^2] = \sigma^2 < \infty$, *and one of the following cases hold.*

- Discrete case. *The law of X_1 is integer valued and, for simplicity, the random walk is aperiodic, i.e. $\mathrm{P}(S_n = 0) > 0$ for large n, say $n \geq n_0$.*
- Continuous case. *The law of X_1 has a density with respect to the Lebesgue measure, and the density of S_n is essentially bounded for some $n \in \mathbb{N}$:*

$$f_n(x) := \frac{\mathrm{P}(S_n \in \mathrm{d}x)}{\mathrm{d}x} \in L^\infty\,.$$

It follows that for large n, say $n \geq n_0$, f_n is bounded and continuous, and $f_n(0) > 0$.

Let us denote by P_n the law of the first n steps of the walk:

$$\mathrm{P}_n := \mathrm{P}\big((S_0, S_1, \ldots, S_n) \in \cdot\,\big)\,. \tag{2.1}$$

Date: September 28, 2018.

2010 *Mathematics Subject Classification.* 82B41, 60K35, 60B10.

Key words and phrases. Random Walk, Bridge, Excursion, Conditioning to Stay Positive, Uniform Integrability, Polymer Model, Pinning Model, Wetting Model, Tightness.

Next we define the laws of the *meander*, *bridge* and *excursion*:

$$\begin{aligned} \mathrm{P}_n^{\mathrm{mea}}(\,\cdot\,) &:= \mathrm{P}\big((S_0, S_1, \dots, S_n) \in \cdot \,\big|\, S_1 > 0, S_2 > 0, \dots, S_n > 0\big)\,, \\ \mathrm{P}_n^{\mathrm{bri}}(\,\cdot\,) &:= \mathrm{P}\big((S_0, S_1, \dots, S_n) \in \cdot \,\big|\, S_n = 0\big)\,, \\ \mathrm{P}_n^{\mathrm{exc}}(\,\cdot\,) &:= \mathrm{P}\big((S_0, S_1, \dots, S_n) \in \cdot \,\big|\, S_1 > 0, S_2 > 0, \dots, S_{n-1} > 0, S_n = 0\big)\,. \end{aligned} \tag{2.2}$$

In Remark 2.5 below we discuss the conditioning on $\{S_n = 0\}$, and periodicity issues.

Our main result concerns the integrability of the absolute maximum of the walk:

$$M_n := \max_{0 \le i \le n} |S_i|\,. \tag{2.3}$$

Theorem 2.2. *Let Assumption 2.1 hold. Then M_n^2/n is uniformly integrable under any of the laws $\mathrm{Q} \in \big\{\mathrm{P}_n, \mathrm{P}_n^{\mathrm{bri}}, \mathrm{P}_n^{\mathrm{mea}}, \mathrm{P}_n^{\mathrm{exc}}\big\}$:*

$$\lim_{K \to \infty} \sup_{n \in \mathbb{N}} \mathrm{E}_{\mathrm{Q}}\bigg[\frac{M_n^2}{n} \mathbb{1}_{\left\{\frac{M_n^2}{n} > K\right\}}\bigg] = 0\,. \tag{2.4}$$

The proof of Theorem 2.2, given in Section 4, comes in three steps. First we exploit *local limit theorems*, to remove the conditioning on $\{S_n = 0\}$ and just deal with P_n, $\mathrm{P}_n^{\mathrm{mea}}$. Then we use *martingale arguments*, to get rid of the maximum M_n and focus on S_n. Finally we use *fluctuation theory*, to perform sharp computations on the law of S_n.

Remark 2.3. *For a symmetric random walk, the bound $M_n^2 \ge X_n^2 \mathbb{1}_{\{S_{n-1} \ge 0,\, X_n \ge 0\}}$ gives*

$$\mathrm{E}\bigg[\frac{M_n^2}{n} \mathbb{1}_{\left\{\frac{M_n^2}{n} > K\right\}}\bigg] \;\ge\; \frac{1}{4}\,\mathrm{E}\bigg[\frac{X_1^2}{n} \mathbb{1}_{\left\{\frac{X_1^2}{n} > K\right\}}\bigg]\,. \tag{2.5}$$

Given $n \in \mathbb{N}$, we can choose the law of X_1 so that the right hand side vanishes as slow as we wish, as $K \to \infty$. Thus (2.4) *cannot be improved, without further assumptions.*

We next introduce the laws of the random walk and bridge *conditioned to avoid zero*:

$$\begin{aligned} \mathrm{P}_n^{\mathrm{mea2}}(\,\cdot\,) &:= \mathrm{P}\big((S_0, S_1, \dots, S_n) \in \cdot \,\big|\, S_1 \neq 0, S_2 \neq 0, \dots, S_n \neq 0\big)\,, \\ \mathrm{P}_n^{\mathrm{exc2}}(\,\cdot\,) &:= \mathrm{P}\big((S_0, S_1, \dots, S_n) \in \cdot \,\big|\, S_1 \neq 0, S_2 \neq 0, \dots, S_{n-1} \neq 0, S_n = 0\big)\,. \end{aligned} \tag{2.6}$$

In the continuous case $\mathrm{P}(S_n \neq 0) = 1$, so we have trivially $\mathrm{P}_n^{\mathrm{mea2}} = \mathrm{P}_n$ and $\mathrm{P}_n^{\mathrm{exc2}} = \mathrm{P}_n^{\mathrm{bri}}$. In the discrete case, however, the conditioning on $\{S_n \neq 0\}$ has a substantial effect: $\mathrm{P}_n^{\mathrm{mea2}}$ and $\mathrm{P}_n^{\mathrm{exc2}}$ are close to "two-sided versions" of $\mathrm{P}_n^{\mathrm{mea}}$ and $\mathrm{P}_n^{\mathrm{exc}}$ (see [Bel72, Kai76]).

We prove the following analogue of Theorem 2.2.

Theorem 2.4. *Let Assumption 2.1 hold. Then M_n^2/n under $\mathrm{P}_n^{\mathrm{exc2}}$ or $\mathrm{P}_n^{\mathrm{mea2}}$ is uniformly integrable.*

Theorem 2.4 is proved in Section 5. We first use local limit theorems to reduce the analysis to $\mathrm{P}_n^{\mathrm{mea2}}$, as for Theorem 2.2, but we can no longer apply martingale techniques. We then exploit direct path arguments to *deduce* Theorem 2.4 from Theorem 2.2.

Remark 2.5. *The laws $\mathrm{P}_n^{\mathrm{bri}}$, $\mathrm{P}_n^{\mathrm{exc}}$, $\mathrm{P}_n^{\mathrm{exc2}}$ are well-defined for $n \ge n_0$ — since $\mathrm{P}(S_n = 0) > 0$ or $f_n(0) > 0$, see Assumption 2.1 — but not obviously for $n < n_0$. This is quite immaterial for our goals, since uniform integrability is essentially an asymptotic property: we can take any definition for these laws for $n < n_0$, as long as we have $M_n \in L^2$.*

We also stress that we require aperiodicity in Assumption 2.1 only for notational convenience. If a discrete random walk has period $T \ge 2$, then Theorems 2.2 and 2.4 still hold, with essentially no change in the proofs, but for the the laws $\mathrm{P}_n^{\mathrm{bri}}$, $\mathrm{P}_n^{\mathrm{exc}}$, $\mathrm{P}_n^{\mathrm{exc2}}$ to be well-defined we have to restrict $n \in T\mathbb{N}$, to ensure that $\mathrm{P}(S_n = 0) > 0$ for large n.

3. Tightness for pinning and wetting models

We prove tightness under diffusive rescaling for *pinning and wetting models*, see Subsection 3.2, exploiting the property that these models have independent excursions[†], conditionally on their zero level set. It is simpler and more transparent to work with general probabilities which enjoy (a generalization of) this property, that we now define.

3.1. **A sharp criterion for tightness based on excursions.** Given $t \in \mathbb{N}$, we use the shorthands

$$[t] := \{0, 1, \dots, t\}\,, \qquad \mathbb{R}^{[t]} = \{x = (x_0, x_1, \dots, x_t) :\ x_i \in \mathbb{R}\} \simeq \mathbb{R}^{t+1}\,.$$

We consider probabilities $\mathbf{P}_N$ on paths $x = (x_0, \dots, x_N) \in \mathbb{R}^{[N]}$ which admit *regeneration epochs in their zero level set*. To define $\mathbf{P}_N$, we need three ingredients:

- the *regeneration law* p_N is a probability on the space of subsets of $[N]$ which contain 0;
- the *bulk excursion laws* P_t^{bulk}, $t \in \mathbb{N}$, are probabilities on $\mathbb{R}^{[t]}$ with $P_t^{\mathrm{bulk}}(x_0 = x_t = 0) = 1$;
- the *final excursion laws* P_t^{fin}, $t \in \mathbb{N}$, are probabilities on $\mathbb{R}^{[t]}$ with $P_t^{\mathrm{fin}}(x_0 = 0) = 1$.

Definition 3.1. *The law $\mathbf{P}_N$ is the probability on $\mathbb{R}^{[N]}$ under which the path $x = (x_0, x_1, \dots, x_N)$ is built as follows.*

(1) *First sample the number n and the locations $0 =: t_1 < \ldots < t_n \le N$ of the* regeneration epochs*, with probabilities $p_N(\{t_1, \dots, t_n\})$.*
(2) *Then write the path x as a concatenation of n* excursions *$x^{(i)}$, with $i = 1, \dots, n$:*

$$x^{(i)} := (x_{t_i}, \dots, x_{t_{i+1}})\,, \qquad \text{with } \ t_{n+1} := N\,.$$

(3) *Finally, given the regeneration epochs, sample the excursions $x^{(i)}$ independently, with marginal laws $P^{\mathrm{bulk}}_{t_{i+1}-t_i}$ for $i = 1, \dots, n-1$ and (in case $t_n < N$) $P^{\mathrm{fin}}_{N-t_n}$ for $i = n$.*

Let $C([0,1])$ be the space of continuous functions $f : [0,1] \to \mathbb{R}$, with the topology of uniform convergence. We define the *diffusive rescaling* operator $\mathcal{R}_N : \mathbb{R}^{[N]} \to C([0,1])$

$$\mathcal{R}_N(x) := \Big\{\text{linear interpolation of } \ \tfrac{1}{\sqrt{N}} x_{Nt} \ \text{ for } t \in \big\{0, \tfrac{1}{N}, \dots, \tfrac{N-1}{N}, 1\big\}\Big\}$$

We give optimal conditions under which the laws $\mathbf{P}_N \circ \mathcal{R}_N^{-1}$, called *diffusive rescalings* of $\mathbf{P}_N$, are tight. Remarkably, we make *no assumption on the regeneration laws p_N*.

Theorem 3.2. *Let $\mathbf{P}_N$ be as in Definition 3.1. The diffusive rescalings $(\mathbf{P}_N \circ \mathcal{R}_N^{-1})_{N\in\mathbb{N}}$ are tight in $C([0,1])$, for an arbitrary choice of the regeneration laws $(p_N)_{N\in\mathbb{N}}$, if and only if the following conditions hold:*

(1) *the diffusive rescalings $(P_t^{\mathrm{bulk}} \circ \mathcal{R}_t^{-1})_{t\in\mathbb{N}}$ and $(P_t^{\mathrm{fin}} \circ \mathcal{R}_t^{-1})_{t\in\mathbb{N}}$ are tight in $C([0,1])$;*
(2) *the bulk excursion law satisfies the following integrability bound:*

$$\sup_{t\in\mathbb{N}} P_t^{\mathrm{bulk}}\Big(\frac{\max_{0\le i\le t}|x_i|}{\sqrt{t}} > a\Big) = o\Big(\frac{1}{a^2}\Big) \qquad \text{as } a \uparrow \infty\,. \tag{3.1}$$

We point out that a slightly weaker version of Theorem 3.2 was proved in [CGZ07b].

To make a link with the previous section, we set $M_t := \max_{0\le i\le t}|x_i|$ and observe that

$$P_t^{\mathrm{bulk}}\Big(\frac{\max_{0\le i\le t}|x_i|}{\sqrt{t}} > a\Big) \le \frac{1}{a^2}\, E_t^{\mathrm{bulk}}\Big[\frac{M_t^2}{t}\, \mathbb{1}_{\left\{\frac{M_t^2}{t} > a\right\}}\Big]\,.$$

Thus condition (2) in Theorem 3.2 is satisfied *if M_t^2/t is uniformly integrable under P_t^{bulk}*. We then obtain the following corollary of Theorems 2.2 and 2.4.

Proposition 3.3. *Condition* (2) *in Theorem 3.2 is satisfied if P_t^{bulk} is chosen among $\{\mathrm{P}_t^{\mathrm{bri}}, \mathrm{P}_t^{\mathrm{exc}}, \mathrm{P}_t^{\mathrm{exc2}}\}$, see* (2.2) *and* (2.6)*, for a random walk satisfying Assumption 2.1.*

[†]In this section the word "excursion" has a more general meaning than in Section 2.

Remark 3.4. *Condition* (1) *in Theorem 3.2 is satisfied too, if* P_t^{bulk} *is chosen among* $\{\mathrm{P}_t^{\mathrm{bri}}, \mathrm{P}_t^{\mathrm{exc}}, \mathrm{P}_t^{\mathrm{exc2}}\}$ *and* P_t^{fin} *is chosen among* $\{\mathrm{P}_n, \mathrm{P}_n^{\mathrm{mea}}, \mathrm{P}_n^{\mathrm{mea2}}\}$*, under Assumption 2.1. Indeed, the diffusive rescalings of* P_n*,* $\mathrm{P}_n^{\mathrm{bri}}$*,* $\mathrm{P}_n^{\mathrm{mea}}$ *and* $\mathrm{P}_n^{\mathrm{exc}}$ *converge weakly to Brownian motion* [Don51]*, bridge* [Lig68, DGZ05]*, meander* [Bol76] *and excursion* [CC13]*; in the discrete case, the diffusive rescalings of* $\mathrm{P}_n^{\mathrm{mea2}}$ *and* $\mathrm{P}_n^{\mathrm{exc2}}$ *converge weakly to two-sided Brownian meander* [Bel72] *and excursion* [Kai76].

3.2. **Pinning and wetting models.** An important class of laws $\mathbf{P}_N$ to which Theorem 3.2 applies is given by pinning and wetting models (see [Gia07, Gia11, Hol09] for background).

Fix a random walk $(S_n)_{n\in\mathbb{N}_0}$ as in Assumption 2.1 and a real sequence $\xi = (\xi_n)_{n\in\mathbb{N}}$ (*environment*). For $N \in \mathbb{N}$, the *pinning model* $\mathbf{P}_N^\xi$ is the law on $\mathbb{R}^{[N]}$ defined as follows.

- *Discrete case.* We define
$$\frac{\mathbf{P}_N^\xi\big((S_0,\dots,S_N)=(s_0,\dots,s_N)\big)}{\mathrm{P}\big((S_1,\dots,S_N)=(s_1,\dots,s_N)\big)} := \frac{e^{\sum_{n=1}^N \xi_n \mathbb{1}_{\{s_n=0\}}}}{Z_N^\xi},$$
where Z_N^ξ is a suitable normalizing constant, called *partition function*.
- *Continuous case.* We assume that $\xi_n \geq 0$ for all $n \in \mathbb{N}$ and we define $\mathbf{P}_N^\xi$ by
$$\mathbf{P}_N^\xi\big((S_0,\dots,S_n)\in(\mathrm{d}s_0,\dots,\mathrm{d}s_n)\big) := \delta_0(\mathrm{d}s_0)\,\frac{\prod_{n=1}^N \big(f(s_n-s_{n-1})\,\mathrm{d}s_n \,+\, \xi_n\,\delta_0(\mathrm{d}s_n)\big)}{Z_N^\xi},$$
where $f(\cdot)$ is the density of S_1 and $\delta_0(\cdot)$ is the Dirac mass at 0.

Note that $\mathbf{P}_N^\xi$ fits Definition 3.1 with regeneration epochs $\{k \in [N] :\ s_k = 0\}$ (the whole zero level set) and $P_t^{\mathrm{bulk}} = \mathrm{P}_t^{\mathrm{exc2}}$, $P_t^{\mathrm{fin}} = \mathrm{P}_t^{\mathrm{mea2}}$ (which means $P_t^{\mathrm{bulk}} = \mathrm{P}_t^{\mathrm{bri}}$, $P_t^{\mathrm{fin}} = \mathrm{P}_t$ in the continuous case).

Another example of law $\mathbf{P}_N$ as in Definition 3.1 is the *wetting model* $\mathbf{P}_N^{\xi,+}$, defined by
$$\mathbf{P}_N^{\xi,+}(\,\cdot\,) := \mathbf{P}_N^\xi(\,\cdot\,|\,s_1 \geq 0,\, s_2 \geq 0,\,\dots,\, s_N \geq 0\,).$$
The bulk excursion law is now $P_t^{\mathrm{bulk}} = \mathrm{P}_t^{\mathrm{exc}}$, while the final excursion law is $P_t^{\mathrm{fin}} = \mathrm{P}_t^{\mathrm{mea}}$.

Finally, *constrained* versions of the pinning and wetting models also fit Definition 3.1:
$$\mathbf{P}_N^{\xi,c}(\,\cdot\,) := \mathbf{P}_N^\xi(\,\cdot\,|\,s_N = 0)\,, \qquad \mathbf{P}_N^{\xi,+,c}(\,\cdot\,) := \mathbf{P}_N^{\xi,+}(\,\cdot\,|\,s_N = 0)\,.$$
The final and bulk excursion laws coincide ($P_t^{\mathrm{fin}} = \mathrm{P}_t^{\mathrm{exc2}}$ for $\mathbf{P}_N^{\xi,c}$, $P_t^{\mathrm{fin}} = \mathrm{P}_t^{\mathrm{exc}}$ for $\mathbf{P}_N^{\xi,+,c}$).

Proposition 3.3 and Remark 3.4 yield immediately the following result.

Theorem 3.5 (Tightness for pinning and wetting models)**.** *Fix a real sequence* $\xi = (\xi_n)_{n\in\mathbb{N}}$*. Under Assumption 2.1, the diffusive rescalings* $(\mathsf{P}_N \circ \mathcal{R}_N^{-1})_{N\in\mathbb{N}}$ *of pinning or wetting models* $\mathsf{P}_N \in \{\mathbf{P}_N^\xi, \mathbf{P}_N^{\xi,+}, \mathbf{P}_N^{\xi,c}, \mathbf{P}_N^{\xi,+,c}\}$ *are tight in* $C([0,1])$*.*

This result fills a gap in the proof of [DGZ05, Lemma 4], which was also used in the works [CGZ06], [CGZ07a]. A recent application of Theorem 3.5 can be found in [DO18].

Pinning and wetting models are challenging models, which display a rich behavior. This complexity is hidden in the regeneration law $p_N = p_N^\xi$. This explains the importance of having criteria for tightness, such as Theorem 3.2, which only looks at excursions.

Remark 3.6. *There are models where regeneration epochs are a strict subset of the zero level set. For instance, in presence of a Laplacian interaction* [BC10, CD08, CD09]*, couples of adjacent zeros are regeneration epochs. Theorem 3.2 can cover these cases.*

4. Proof of Theorem 2.2

We fix a random walk $(S_n)_{n\in\mathbb{N}_0}$ which satisfies Assumption 2.1, for simplicity with $\sigma^2 = 1$. We split the proof of Theorem 2.2 in three steps. To prove (2.4) we may take $n \geq n_0$, with n_0 as in Assumption 2.1, because (2.4) holds for any fixed n, by $M_n \in L^2$.

Step 1. We use the shorthand UI for "uniformly integrable". In this step assume that

$$\tfrac{M_n^2}{n} \text{ under } \mathrm{P}_n \text{ (resp. under } \mathrm{P}^{\mathrm{mea}}) \text{ is UI}\,, \tag{4.1}$$

and we show that

$$\tfrac{M_n^2}{n} \text{ under } \mathrm{P}_n^{\mathrm{bri}} \text{ (resp. under } \mathrm{P}_n^{\mathrm{exc}}) \text{ is UI}\,. \tag{4.2}$$

Let us set $M_{[a,b]} := \max_{a\leq i\leq b} |S_i|$. Since $M_n \leq \max\{M_{[0,n/2]}, M_{[n/2,n]}\}$, it suffices to prove that $M^2_{[0,n/2]}/n$ and $M^2_{[n/2,n]}/n$ are UI. By symmetry, (4.2) is equivalent to

$$\tfrac{M_{n/2}^2}{n} \text{ under } \mathrm{P}_n^{\mathrm{bri}} \text{ (resp. under } \mathrm{P}_n^{\mathrm{exc}}) \text{ is UI}\,. \tag{4.3}$$

We take n even (for simplicity). We show that the laws of $\boldsymbol{V}_{n/2} := (S_1, \dots, S_{n/2})$ under $\mathrm{P}_n^{\mathrm{bri}}$ (resp. $\mathrm{P}_n^{\mathrm{exc}}$) and under P_n (resp. $\mathrm{P}_n^{\mathrm{mea}}$) *have a bounded Radon-Nikodym density*:

$$\sup_{n\geq n_0} \sup_{\boldsymbol{z}\in\mathbb{R}^{n/2}} \frac{\mathrm{P}_n^{\mathrm{bri}}(\boldsymbol{V}_{n/2} \in \mathrm{d}\boldsymbol{z})}{\mathrm{P}_n(\boldsymbol{V}_{n/2} \in \mathrm{d}\boldsymbol{z})} < \infty \qquad \left(\text{resp. } \sup_{n\geq n_0} \sup_{\boldsymbol{z}\in\mathbb{R}^{n/2}} \frac{\mathrm{P}_n^{\mathrm{exc}}(\boldsymbol{V}_{n/2} \in \mathrm{d}\boldsymbol{z})}{\mathrm{P}_n^{\mathrm{mea}}(\boldsymbol{V}_{n/2} \in \mathrm{d}\boldsymbol{z})} < \infty\right). \tag{4.4}$$

Since $M_{n/2}$ is a function of $\boldsymbol{V}_{n/2}$, it follows that (4.1) implies (4.3) (note that $M_{n/2} \leq M_n$).

It remains to prove (4.4). By Gnedenko's local limit theorem, in the discrete case

$$\forall n \geq n_0 : \qquad \mathrm{P}(S_n = 0) \geq \tfrac{c}{\sqrt{n}}\,, \qquad \sup_{x\in\mathbb{Z}} \mathrm{P}(S_n = x) \leq \tfrac{C}{\sqrt{n}}\,, \tag{4.5}$$

hence

$$\frac{\mathrm{P}_n^{\mathrm{bri}}(\boldsymbol{V}_{n/2} = \boldsymbol{z})}{\mathrm{P}_n(\boldsymbol{V}_{n/2} = \boldsymbol{z})} = \frac{\mathrm{P}(S_{n/2} = -z_{n/2})}{\mathrm{P}(S_n = 0)} \leq \frac{C}{c} < \infty\,,$$

which proves the first relation in (4.4) in the discrete case. The continuous case is similar, since $f_n(0) \geq \frac{c}{\sqrt{n}}$ and $\sup_{x\in\mathbb{R}} f_n(x) \leq \frac{C}{\sqrt{n}}$ for $n \geq n_0$, under Assumption 2.1.

To prove the second relation in (4.4), in the discrete case we compute

$$\frac{\mathrm{P}_n^{\mathrm{exc}}(\boldsymbol{V}_{n/2} = \boldsymbol{z})}{\mathrm{P}_n^{\mathrm{mea}}(\boldsymbol{V}_{n/2} = \boldsymbol{z})} = \frac{\mathrm{P}_0(S_1 > 0, \dots, S_n > 0)\, \mathrm{P}_{z_{n/2}}(S_1 > 0, \dots, S_{n/2-1} > 0, S_{n/2} = 0)}{\mathrm{P}_0(S_1 > 0, \dots, S_{n-1} > 0, S_n = 0)\, \mathrm{P}_{z_{n/2}}(S_1 > 0, \dots, S_{n/2} > 0)}\,,$$

where P_x is the law of the random walk started at $S_0 = x$. For some $c_1 < \infty$ we have

$$\mathrm{P}_0(S_1 > 0, \dots, S_n > 0) \leq \tfrac{c_1}{\sqrt{n}}\,,$$

by [Fel71, Th.1 in §XII.7, Th.1 in §XVIII.5]. Next we apply [CC13, eq. (4.5) in Prop. 4.1] (with $a_n = \sqrt{n}\,(1 + o(1))$), which summarizes [AD99, VV09]: for some $c_2 \in (0, \infty)$

$$\mathrm{P}_0(S_1 > 0, \dots, S_{n-1} > 0, S_n = 0) \geq \tfrac{c_2}{n^{3/2}}\,.$$

As a consequence, if we rename $n/2$ as n and $z_{n/2}$ as x, it remains to show that

$$\sup_{n\geq n_0} \sup_{x\geq 0} \frac{n\, \mathrm{P}_x(S_1 > 0, \dots, S_{n-1} > 0, S_n = 0)}{\mathrm{P}_x(S_1 > 0, \dots, S_n > 0)} < \infty\,. \tag{4.6}$$

By contradiction, if (4.6) does not hold, there are subsequences $n = n_k \in \mathbb{N}$, $x = x_k \geq 0$, for $k \in \mathbb{N}$, such that *the ratio in* (4.6) *diverges as* $k \to \infty$. We distinguish two cases: either $\liminf_{k\to\infty} x_k/\sqrt{n_k} = \eta > 0$ (case 1), or $\liminf_{k\to\infty} x_k/\sqrt{n_k} = 0$, i.e. there is a subsequence k_ℓ with $x_{k_\ell} = o(\sqrt{n_{k_\ell}})$ (case 2).

In case 1, i.e. for $x \geq \eta\sqrt{n}$, the denominator in (4.6) is bounded away from zero:

$$\mathrm{P}_x(S_1 > 0, \dots, S_n > 0) \geq \mathrm{P}_{\lfloor\eta\sqrt{n}\rfloor}(S_1 > 0, \dots, S_n > 0) \xrightarrow[n\to\infty]{} \mathrm{P}_\eta(B_t > 0\ \forall t \in [0,1]) > 0\,,$$

by Donsker's invariance principle [Don51] ($B = (B_t)_{t\geq 0}$ is Brownian motion started at η). For the numerator, by [CC13, eq. (4.4) in Prop. 4.1] which summarizes [Car05, VV09],

$$\mathrm{P}_x(S_1 > 0, \dots, S_{n-1} > 0, S_n = 0) \leq \tfrac{c_3}{\sqrt{n}}\, \mathrm{P}(S_1 < 0, \dots, S_n < 0) \leq \tfrac{c_3'}{n}\,,$$

for suitable $c_3, c_3' \in (0,\infty)$. Then the ratio in (4.6) is bounded, which is a contradiction.

In case 2, i.e. for $x = o(\sqrt{n})$, by [CC13, eq. (4.5) in Prop. 4.1] we have

$$\mathrm{P}_x(S_1 > 0, \dots, S_{n-1} > 0, S_n = 0) \underset{n\to\infty}{\sim} \underline{V}^-(x)\, \tfrac{c_4}{n^{3/2}}\,, \tag{4.7}$$

for a suitable $\underline{V}^-(x)$. Since $\{S_1 > 0, \dots, S_n > 0\} = \bigcup_{m>n}\{S_1 > 0, \dots, S_{m-1} > 0, S_m = 0\}$ a.s. (note that the random walk is recurrent), we get

$$\mathrm{P}_x(S_1 > 0, \dots, S_{n-1} > 0, S_n > 0) \underset{n\to\infty}{\sim} \underline{V}^-(x)\, \tfrac{2\,c_4}{\sqrt{n}}\,,$$

see also [Don12, Cor. 3]. Thus the ratio in (4.6) is bounded, which is the desired contradiction.

This completes the proof of the second relation in (4.4) in the discrete case. The continuous case is dealt with with identical arguments, exploiting [CC13, Th. 5.1]. □

Step 2. In this step we assume that

$$\tfrac{S_n^2}{n} \text{ under } \mathrm{P}_n \text{ (resp. under } \mathrm{P}_n^{\mathrm{mea}}) \text{ is UI}\,, \tag{4.8}$$

and we deduce that

$$\tfrac{M_n^2}{n} \text{ under } \mathrm{P}_n \text{ (resp. under } \mathrm{P}_n^{\mathrm{mea}}) \text{ is UI}\,. \tag{4.9}$$

Observe that $(|S_i|)_{0\leq i\leq n}$ is a submartingale under P_n. Let us show that $(|S_i|)_{0\leq i\leq n}$ is a submartingale also under $\mathrm{P}_n^{\mathrm{mea}}$ (for every fixed $n \in \mathbb{N}$). We set for $m \in \mathbb{N}$ and $x \in \mathbb{R}$

$$q_m(x) := \mathrm{P}(x + S_1 > 0, x + S_2 > 0, \dots, x + S_m > 0)\,,$$

with $q_0(x) := 1$. Then we can write, for any $n \in \mathbb{N}$, $i \in \{0, 1, \dots, n-1\}$ and $x \geq 0$,

$$\mathrm{P}_n^{\mathrm{mea}}\big[S_{i+1} \in \mathrm{d}y \,\big|\, S_i = x\big] = \frac{\mathbb{1}_{(0,\infty)}(y)\, q_{n-(i+1)}(y)}{q_{n-i}(x)}\, \mathrm{P}(X_1 \in \mathrm{d}y - x)\,.$$

Since $y \mapsto (y - x)$ and $y \mapsto \mathbb{1}_{(0,\infty)}(y)\, q_{n-(i+1)}(y)$ are non-decreasing functions, it follows by the Harris inequality (a special case of the FKG inequality) and $\mathrm{E}[X_1] = 0$ that

$$\mathrm{E}_n^{\mathrm{mea}}\big[S_{i+1} - S_i \,\big|\, S_i = x\big] \;\geq\; \int_{\mathbb{R}} (y - x)\, \mathrm{P}(X_1 \in \mathrm{d}y - x) \cdot \int_0^\infty \frac{q_{n-(i+1)}(y)}{q_{n-i}(x)}\, \mathrm{P}(X_1 \in \mathrm{d}y - x) \;=\; 0\,.$$

Since $(|S_i|)_{0\leq i\leq n}$ is a submartingale, also $(Z_i := (|S_i| - K)^+)_{0\leq i\leq n}$ is a submartingale, for any $K \in (0,\infty)$. Doob's L^2 inequality yields, for $\mathsf{P}_n = \mathrm{P}_n$ or $\mathsf{P}_n = \mathrm{P}_n^{\mathrm{mea}}$ (recall (2.3)),

$$\mathsf{E}_n\big[(M_n - K)^2\, \mathbb{1}_{\{M_n > K\}}\big] = \mathsf{E}_n\Big[\Big(\max_{0\leq i\leq n} Z_i\Big)^2\Big] \;\leq\; 4\, \mathsf{E}_n\big[Z_n^2\big] = 4\, \mathsf{E}_n\big[(S_n - K)^2\, \mathbb{1}_{\{S_n > K\}}\big]\,.$$

For $M_n > 2K$ we can bound $M_n^2 \leq 4(M_n - K)^2$. Since $(S_n - K)^2 \leq S_n^2$ for $S_n > K$, we get

$$\mathsf{E}_n\big[M_n^2\, \mathbb{1}_{\{M_n > 2K\}}\big] \leq 16\, \mathsf{E}_n\big[S_n^2\, \mathbb{1}_{\{S_n > K\}}\big]\,.$$

We finally choose $K = \frac{1}{2}\sqrt{tn}$, for $t \in (0,\infty)$, to obtain

$$\mathsf{E}_n\Big[\tfrac{M_n^2}{n}\, \mathbb{1}_{\{\frac{M_n^2}{n} > t\}}\Big] \leq 16\, \mathsf{E}_n\Big[\tfrac{S_n^2}{n}\, \mathbb{1}_{\{\frac{S_n^2}{n} > \frac{t}{2}\}}\Big]\,, \qquad \forall t > 0\,.$$

This relation for $\mathsf{P}_n = \mathrm{P}_n$ (resp. $\mathsf{P}_n = \mathrm{P}_n^{\mathrm{mea}}$) shows that (4.8) implies (4.9). □

Step 3. In this step we prove that (4.8) holds, completing the proof of Theorem 2.2. We are going to apply the following standard result, proved below.

Proposition 4.1. *Let $(Y_n)_{n\in\mathbb{N}}$, Y be random variables in L^1, such that $Y_n \to Y$ in law. Then $(Y_n)_{n\in\mathbb{N}}$ is UI if and only if $\lim_{n\to\infty} \mathrm{E}[|Y_n|] = \mathrm{E}[|Y|]$.*

Let us define

$$Y_n := \tfrac{S_n^2}{n}\,.$$

Since $S_n/\sqrt{n}$ under P_n converges in law to $Z \sim N(0,1)$, we have $Y_n \to Z^2$ in law. Since $\mathrm{E}_n[|Y_n|] = 1 = \mathrm{E}[Z^2]$ for all $n \in \mathbb{N}$, relation (4.8) under P_n follows by Proposition 4.1.

Next we focus on $\mathrm{P}_n^{\mathrm{mea}}$. It is known [Bol76] that $S_n/\sqrt{n}$ under $\mathrm{P}_n^{\mathrm{mea}}$ converges in law toward the Brownian meander at time 1, that is a random variable V with law $\mathrm{P}(V \in \mathrm{d}x) := x\, e^{-x^2/2}\, \mathbb{1}_{(0,\infty)}(x)\, \mathrm{d}x$. Therefore $Y_n \to V^2$ in law, under $\mathrm{P}_n^{\mathrm{mea}}$. Since $\mathrm{E}[V^2] = 2$, relation (4.8) under $\mathrm{P}_n^{\mathrm{mea}}$ is proved once we show that

$$\lim_{n\to\infty} \mathrm{E}_n^{\mathrm{mea}}\Big[\tfrac{S_n^2}{n}\Big] = 2\,. \tag{4.10}$$

To evaluate this limit, we express the law of $S_n/\sqrt{n}$ under $\mathrm{P}_n^{\mathrm{mea}}$ using fluctuation theory for random walks. By [Car05, equations (3.1) and (2.6)], as $n \to \infty$

$$\mathrm{P}_n^{\mathrm{mea}}\Big(\tfrac{S_n}{\sqrt{n}} \in \mathrm{d}x\Big) = \big(\sqrt{2\pi} + o(1)\big) \int_0^1 \int_0^\infty \mathrm{P}\Big(\tfrac{S_{\lfloor n(1-\alpha)\rfloor}}{\sqrt{n}} \in \mathrm{d}x - \beta\Big)\, \mathbb{1}_{[0,x)}(\beta)\, \mathrm{d}\mu_n(\alpha,\beta)\,,$$

where μ_n is a finite measure on $[0,1) \times [0,\infty)$, defined in [Car05, eq. (3.2)]. Then

$$\mathrm{E}_n^{\mathrm{mea}}\Big[\tfrac{S_n^2}{n}\Big] = \big(\sqrt{2\pi} + o(1)\big) \int_0^1 \int_0^\infty \Big\{ \mathrm{E}\Big[\tfrac{(S^+_{\lfloor n(1-\alpha)\rfloor})^2}{n}\Big] + 2\beta\, \mathrm{E}\Big[\tfrac{S^+_{\lfloor n(1-\alpha)\rfloor}}{\sqrt{n}}\Big] + \beta^2\, \mathrm{P}\Big(\tfrac{S_{\lfloor n(1-\alpha)\rfloor}}{\sqrt{n}} > 0\Big)\Big\}\, \mathrm{d}\mu_n\,.$$

By the convergence in law (under P) $S_n/\sqrt{n} \to Z \sim N(0,1)$, together with the uniform integrability of $(S_n/\sqrt{n})^2$ that we already proved, we have as $n \to \infty$

$$\mathrm{E}\Big[\tfrac{(S^+_{\lfloor n(1-\alpha)\rfloor})^2}{n}\Big] \longrightarrow (1-\alpha)\, \mathrm{E}[(Z^+)^2] = \frac{1-\alpha}{2}\,,$$
$$\mathrm{E}\Big[\tfrac{S^+_{\lfloor n(1-\alpha)\rfloor}}{\sqrt{n}}\Big] \longrightarrow \sqrt{1-\alpha}\, \mathrm{E}[Z^+] = \tfrac{\sqrt{1-\alpha}}{\sqrt{2\pi}}\,, \qquad \mathrm{P}\Big(\tfrac{S_{\lfloor n(1-\alpha)\rfloor}}{\sqrt{n}} > 0\Big) \longrightarrow \mathrm{P}(Z>0) = \tfrac{1}{2}\,,$$

uniformly for $\alpha \in [0, 1-\delta]$, for $\delta > 0$. By [Car05, Prop. 5] we have the weak convergence

$$\mu_n(\mathrm{d}\alpha, \mathrm{d}\beta) \Longrightarrow \mu(\mathrm{d}\alpha, \mathrm{d}\beta) := \frac{\beta}{\sqrt{2\pi}\, \alpha^{3/2}}\, e^{-\frac{\beta^2}{2\alpha}}\, \mathrm{d}\alpha\, \mathrm{d}\beta\,,$$

and note that μ is a finite measure on $[0,1) \times [0,\infty)$. Then $\lim_{n\to\infty} \mathrm{E}_n^{\mathrm{mea}}\Big[\tfrac{S_n^2}{n}\Big]$ equals

$$\int_0^1 \Big(\int_0^\infty \Big\{ \tfrac{1-\alpha}{2} + 2\beta\, \tfrac{\sqrt{1-\alpha}}{\sqrt{2\pi}} + \tfrac{1}{2}\beta^2 \Big\}\, \tfrac{\beta}{\alpha^{3/2}}\, e^{-\frac{\beta^2}{2\alpha}}\, \mathrm{d}\beta \Big) \mathrm{d}\alpha = \int_0^1 \Big\{ \tfrac{1-\alpha}{2\sqrt{\alpha}} + \sqrt{1-\alpha} + \sqrt{\alpha} \Big\}\, \mathrm{d}\alpha = 2\,,$$

which completes the proof of (4.10). □

Proof of Proposition 4.1. We assume that $Y_n \to Y$ a.s., by Skorokhod's representation theorem. If $(Y_n)_{n\in\mathbb{N}}$ is UI, then $Y_n \to Y$ in L^1, hence $\mathrm{E}[|Y_n|] \to \mathrm{E}[|Y|]$.

Assume now that $\lim_{n\to\infty} \mathrm{E}[|Y_n|] = \mathrm{E}[|Y|] < \infty$. Since $Y_n \to Y$ a.s., dominated convergence yields $\lim_{n\to\infty} \mathrm{E}[|Y_n| \mathbb{1}_{\{|Y_n| \le T\}}] = \mathrm{E}[|Y| \mathbb{1}_{\{|Y| \le T\}}]$ for $T \in (0,\infty)$ with $\mathrm{P}(|Y| = T) = 0$. Then

$$\lim_{n\to\infty} \mathrm{E}[|Y_n| \mathbb{1}_{\{|Y_n| > T\}}] = \lim_{n\to\infty} \big(\mathrm{E}[|Y_n|] - \mathrm{E}[|Y_n| \mathbb{1}_{\{|Y_n| \le T\}}] \big) = \mathrm{E}[|Y|\, \mathbb{1}_{\{|Y| > T\}}]\,.$$

Since $\lim_{T\to\infty} \mathrm{E}[|Y|\, \mathbb{1}_{\{|Y| > T\}}] = 0$, this shows that $(Y_n)_{n\in\mathbb{N}}$ is UI. □

5. Proof of Theorem 2.4

We fix a random walk $(S_n)_{n\in\mathbb{N}_0}$ which satisfies Assumption 2.1 in the discrete case (the continuous case is covered by Theorem 2.2), with $\sigma^2 = 1$. We proceed in two steps.

Step 1. We assume that M_n^2/n under $\mathrm{P}_n^{\mathrm{mea2}}$ is UI and we prove that M_n^2/n under $\mathrm{P}_n^{\mathrm{exc2}}$ is UI. As in Section 4, it suffices to show that, with $\boldsymbol{V}_{n/2} := (S_1, \dots, S_{n/2})$ and n_0 as in Assumption 2.1,

$$\sup_{n\geq n_0} \sup_{\boldsymbol{z}\in\mathbb{Z}^{n/2}} \frac{\mathrm{P}_n^{\mathrm{exc2}}(\boldsymbol{V}_{n/2} = \boldsymbol{z})}{\mathrm{P}_n^{\mathrm{mea2}}(\boldsymbol{V}_{n/2} = \boldsymbol{z})} < \infty\,. \tag{5.1}$$

If we define $T := \min\{n \in \mathbb{N} :\ S_n = 0\}$, we can compute (recall (2.6))

$$\frac{\mathrm{P}_n^{\mathrm{exc2}}(\boldsymbol{V}_{n/2} = \boldsymbol{z})}{\mathrm{P}_n^{\mathrm{mea2}}(\boldsymbol{V}_{n/2} = \boldsymbol{z})} = \frac{\mathrm{P}(T > n)\,\mathrm{P}_{z_{n/2}}(T = n/2)}{\mathrm{P}(T = n)\,\mathrm{P}_{z_{n/2}}(T > n/2)}\,,$$

where P_x is the law of the random walk started at $S_0 = x$. By [Kes63], as $n \to \infty$

$$\mathrm{P}(T = n) = \frac{\sigma}{\sqrt{2\pi}\, n^{3/2}} \big(1 + o(1)\big)\,, \tag{5.2}$$

hence, summing over n, we get $\mathrm{P}(T > n) = 2n\,\mathrm{P}(T = n)\,(1 + o(1))$. Then (5.1) reduces to

$$\sup_{n\geq n_0} \sup_{x\geq 0} \frac{n\,\mathrm{P}_x(T = n)}{\mathrm{P}_x(T > n)} < \infty\,. \tag{5.3}$$

Arguing as in the lines after (4.6), we need to show that the ratio in (5.3) is bounded in two cases: when $x \geq \eta\sqrt{n}$ for fixed $\eta > 0$ (*case 1*) and when $x = x_n = o(\sqrt{n})$ (*case 2*).

In case 1, i.e. for $x \geq \eta\sqrt{n}$, the denominator in (5.3) is bounded away from zero:

$$\mathrm{P}_x(T > n) \geq \mathrm{P}_{\lfloor\eta\sqrt{n}\rfloor}(S_1 > 0, \dots, S_n > 0) \xrightarrow[N\to\infty]{} \mathrm{P}_\eta(B_t > 0\ \forall t \in [0,1]) > 0\,,$$

where $(B_t)_{t\geq 0}$ is a Brownian motion [Don51]. Then the ratio in (5.3) is bounded because $\sup_{x\in\mathbb{Z}} \mathrm{P}_x(T = n) \leq \frac{c'}{n}$ for some $c' \in (0,\infty)$, by [Kai75, Cor. 1].

In case 2, i.e. for $x = o(\sqrt{n})$, we apply [Uch11, Thm. 1.1], which generalizes (5.2):

$$\mathrm{P}_x(T = n) = a^*(x)\,\frac{\sigma}{\sqrt{2\pi}\, n^{3/2}} \big(1 + o(1)\big) \qquad \text{as } n \to \infty\,,\ \text{uniformly in } x \in \mathbb{Z}\,,$$

for a suitable $a^*(x)$ (the potential kernel of the walk). Then $\mathrm{P}_x(T > n) = 2n\,\mathrm{P}_x(T = n)\,(1 + o(1))$, hence the ratio in (5.3) is bounded. This completes the proof of (5.1). □

Step 2. We prove that M_n^2/n under $\mathrm{P}_n^{\mathrm{mea2}}$ is UI. We argue by contradiction: if this does not hold, then there are $\eta > 0$ and $(n_i)_{i\in\mathbb{N}}$, $(K_i)_{i\in\mathbb{N}}$, with $\lim_{i\to\infty} K_i = \infty$, such that

$$\mathrm{E}_{n_i}^{\mathrm{mea2}}\Big[\tfrac{M_{n_i}^2}{n_i}\, \mathbb{1}_{\{\frac{M_{n_i}^2}{n_i} > K_i\}}\Big] \geq \eta\,, \qquad \forall i \in \mathbb{N}\,. \tag{5.4}$$

We are going to deduce that M_n^2/n under P_n is not UI, which contradicts Theorem 2.2.

We show below that we can strengthen (5.4), replacing $\mathrm{E}_{n_i}^{\mathrm{mea2}}$ by $\mathrm{E}_m^{\mathrm{mea2}}$ for any $m \in \{n_i, \dots, 2n_i\}$: more precisely, there exists $\eta' > 0$ such that

$$\mathrm{E}_m^{\mathrm{mea2}}\Big[\tfrac{M_{n_i}^2}{n_i}\, \mathbb{1}_{\{\frac{M_{n_i}^2}{n_i} > K_i\}}\Big] \geq \eta'\,, \qquad \forall i \in \mathbb{N}\,,\ \forall m \in \{n_i, \dots, 2n_i\}\,. \tag{5.5}$$

To exploit (5.5), we work on the time horizon $2n$, for fixed $n \in \mathbb{N}$. We split any path $S = (S_0, \dots, S_{2n})$ with $S_0 = 0$ in two parts $\tilde{S} = (S_0, S_1, \dots, S_\sigma)$ and $\hat{S} = (S_\sigma, S_{\sigma+1}, \dots, S_{2n})$, where $\sigma := \sigma_{2n} := \max\{i \in \{0, \dots, 2n\} :\ S_i = 0\}$. If S is chosen according to the unconditioned law P_{2n}, then $\hat{S}$ has

law $\mathrm{P}^{\mathrm{mea2}}_{2n-\sigma}$, conditionally on σ. If we set $\hat{M}_{2n} := \max|\hat{S}| = \max_{\sigma\le i\le 2n}|S_i|$, the bound $M_{2n}\ge \hat{M}_{2n}$ gives

$$\mathrm{E}\Big[\tfrac{(M_{2n})^2}{2n}\,\mathbb{1}_{\{\frac{(M_{2n})^2}{2n}>\frac{K}{2}\}}\Big] \ge \mathrm{E}\Big[\tfrac{(\hat{M}_{2n})^2}{2n}\,\mathbb{1}_{\{\frac{(\hat{M}_{2n})^2}{2n}>\frac{K}{2}\}}\Big] = \sum_{r=0}^{2n}\mathrm{E}\Big[\mathrm{E}^{\mathrm{mea2}}_{2n-r}\Big[\tfrac{M^2_{2n-r}}{2n}\,\mathbb{1}_{\{\frac{M^2_{2n-r}}{n}>K\}}\Big]\mathbb{1}_{\{\sigma=r\}}\Big].$$

We now restrict the sum to $r\le n$, so that $M^2_{2n-r}\ge M^2_n$, to get

$$\mathrm{E}\Big[\tfrac{(M_{2n})^2}{2n}\,\mathbb{1}_{\{\frac{(M_{2n})^2}{2n}>\frac{K}{2}\}}\Big] \ge \tfrac12\,\mathrm{P}(\sigma_{2n}\le n)\inf_{n\le m\le 2n}\mathrm{E}^{\mathrm{mea2}}_m\Big[\tfrac{M^2_n}{n}\,\mathbb{1}_{\{\frac{M^2_n}{n}>K\}}\Big].$$

Note that $\lim_{n\to\infty}\mathrm{P}(\sigma_{2n}\le n)=\mathrm{P}(B_t\neq 0\ \forall t\in(\frac12,1])=:p>0$ (actually $p=\frac12$, by the arcsine law), hence $\gamma:=\inf_{n\in\mathbb{N}}\mathrm{P}(\sigma_{2n}\le n)>0$. If we take $n=2n_i$ and $K=K_i$, by (5.5)

$$\liminf_{K\to\infty}\sup_{n\in\mathbb{N}}\mathrm{E}\Big[\tfrac{(M_n)^2}{n}\,\mathbb{1}_{\{\frac{(M_n)^2}{n}>\frac{K}{2}\}}\Big] \ge \inf_{i\in\mathbb{N}}\mathrm{E}\Big[\tfrac{(M_{2n_i})^2}{2n_i}\,\mathbb{1}_{\{\frac{(M_{2n_i})^2}{2n_i}>\frac{K_i}{2}\}}\Big] \ge \frac{\gamma\,\eta'}{2}>0\,.$$

This means that M_n^2/n under P_n is not UI, which contradicts Theorem 2.2.

It remains to prove (5.5). We fix $\mathsf{C}\in(0,\infty)$, to be determined later. We may assume that $K_i\ge\mathsf{C}$ for all $i\in\mathbb{N}$. To deduce (5.5) from (5.4), we show that for some $c>0$

$$\inf_{n\in\mathbb{N},\ m\in\{n,\dots,2n\},\ z\in\mathbb{Z}:\,z\ge\mathsf{C}\sqrt{n}}\frac{\mathrm{P}^{\mathrm{mea2}}_m(M_n=z)}{\mathrm{P}^{\mathrm{mea2}}_n(M_n=z)}\ge c\,. \tag{5.6}$$

Fix $m\ge n$ and $z>0$. If we sum over the last $\ell\le n$ for which $M_n=|S_\ell|$, we can write

$$\mathrm{P}^{\mathrm{mea2}}_m(M_n=z)=\sum_{\ell=1}^{n}\mathrm{P}^{\mathrm{mea2}}_m(M_{\ell-1}\le z,\,|S_\ell|=z,\,|S_i|<z\ \forall i=\ell+1,\dots,n)\,.$$

We write $\mathrm{P}^{\mathrm{mea2}}_m(\,\cdot\,)=\mathrm{P}(\,\cdot\,|\,E_m)$, with $E_m:=\{S_1\neq0,\dots,S_m\neq0\}$, and we apply the Markov property at time ℓ. The cases $S_\ell=z$ and $S_\ell=-z$ give a similar contribution and we do not distinguish between them (e.g. assume that the walk is symmetric). Then

$$\mathrm{P}^{\mathrm{mea2}}_m(M_n=z)=\frac{1}{\mathrm{P}(T>m)}\sum_{\ell=1}^{n}\mathrm{P}(M_{\ell-1}\le z,\,|S_\ell|=z,\,E_\ell)\underbrace{\mathrm{P}_z(|S_i|<z\ \forall 1\le i\le n-\ell,\ E_{m-\ell})}_{A}.$$

The same expression holds if we replace $\mathrm{P}^{\mathrm{mea2}}_m$ by $\mathrm{P}^{\mathrm{mea2}}_n$, namely

$$\mathrm{P}^{\mathrm{mea2}}_n(M_n=z)=\frac{1}{\mathrm{P}(T>n)}\sum_{\ell=1}^{n}\mathrm{P}(M_{\ell-1}\le z,\,|S_\ell|=z,\,E_\ell)\underbrace{\mathrm{P}_z(|S_i|<z\ \forall 1\le i\le n-\ell,\ E_{n-\ell})}_{B}.$$

Since $\mathrm{P}(T>m)\le\mathrm{P}(T>n)$, to prove (5.6) we show that $A\ge c\,B$, with $c>0$. We bound

$$B\le\mathrm{P}_z(S_i<z\ \forall i=1,\dots,n-\ell)=\mathrm{P}_0(E^-_{n-\ell})\,,$$

where we set $E^-_k:=\{S_1<0,\dots,S_k<0\}$. Similarly, for $z\ge\mathsf{C}\sqrt{n}$ we bound

$$\begin{aligned}A&\ge\mathrm{P}_z(S_i<z\ \forall i=1,\dots,n-\ell,\ S_i>0\ \forall i=1,\dots,m-\ell)\\&=\mathrm{P}_0(E^-_{n-\ell},\ S_i>-z\ \forall i=1,\dots,m-\ell)\ge\mathrm{P}_0(E^-_{n-\ell})\underbrace{\mathrm{P}_0\big((-S_i)<\mathsf{C}\sqrt{n}\ \forall i=1,\dots,m-\ell\,\big|\,E^-_{n-\ell}\big)}_{D}.\end{aligned}$$

It remains to show that $D\ge c$. Let us set $\tilde{S}_i:=-S_i$ and $\tilde{E}^+_k:=E^-_k=\{\tilde{S}_1>0,\dots,\tilde{S}_k>0\}$. If we write $r:=n-\ell$, for $m\in\{n,\dots,2n\}$, we have $m-\ell=r+(m-n)\le r+n$, hence

$$D\ge\mathrm{P}\big(\tilde{S}_i<\tfrac12\mathsf{C}\sqrt{r}\ \forall i=1,\dots,r\,\big|\,\tilde{E}^+_r\big)\cdot\mathrm{P}\big(\tilde{S}_i<\tfrac12\mathsf{C}\sqrt{n}\ \forall i=1,\dots,n\,\big)\,, \tag{5.7}$$

by the Markov property, since $(\tilde{S}_j)_{j\ge r}$ under $\mathrm{P}(\,\cdot\,|\tilde{E}^+_r)$ is the random walk $\tilde{S}$ started at $\tilde{S}_r$.

By [Bol76, Don51], as $r\to\infty$ the two probabilities in the right hand side of (5.7) converge respectively to $\mathrm{P}(\sup_{t\in[0,1]}m_t<\frac12\mathsf{C})$ and $\mathrm{P}(\sup_{t\in[0,1]}B_t<\frac12\mathsf{C})$, where $B=(B_t)_{t\ge0}$ is Brownian

motion and $m = (m_t)_{t\in[0,1]}$ is Brownian meander. Then, if we fix $\mathsf{C} > 0$ large enough, the right hand side of (5.7) is $\geq c > 0$ for all $r, n \in \mathbb{N}_0$. □

6. Proof of Theorem 3.2

Let us set $\mathbf{Q}_N := \mathbf{P}_N \circ \mathcal{R}_N^{-1}$. We prove that conditions (1) and (2) in Theorem 3.2 are necessary and sufficient for the tightness of $(\mathbf{Q}_N)_{N\in\mathbb{N}}$.

Necessity. The necessity of condition (1) is clear: just note that, by Definition 3.1, the law $\mathbf{P}_N$ coincides with P_N^{fin} (resp. with P_N^{bulk}) if we choose the regeneration law p_N to be concentrated on the single set $\{0\}$ (resp. on the single set $\{0, N\}$).

To prove necessity of condition (2), we assume by contradiction that (2) fails. Then there exists $\eta > 0$ and two sequences $(t_n)_{n\in\mathbb{N}}$, $(a_n)_{n\in\mathbb{N}}$, with $\lim_{n\to\infty} a_n = \infty$, such that

$$P_{t_n}^{\mathrm{bulk}}\left(\frac{M_{t_n}}{\sqrt{t_n}} > a_n\right) \geq \frac{\eta}{a_n^2}\,, \qquad \forall n \in \mathbb{N}\,, \qquad \text{where} \quad M_t := \max_{0\leq i\leq t} |x_i|\,. \tag{6.1}$$

We may assume that $a_n \in \mathbb{N}$ (otherwise consider $\lfloor a_n \rfloor$ and redefine η).

Define $N_n := t_n\, a_n^2$ and let p_{N_n} be the regeneration law concentrated on the single set $\{0, t_n, 2t_n, \ldots, N_n - t_n, N_n\}$. Let $\mathbf{P}_{N_n}$ be the corresponding probability on $\mathbb{R}^{[N_n]}$, see Definition 3.1. We now show that $\mathbf{Q}_N = \mathbf{P}_N \circ \mathcal{R}_N^{-1}$ *is not tight on* $C([0,1])$.

Any path $f(t)$ under $\mathbf{Q}_{N_n}$ vanishes for $t \in \{0, \frac{1}{a_n^2}, \frac{2}{a_n^2}, \ldots, 1 - \frac{1}{a_n^2}, 1\}$, which becomes dense in $[0,1]$ as $n \to \infty$. Then, if $(\mathbf{Q}_{N_n})_{n\in\mathbb{N}}$ were tight, it would converge weakly to the law concentrated on the single path $(f_t \equiv 0)_{t\in[0,1]}$. We rule this out by showing that

$$\liminf_{n\to\infty} \mathbf{Q}_{N_n}\left(\sup_{t\in[0,1]} |f_t| > 1\right) \geq 1 - e^{-\eta} > 0\,. \tag{6.2}$$

For $x \in \mathbb{R}^{[N_n]}$ and $j = 1, \ldots, a_n^2$ we define $M_{t_n}^{(j)} := \max_{i\in\{(j-1)t_n, \ldots, jt_n\}} |x_i|$, so that

$$\mathbf{Q}_{N_n}\left(\sup_{t\in[0,1]} |f_t| > 1\right) = \mathbf{P}_{N_n}\left(\max_{i=0,1,\ldots,N_n} |x_i| > \sqrt{N_n}\right) = \mathbf{P}_{N_n}\left(\max_{j=1,\ldots,a_n^2} \frac{M_{t_n}^{(j)}}{\sqrt{t_n}} > a_n\right).$$

The random variables $M_{t_n}^{(j)}$ for $j = 1, \ldots, a_n^2$ are independent and identically distributed, because they refer to different excursions. Then we conclude by (6.1):

$$\mathbf{Q}_{N_n}\left(\sup_{t\in[0,1]} |f_t| > 1\right) = 1 - \left(1 - P_{t_n}^{\mathrm{bulk}}\left(\frac{M_{t_n}}{\sqrt{t_n}} > a_n\right)\right)^{a_n^2} \geq 1 - \left(1 - \frac{\eta}{a_n^2}\right)^{a_n^2} \xrightarrow[n\to\infty]{} 1 - e^{-\eta}\,.$$

Sufficiency. We assume that conditions (1) and (2) in Theorem 3.2 hold and we prove that $(\mathbf{Q}_N)_{N\in\mathbb{N}}$ is tight in $C([0,1])$, that is

$$\forall \eta > 0: \qquad \lim_{\delta\downarrow 0} \sup_{N\in\mathbb{N}} \mathbf{Q}_N\big(\Gamma(\delta) > \eta\big) = 0\,, \tag{6.3}$$

where $\Gamma(\delta)(f) := \sup_{|t-s|\leq\delta} |f_t - f_s|$ denotes the continuity modulus of $f \in C([0,1])$.

Given a finite subset $U = \{u_1 < \ldots < u_n\} \subseteq [0,1]$ and points $s, t \in [0,1]$, we write $s \sim_U t$ iff no point $u_i \in U$ lies between s and t. Then we define

$$\tilde{\Gamma}_U(\delta)(f) := \sup_{s,t\in[0,1]:\ s\sim_U t,\ |t-s|\leq\delta} |f_t - f_s|\,.$$

Plainly, if $f(u_i) = 0$ for all $u_i \in U$, then $\Gamma(\delta)(f) \leq 2\,\tilde{\Gamma}_U(\delta)(f)$. This means that in (6.3) we can replace $\Gamma(\delta)(f)$ by $\tilde{\Gamma}_U(\delta)(f)$, where U is any subset of $[0,1]$ on which f vanishes. We fix $U = \{\frac{t_1}{N}, \ldots, \frac{t_n}{N}\}$, where t_i are the regeneration epochs of $\mathbf{P}_N$. It remains to show that

$$\forall \eta > 0: \qquad \lim_{\delta\downarrow 0} \sup_{N\in\mathbb{N}} \mathbf{Q}_N\big(\tilde{\Gamma}_U(\delta) > \eta\big) = 0\,. \tag{6.4}$$

We set for short $Q_t^{\rm fin} := P_t^{\rm fin} \circ \mathcal{R}_t^{-1}$ and $Q_t^{\rm bulk} := P_t^{\rm bulk} \circ \mathcal{R}_t^{-1}$. By Definition 3.1

$$\mathbf{Q}_N\big(\tilde\Gamma_U(\delta) \le \eta\big) = \sum_{n=1}^{N+1} \sum_{0=t_1<\ldots<t_n\le N} p_N(\{t_1,\ldots,t_n\}) \prod_{i=1}^{n-1} Q^{\rm bulk}_{t_{i+1}-t_i}\Big(\Gamma(\tfrac{N}{t_{i+1}-t_i}\delta) \le \eta\sqrt{\tfrac{N}{t_{i+1}-t_i}}\Big) \times \\ \times\, Q^{\rm fin}_{N-t_n}\Big(\Gamma(\tfrac{N}{N-t_n}\delta) \le \eta\sqrt{\tfrac{N}{N-t_n}}\Big)\,.$$

Note that we have the original continuity modulus Γ. Let us set

$$g_\eta^{\rm bulk}(\delta) := \inf_{\substack{N\in\mathbb{N},\ 2\le n\le N+1,\\ 0=t_1<\ldots<t_n\le N}} \prod_{i=1}^{n-1} Q^{\rm bulk}_{t_{i+1}-t_i}\Big(\Gamma(\tfrac{N}{t_{i+1}-t_i}\delta) \le \eta\sqrt{\tfrac{N}{t_{i+1}-t_i}}\Big) \tag{6.5}$$

$$g_\eta^{\rm fin}(\delta) := \inf_{N\in\mathbb{N},\ 1\le t<N} Q^{\rm fin}_{N-t}\Big(\Gamma(\tfrac{N}{N-t}\delta) \le \eta\sqrt{\tfrac{N}{N-t}}\Big)\,,$$

so that we can bound $\mathbf{Q}_N\big(\tilde\Gamma_U(\delta)\le\eta\big) \ge g_\eta^{\rm bulk}(\delta)\, g_\eta^{\rm fin}(\delta)$ (we recall that $p_N(\cdot)$ is a probability). We complete the proof of (6.4) by showing that

$$\forall \eta>0: \qquad \lim_{\delta\downarrow 0} g_\eta^{\rm bulk}(\delta)\, g_\eta^{\rm fin}(\delta) \ge 1\,.$$

We first show that $\lim_{\delta\downarrow0} g_\eta^{\rm fin}(\delta)\ge 1$, for every $\eta>0$. We fix $\theta\in(0,1)$ and consider two regimes. For $t<(1-\theta)N$ we can bound (recall that $(Q_\ell^{\rm fin})_{\ell\in\mathbb{N}}$ is tight by assumption)

$$\inf_{N\in\mathbb{N},\ 1\le t<(1-\theta)N} Q^{\rm fin}_{N-t}\Big(\Gamma(\tfrac{N}{N-t}\delta) \le \eta\sqrt{\tfrac{N}{N-t}}\Big) \ge \inf_{\ell\in\mathbb{N}} Q_\ell^{\rm fin}\big(\Gamma(\tfrac{\delta}{\theta})\le\eta\big) \xrightarrow[\delta\downarrow0]{} 1\,.$$

On the other hand, for $t\ge(1-\theta)N$ we can bound

$$\inf_{N\in\mathbb{N},\ (1-\theta)N\le t<N} Q^{\rm fin}_{N-t}\Big(\Gamma(\tfrac{N}{N-t}\delta) \le \eta\sqrt{\tfrac{N}{N-t}}\Big) \ge \inf_{\ell\in\mathbb{N}} Q_\ell^{\rm fin}\Big(\max_{s\in[0,1]}|f_s|\le\tfrac12\tfrac{\eta}{\sqrt\theta}\Big) =: h_\eta(\theta)\,.$$

For any $\eta>0$, we have $\lim_{\delta\downarrow0} g_\eta^{\rm fin}(\delta) \ge \lim_{\theta\downarrow0} h_\eta(\theta) = 1$, by the tightness of $(Q_\ell^{\rm fin})_{\ell\in\mathbb{N}}$.

To complete the proof, we show that $\lim_{\delta\downarrow0} g_\eta^{\rm bulk}(\delta)\ge 1$, for every $\eta>0$. Note that

$$\inf_{t\in\mathbb{N}} Q_t^{\rm bulk}\Big(\max_{s\in[0,1]}|f_s|\le a\Big) = \inf_{t\in\mathbb{N}} P_t^{\rm bulk}\Big(\max_{i=0,\ldots,t}|x_i|\le a\sqrt{t}\Big) \ge 1-\frac{\epsilon(a)}{a^2}\,,$$

where $\lim_{a\uparrow\infty}\epsilon(a)=0$, by assumption (2). We may assume that $a\mapsto\epsilon(a)$ is non increasing. Fix $\theta\in(0,1)$. Given a family of epochs $0\le t_1<\ldots<t_n\le N$, we distinguish two cases.

- For $\theta N < t_{i+1}-t_i \le N$ we can bound

$$Q^{\rm bulk}_{t_{i+1}-t_i}\Big(\Gamma(\tfrac{N}{t_{i+1}-t_i}\delta) \le \eta\sqrt{\tfrac{N}{t_{i+1}-t_i}}\Big) \ge \inf_{t\in\mathbb{N}} Q_t^{\rm bulk}\Big(\Gamma(\tfrac{\delta}{\theta})\le\eta\Big) =: F_{\eta,\theta}(\delta)\,,$$

  and note that for fixed η,θ we have $\lim_{\delta\downarrow0} F_{\eta,\theta}(\delta)=1$, because $(Q_t^{\rm bulk})_{t\in\mathbb{N}}$ is tight.
- For $t_{i+1}-t_i\le\theta N$ we can bound

$$Q^{\rm bulk}_{t_{i+1}-t_i}\Big(\Gamma(\tfrac{N}{t_{i+1}-t_i}\delta) \le \eta\sqrt{\tfrac{N}{t_{i+1}-t_i}}\Big) \ge Q^{\rm bulk}_{t_{i+1}-t_i}\Big(\max_{s\in[0,1]}|f_s|\le\tfrac{\eta}{2}\sqrt{\tfrac{N}{t_{i+1}-t_i}}\Big) \\ \ge 1-\tfrac{4(t_{i+1}-t_i)}{\eta^2\,N}\,\epsilon\big(\tfrac{\eta}{2\sqrt\theta}\big) \ge \exp\Big(-\tfrac{8(t_{i+1}-t_i)}{\eta^2\,N}\,\epsilon\big(\tfrac{\eta}{2\sqrt\theta}\big)\Big)\,,$$

  where the last inequality holds for $\theta>0$ small, by $1-z\ge e^{-2z}$ for $z\in[0,\frac12]$.

We can have $t_{i+1}-t_i>\theta N$ for at most $\lfloor 1/\theta\rfloor$ values of i, hence

$$g_\eta^{\rm bulk}(\delta) \ge F_{\eta,\theta}(\delta)^{\frac1\theta} \prod_{i=1}^{n-1}\exp\Big(-\tfrac{8(t_{i+1}-t_i)}{\eta^2\,N}\,\epsilon\big(\tfrac{\eta}{2\sqrt\theta}\big)\Big) \ge F_{\eta,\theta}(\delta)^{\frac1\theta}\,\exp\Big(-\tfrac{8}{\eta^2}\,\epsilon\big(\tfrac{\eta}{2\sqrt\theta}\big)\Big)\,.$$

Given $\eta > 0$ and $\epsilon > 0$, we first fix $\theta > 0$ small enough, so that the exponential is greater than $1 - \epsilon$; then we let $\delta \to 0$, so that $F_{\eta,\theta}(\delta)^{\frac{1}{\theta}} \to 1$. This yields $\lim_{\delta \downarrow 0} g_\eta^{\mathrm{bulk}}(\delta) \geq 1 - \epsilon$. As $\epsilon > 0$ was arbitrary, we get $\lim_{\delta \downarrow 0} g_\eta^{\mathrm{bulk}}(\delta) \geq 1$. □

Acknowledgements

I thank Tal Orenshtein for reviving the interest on the problem and for stimulating email exchanges. I also thank Denis Denisov, Ron Doney, Giambattista Giacomin, Vitali Wachtel and Lorenzo Zambotti for discussions and references. This work is supported by the PRIN Grant 20155PAWZB "Large Scale Random Structures".

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