A NOTE ON DIRECTLY RIEMANN INTEGRABLE FUNCTIONS

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Abstract. A non-negative function f, defined on the real line or on a half-line, is said to be *directly Riemann integrable* (d.R.i.) if the upper and lower Riemann sums of f over the whole (unbounded) domain converge to the same finite limit, as the mesh of the partition vanishes. In this note we show that, for a Lebesgue-integrable function f, very mild conditions are enough to ensure that some n-fold convolution of f with itself is d.R.i.. Applications to renewal theory and to local limit theorems are discussed.

1. Introduction

1.1. The problem. A non-negative function $g : \mathbb{R} \to [0, \infty)$ is said to be *directly Riemann integrable* (d.R.i.) if its upper and lower Riemann sums *over the whole real line* converge to the same finite limit, as the mesh of the partition vanishes:

$$\lim_{\delta \downarrow 0} \sum_{m \in \mathbb{Z}} \delta \left(\sup_{z \in [m\delta, (m+1)\delta)} g(z) \right) = \lim_{\delta \downarrow 0} \sum_{m \in \mathbb{Z}} \delta \left(\inf_{z \in [m\delta, (m+1)\delta)} g(z) \right) \in (-\infty, +\infty) . \qquad (1.1)$$

If the function g may also take negative values, it is said to be d.R.i. if both its positive and negative parts g^+ and g^- are so. We refer to [1, §V.4] and [8, §XI.1] for more details.

Every d.R.i. function is necessarily in $L^1(\mathbb{R}, Leb)$ and vanishes at infinity, but the converse might fail, even for continuous functions, because of the possible oscillations at infinity. The aim of this note is to show that very mild conditions on $f \in L^1(\mathbb{R}, Leb)$ are enough to ensure that some convolution of f with itself is d.R.i., cf. Theorem 1.1 below.

Beyond its intrinsic interest, our main motivations for such a result come from local limit theorems and renewal theory, where d.R.i. functions play an important role. In particular, we suggest to keep in mind the special case when f is a probability density function on $\mathbb{R}$.

1.2. The main result. Given a Lebesgue-integrable function $f \in L^1(\mathbb{R}, Leb)$, let us denote by $f_k(\cdot) = f^{*k}(\cdot)$ the k-fold convolution of f with itself, that is

$$f_1(x) := f(x) , \qquad f_{k+1}(x) := (f_k * f)(x) = \int_{\mathbb{R}} f_k(x-y)\, f(y)\, \mathrm{d}y , \quad \forall k \in \mathbb{N} . \qquad (1.2)$$

Our main result, that we prove in section 3, reads as follows.

Theorem 1.1. *Let $f \in L^1(\mathbb{R}, Leb)$ satisfy the following assumptions:*

(1) $f_{k_0} \in L^\infty(\mathbb{R}, Leb)$ *for some* $k_0 \in \mathbb{N}$*;*
(2) $\int_{\mathbb{R}} |x|^\varepsilon |f(x)|\, \mathrm{d}x < \infty$ *for some* $\varepsilon > 0$.

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Then there exists $k_1 \in \mathbb{N}$ such that for every $k \ge k_1$ the function $x \mapsto (1+|x|^\varepsilon) f_k(x)$ is bounded, continuous and directly Riemann integrable. In particular, f_k itself is bounded, continuous and directly Riemann integrable, for every $k \ge k_1$.

1.3. **Organization of the paper.** The rest of the introduction is devoted to discussing the role of the two assumptions of Theorem 1.1. In section 2 we present some applications of Theorem 1.1 to renewal theory (and, more generally, to local limit theorems, cf. §2.3). We mention in particular Proposition 2.2, which provides a local version of the renewal theorem for heavy-tailed renewal processes. The proof of Theorem 1.1 is contained in section 3, while some technical points are deferred to the appendix.

Henceforth we write $L^p := L^p(\mathbb{R}, Leb)$ for short.

1.4. **Discussion.** A d.R.i. function is necessarily bounded, since otherwise every upper or lower Riemann sum would be infinite. Therefore *assumption* (1) *is necessary for Theorem 1.1 to hold.* Let us now give a standard and more concrete reformulation of this assumption in terms of the Fourier transform $\widehat{f}(\vartheta) := \int_{\mathbb{R}} e^{i\vartheta x} f(x)\, \mathrm{d}x$ of f.

Lemma 1.2. *A function $f \in L^1$ satisfies assumption* (1) *of Theorem 1.1 if and only if $\widehat{f} \in L^p$ for some $p \in [1,\infty)$.*

From this, we can deduce a very practical sufficient condition.

Lemma 1.3. *Assumption* (1) *of Theorem 1.1 is satisfied if $f \in L^1 \cap L^p$, for some $p \in (1,\infty]$.*

The (quite standard) proofs of these two lemmas are given in §A.1 and §A.2 below.

Remark 1.4. For a function $f \in L^1$, the condition $f \in L^p$ for some $p > 1$ is very mild and shows that assumption (1) is typically verified in concrete situations. Let us mention, however, that (somewhat pathological) examples of functions $f \in L^1$ not satisfying assumption (1) do exist, cf. Examples (a), (b), (c) in [8, §XV.5].

Remark 1.5. When f is a probability density with zero mean and unit variance, assumption (1) is a necessary and sufficient condition for the Local Central Limit Theorem, that is, for $\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |\sqrt{n} f_n(\sqrt{n}x) - \frac{1}{\sqrt{2\pi}} e^{-x^2/2}| = 0$, cf. [8, Theorem 2] and the following lines.

Let us now discuss assumption (2). This is *not* necessary for Theorem 1.1 to hold, as the following example shows: the function $f(x) := \frac{1}{x\,(\log x)^2} \mathbb{1}_{[e,\infty)}(x)$ is in L^1, it does not satisfy assumption (2), for any $\varepsilon > 0$, but f is d.R.i.. Only the latter statement requires a proof. Being Riemann integrable on every compact interval (it is continuous except at the point e), it suffices to verify that an upper Riemann sum of f is finite, by Lemma 3.1 below. Since $f(x) = 0$ for $x < e$ and f is *decreasing* on $[e,\infty)$, we have

$$\sum_{m\in\mathbb{Z}} \sup_{z\in[m,m+1)} f(z) \;=\; \sum_{m\in\mathbb{N},\, m\ge 2} \sup_{z\in[m,m+1)} f(z) \;\le\; \frac{2}{e} + \int_3^\infty f(z)\, \mathrm{d}z \;<\; \infty \,,$$

where $\frac{2}{e}$ accounts for the terms $m = 2, 3$ of the sum, while for $m \ge 4$ we have used the estimate $\sup_{z\in[m,m+1)} f(z) \le \int_{m-1}^m f(z)\, \mathrm{d}z$, by monotonicity of f.

Remark 1.6. The same argument shows that if some convolution f_k is bounded, continuous and dominated in absolute value by $g \in L^1$, with g non-increasing in a neighborhood of infinity, then f_k is d.R.i.. Such a condition, however, seems difficult to check in terms of f.

Remark 1.7. Although not necessary, assumption (2) is very mild and easily satisfied in most situations. For instance, when f is the probability density of a random variable X, we can write, for every $\varepsilon > 0$,

$$\int_{\mathbb{R}} |x|^\varepsilon f(x)\,\mathrm{d}x \;=\; \mathrm{E}(|X|^\varepsilon) \;=\; \int_0^\infty \mathrm{P}(|X|^\varepsilon > s)\,\mathrm{d}s \;=\; \varepsilon \int_0^\infty \frac{\mathrm{P}(|X| > t)}{t^{1-\varepsilon}}\,\mathrm{d}t\,. \tag{1.3}$$

It follows, in particular, that *assumption* (2) *always holds for the density f of a random variable X in the domain of attraction of a stable law, of any index $\alpha \in (0,2]$*, because it is well-known that in this case $\mathrm{E}(|X|^\varepsilon) < \infty$ for every $0 < \varepsilon < \alpha$.

Remark 1.8. We don't know whether assumption (2) can be completely eliminated from Theorem 1.1. In other terms, we are not aware of examples of functions $f \in L^1$ satisfying assumption (1) — and necessarily *not* satisfying assumption (2) — such that no convolution f_k is d.R.i., for any $k \in \mathbb{N}$. We point out that, if they exist, such counterexamples can be found in the class of bounded and continuous functions that vanish at infinity, because assumption (1) entails that f_{k_0+1} has these properties, by Lemma 3.2 below.

2. Applications to renewal theory

If $\{X_n\}_{n\in\mathbb{N}}$ are independent, identically distributed *non-negative* random variables, the associated random walk started at zero, that is $S_0 := 0$ and $S_n := X_1 + \ldots + X_n$ for every $n \in \mathbb{N}$, is called (undelayed) *renewal process*. The corresponding renewal measure $U(\cdot)$ is the σ-finite Borel measure on $[0,\infty)$ defined by

$$U(A) \;:=\; \mathrm{E}\big(\#\{n \in \mathbb{N}_0 :\; S_n \in A\}\big) \;=\; \sum_{n\in\mathbb{N}_0} \mathrm{P}(S_n \in A)\,, \quad \text{for Borel } A \subseteq [0,\infty)\,. \tag{2.1}$$

When $\mu := \mathrm{E}(X_1) \in (0,\infty)$ and the law of X_1 is non-lattice, Blackwell's renewal theorem states that, for every fixed $\delta > 0$,

$$\lim_{x\to+\infty} U\big([x, x+\delta)\big) \;=\; \frac{\delta}{\mu}\,. \tag{2.2}$$

This means that, roughly speaking, the measure $U(\cdot)$ is asymptotically close to $\frac{1}{\mu}$ times the Lebesgue measure. It is therefore natural to conjecture that, for suitable $g : [0,\infty) \to \mathbb{R}$,

$$\lim_{x\to+\infty} \int_{[0,\infty)} g(x-z)\,U(\mathrm{d}z) \;=\; \frac{1}{\mu}\left(\int_{[0,\infty)} g(z)\,\mathrm{d}z\right). \tag{2.3}$$

This relation indeed holds *whenever g is d.R.i.* (but can fail for general $g \in L^1$) and is known as the *key renewal theorem*. This is how d.R.i. functions appear in renewal theory.

Note that taking $g = \mathbb{1}_{(0,\delta]}$ one recovers Blackwell's renewal theorem (2.2). We point out that relations (2.2), (2.3) hold also when $\mu = +\infty$, the right hand side being interpreted as zero, but of course they give much less information (we come back on this point below). We refer to [1, §V] for more details on renewal theory.

2.1. **On the renewal density theorem.** Consider now the case when the law of X_1 is absolutely continuous, with density f, and always assume that $\mu = \mathrm{E}(X_1) \in (0,\infty)$. Then

$$U(\mathrm{d}x) \;=\; \delta_0(\mathrm{d}x) \;+\; \left(\sum_{n=1}^\infty f_n(x)\right)\mathrm{d}x \;=:\; \delta_0(\mathrm{d}x) \;+\; u(x)\,\mathrm{d}x\,, \tag{2.4}$$

where we recall that $f_n = f^{*n}$ denotes the n-fold convolution of f with itself, cf. (1.2). Therefore, excluding the Dirac mass at zero due to S_0, the renewal measure $U(\cdot)$ is absolutely continuous, with density $u(x)$. It is tempting to deduce from (2.2) the corresponding relation for the density, namely

$$\lim_{x\to+\infty} u(x) \;=\; \frac{1}{\mu}\,, \tag{2.5}$$

sometimes called *renewal density theorem*. However, additional conditions on f are needed for (2.5) to hold: for instance, *it is necessary that* $\lim_{x\to+\infty} f(x) = 0$, as proved by Smith [12, §4] (generalizing an earlier result by Feller [7]). This is rather intuitive, because any fixed term in the sum (2.1) gives no contribution to the asymptotic behavior of $U([x, x+\delta))$ — since $\lim_{x\to+\infty} \mathrm{P}(S_n \in [x, x+\delta)) = 0$ for every fixed $n \in \mathbb{N}$ — while this is not the case for $u(x)$ if the density f does not vanish at infinity, cf. (2.4).

Sharp necessary and sufficient conditions on f for the validity of the renewal density theorem (2.5) are known [12], but they are quite involved and implicit. A natural sufficient condition [10, 11] is simply that $f \in L^p$ for some $p \in (1,\infty]$ (in addition to $\mu = \int_{\mathbb{R}} x\, f(x)\, \mathrm{d}x \in (0,\infty)$ and $\lim_{x\to+\infty} f(x) = 0$). It is worth noting that the sufficiency of these conditions is an immediate corollary of our Theorem 1.1: in fact, if $\mu < \infty$, assumption (2) is satisfied with $\varepsilon = 1$, and if $f \in L^p$ with $p > 1$, assumption (1) is also satisfied, by Lemma 1.3; it follows that f_k is d.R.i. for some $k \in \mathbb{N}$, and by the key renewal theorem (2.3) we have

$$\lim_{x\to+\infty} \int_{\mathbb{R}} f_k(x-z)\, U(\mathrm{d}z) \;=\; \lim_{x\to+\infty} \left(\sum_{n=k}^{\infty} f_n(x) \right) \;=\; \frac{1}{\mu} \int_{\mathbb{R}} f_k(z)\, \mathrm{d}z \;=\; \frac{1}{\mu}\,, \tag{2.6}$$

where in the second equality we have used (2.4) and the fact that $f_i * f_j = f_{i+j}$. We can rewrite this relation as

$$\lim_{x\to+\infty} \left(u(x) \;-\; \sum_{n=1}^{k-1} f_n(x) \right) \;=\; \frac{1}{\mu}\,. \tag{2.7}$$

It follows easily by (1.2) that

$$f_{n+1}(x) \;\le\; \sup_{y \ge x/2} \big(f(y) + f_n(y) \big)\,, \qquad \forall x \in [0,\infty),\ \forall n \in \mathbb{N}\,. \tag{2.8}$$

If $\lim_{x\to+\infty} f(x) = 0$, relation (2.8) shows by induction that also $\lim_{x\to+\infty} f_n(x) = 0$, for every fixed $n \in \mathbb{N}$, hence relation (2.5) follows from (2.7).

Remark 2.1. The idea of deriving the renewal density theorem (2.5) from the key renewal theorem (2.3) is a classical one, dating back at least to Feller, who proved the validity of (2.7) with $k = 2$ when $f \in L^\infty$, showing that f_2 is d.R.i., cf. Theorem 2a in [8, §XV.3]. Feller's proof is based on the simple observation that, by (1.2) and a symmetry argument,

$$f_2(x) \;=\; 2 \int_{z > x/2} f(z)\, f(x-z)\, \mathrm{d}z \;\le\; 2\, \|f\|_\infty \int_{z > x/2} f(z) \;=\; 2\, \|f\|_\infty\, \mathrm{P}\left(X_1 > \tfrac{x}{2}\right). \tag{2.9}$$

Since by assumption

$$\int_0^\infty \mathrm{P}\left(X_1 > \tfrac{x}{2}\right) \mathrm{d}x \;=\; 2\,\mathrm{E}(X_1) \;\in\; (0,\infty)\,,$$

the function f_2 is dominated by a non-increasing, integrable function, hence it is d.R.i..

We point out that the generalization that we sketched above, in which the assumption $f \in L^\infty$ is relaxed to $f \in L^p$ for some $p > 1$, is a rather elementary upgrade: since a convolution f_k of f is bounded, thanks to Lemma 1.3, applying relation (2.9) to f_k yields immediately that f_{2k} is d.R.i., allowing to deduce (2.7) (with k replaced by $2k$) from the key renewal theorem (2.3). Let us stress, however, that to deduce direct Riemann integrability from (2.9), the *finiteness of the mean* $\mathrm{E}(X_1)$ is essential.

2.2. **The heavy-tailed case.** The novelty of Theorem 1.1 is that assumption (2) only requires the finiteness of an arbitrarily small moment, allowing in particular to deal with cases when the mean is infinite. This is especially interesting from the viewpoint of heavy-tailed renewal theory. More precisely, keeping the notation of the beginning of this section, assume that the law of X_1 is non-lattice and satisfies the following relation, for some $\alpha \in (0, 1]$:

$$\mathrm{P}(X_1 > x) \sim \frac{L(x)}{x^\alpha} \qquad \text{as } x \to +\infty\,, \tag{2.10}$$

where $L(\cdot)$ is a slowly varying function [2]. For $\alpha < 1$, relation (2.10) is the same as requiring that X_1 is in the domain of attraction of the (unique up to multiples) positive stable law of index α (cf. Theorem 8.3.1 in [2]), while for $\alpha = 1$ relation (2.10) implies that X_1 is relatively stable (cf. Theorem 8.8.1, Corollary 8.1.7 and the following lines in [2]).

When $\mu = \mathrm{E}(X_1) = +\infty$ (in particular, for every $\alpha < 1$), Blackwell's renewal theorem (2.2) only says that $\lim_{x\to+\infty} U([x, x+\delta)) = 0$. Sharpenings of this relation have been proved by Erickson [6, Theorems 1–4], extending the corresponding results for lattice distributions obtained by Garsia and Lamperti [9]. Introducing the truncated mean function

$$m(x) := \int_0^x \mathrm{P}(X_1 > y)\,\mathrm{d}y = x\,\mathrm{P}(X_1 > x) + \mathrm{E}(X_1 1_{\{X_1 < x\}})\,, \qquad \forall x \geq 0\,, \tag{2.11}$$

it follows from (2.10) that $m(x) \sim \frac{1}{1-\alpha} L(x) x^{1-\alpha}$ as $x \to +\infty$. The generalized version of Blackwell's renewal theorem then reads as follows: for every fixed $\delta > 0$

$$\liminf_{x\to+\infty} m(x)\, U\big([x, x+\delta)\big) = \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)}\,\delta\,, \tag{2.12}$$

and when $\alpha \in (\frac{1}{2}, 1]$ the lim inf in this relation can be upgraded to a true limit. A generalized version of the key renewal theorem is also available: for every d.R.i. function $g : [0, \infty) \to \mathbb{R}$

$$\liminf_{x\to+\infty} m(x) \int_{[0,\infty)} g(x-z)\, U(\mathrm{d}z) = \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} \left(\int_{[0,\infty)} g(y)\,\mathrm{d}y \right). \tag{2.13}$$

Furthermore, when $\alpha \in (\frac{1}{2}, 1]$ and $g(x) = O(1/x)$ as $x \to +\infty$, also in this relation the lim inf can be upgraded to a true limit. The reason for the presence of lim inf instead of lim is discussed in Remark 2.4 below. Apart from that, note that relations (2.12), (2.13) match perfectly with (2.2), (2.3), because $\mu = \mathrm{E}(X_1) = m(\infty)$.

Assume now that X_1 is absolutely continuous, with a density f. The density u of the renewal measure U is always defined by (2.4), and it is natural to ask whether the density version of (2.12) holds true. As a corollary of Theorem 1.1, we obtain the following result, which seems to be new.

Proposition 2.2. *Let X_1 be a non-negative random variable satisfying* (2.10)*, for $\alpha \in (0, 1]$ and $L(\cdot)$ slowly varying. Assume that the law of X_1 is absolutely continuous, with a density f such that $f_k \in L^\infty$ for some $k \in \mathbb{N}$ (cf. Lemmas 1.2 and 1.3). Then, recalling the definitions* (2.4) *of $u(\cdot)$ and* (2.11) *of $m(\cdot)$, there exists $\overline{k} \in \mathbb{N}$ such that*

$$\liminf_{x\to+\infty} m(x) \left(u(x) - \sum_{n=1}^{\overline{k}-1} f_n(x) \right) = \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)}\,. \tag{2.14}$$

If furthermore $\lim_{x\to+\infty} f(x) = 0$, then

$$\liminf_{x\to+\infty} m(x)\, u(x) = \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)}\,. \tag{2.15}$$

Finally, if $\alpha \in (\frac{1}{2}, 1]$ *and* $f_k(x) = O(1/x)$ *as* $x \to +\infty$*, for some* $k \in \mathbb{N}$*, relations* (2.14) *and* (2.15) *hold with* lim *instead of* lim inf*.*

Proof. The density f satisfies the assumptions of Theorem 1.1, hence there exists $k_1 \in \mathbb{N}$ such that f_k is d.R.i. for every $k \geq k_1$. Choosing $\overline{k} = k_1$ and applying (2.13) with $g = f_{\overline{k}}$, we obtain immediately (2.14). We have already remarked that, by (2.8), if $\lim_{x\to+\infty} f(x) = 0$, then also $\lim_{x\to+\infty} f_n(x) = 0$, for every fixed $n \in \mathbb{N}$, hence relation (2.15) follows from (2.14). Finally, if $\alpha \in (\frac{1}{2}, 1]$ and $f_k(x) = O(1/x)$ for some $k \in \mathbb{N}$, then also $f_{nk}(x) = O(1/x)$ for every $n \in \mathbb{N}$, as it follows by induction from (2.8) applied to f_k. If n is large enough so that $\overline{k} := nk \geq k_1$, the function $f_{\overline{k}}(x)$ is both d.R.i. and $O(1/x)$, hence relation (2.13) holds with lim instead of lim inf and consequently the same is true for (2.14), (2.15). □

Remark 2.3. It is not clear whether the additional condition $g(x) = O(1/x)$, for the validity of the generalized key renewal theorem (2.13) with lim instead of lim inf when $\alpha \in (\frac{1}{2}, 1]$, formulated in [6, Theorem 3], is substantial or just technical. In any case, if that condition can be removed or relaxed, the same applies immediately to Proposition 2.2.

Remark 2.4. The presence of lim inf instead of lim, in relations (2.12) and (2.13) when $\alpha \in (0, \frac{1}{2}]$ but not when $\alpha \in (\frac{1}{2}, 1]$, may appear strange, but can be explained as follows. By (2.10), $\mathrm{P}(X_1 \in [x, x + \delta)) \leq \mathrm{P}(X_1 \geq x) \sim L(x)/x^\alpha$ as $x \to \infty$, and a similar estimate (up to a constant) holds for $\mathrm{P}(S_n \in [x, x + \delta))$, for every $n \in \mathbb{N}$. Since $m(x) \sim \frac{1}{1-\alpha} L(x)x^{1-\alpha}$ as $x \to +\infty$, when $\alpha < \frac{1}{2}$ we have $\mathrm{P}(S_n \in [x, x + \delta)) \ll 1/m(x)$. Recalling (2.12), this means that any term $\mathrm{P}(S_n \in [x, x + \delta))$, for fixed $n \in \mathbb{N}$, gives a negligible contribution to the asymptotic behavior of $U([x, x + \delta))$. This is no longer true when $\alpha > \frac{1}{2}$, as one can build examples of laws of X_1 satisfying (2.10) but such that $\mathrm{P}(X_1 \in [x, x + \delta))$ is anomalously close to $\mathrm{P}(X_1 \geq x) \sim L(x)/x^\alpha$, for (rare but) arbitrarily large values of x; in this way, the contribution of the single term $\mathrm{P}(X_1 \in [x, x+\delta))$ can be made much larger than the "typical" behavior of $U([x, x + \delta))$, that is $1/m(x)$, by (2.12). For more details, we refer to [9], [16].

In view of these considerations, it is natural to conjecture that, under some additional regularity assumptions on the distribution of X_1, it should be possible to upgrade the lim inf in the generalized Blackwell's renewal theorem (2.12), or in its density form (2.15), to a true lim also for $\alpha \in (0, \frac{1}{2}]$. This is indeed the case, as shown recently by Topchii in [13, Theorem 8.3] (generalizing the analogous results for lattice distributions by Doney [4, Theorem B]). More precisely, assume that X_1 satisfies (2.10); that it has an absolutely continuous law with density f, such that $f_k \in L^\infty$ for some $k \in \mathbb{N}$ (that is, assumption (1) in Theorem 1.1); and furthermore that there exist positive constants C, x_0 such that $f(x) \leq CL(x)/x^{1+\alpha}$ for all $x \geq x_0$, where $L(\cdot)$ is the same slowly varying function appearing in (2.10); then the generalized renewal density theorem (2.15), and consequently also the generalized Blackwell's theorem (2.12), holds with lim instead of lim inf.

2.3. **Application to local limit theorems.** Beyond renewal theory, Theorem 1.1 can be used to derive local limit theorems *for the density* of a random walk, even under conditioning, when the corresponding local limit theorems *à la Stone* are available.

For instance, let $\{X_n\}_{n\in\mathbb{N}}$ be independent, identically distributed real random variables, in the domain of attraction of a stable law, and denote by $(S = \{S_n\}_{n\in\mathbb{N}}, \mathrm{P}_x)$ the associated random walk started at $x \in \mathbb{R}$, that is $\mathrm{P}_x(S_0 = x) = 1$ and $S_n := S_{n-1} + X_n$ for all $n \in \mathbb{N}$. Let us also set $C_n := [0, \infty)^n \subseteq \mathbb{R}^n$. When the law of X_1 is non-lattice, local limit theorems in the Stone form — that is, for the probabilities of small intervals — for the law of S_n on the event $\{(S_1, \ldots, S_n) \in C_n\}$ are available, cf. [15, 5]. For example, there exist diverging sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and real functions φ, ψ such that, for any *fixed* $\delta > 0$, as $n \to \infty$

$$\mathrm{P}_x\left(S_n \in [y, y+\delta)\,,\,(S_1,\ldots,S_n) \in C_n\right) \;=\; \delta\,\frac{1}{b_n}\,\psi(x)\,\varphi\!\left(\frac{y}{a_n}\right)\big(1+o(1)\big)\,, \tag{2.16}$$

uniformly when $x/a_n \to 0$ and y/a_n is bounded away both from 0 and ∞.

Assume now that the law of X_1 is absolutely continuous, with density f, and denote by f_n^+ the density of S_n on the event $\{(S_1,\ldots,S_n) \in C_n\}$, i.e.

$$f_n^+(x,y) \;:=\; \frac{\mathrm{P}_x\left(S_n \in \mathrm{d}y\,,\,(S_1,\ldots,S_n) \in C_n\right)}{\mathrm{d}y}\,.$$

It is then very natural to conjecture that the density version of (2.16) holds, namely

$$f_n^+(x,y) \;=\; \frac{1}{b_n}\,\psi(x)\,\varphi\!\left(\frac{y}{a_n}\right)\big(1+o(1)\big)\,, \tag{2.17}$$

but some care is needed in order to interchange the limits $\delta \downarrow 0$ and $n \to \infty$. In fact, (2.17) is not true in general, as the density of S_n, and hence $f_n^+(x,y)$, might be unbounded for every $n \in \mathbb{N}$. However, this turns out to be the only possible pathology.

If we assume that the density of S_n is bounded for some $n \in \mathbb{N}$, then Theorem 1.1 may be applied (note that assumption (2) is automatically satisfied, since we assume that X_1 is in the domain of attraction of a stable law, as we already remarked). It follows that the density f_k of S_k, and hence $z \mapsto f_k^+(z,y)$, is d.R.i. for some $k \in \mathbb{N}$, and this allows to rigorously derive (2.17) from (2.16). We refer to [3, §5] for the technical details, but let us sketch the main idea, which is quite simple. For fixed $k \in \mathbb{N}$ and $n \geq k$ we can write

$$f_n^+(x,y) \;=\; \int_{[0,\infty)} \mathrm{d}z\, f_{n-k}^+(x,z)\, f_k^+(z,y)\,. \tag{2.18}$$

Since $x \mapsto f_k^+(x,y)$ is d.R.i., we can effectively approximate it with a step function, piecewise constant over disjoint intervals. The integral in the right hand side of (2.18) then becomes a sum of terms, each of which is like the left hand side of (2.16), with $n-k$ instead of n. Since k is fixed, as $n \to \infty$ relation (2.16) holds and (2.17) can be recovered from (2.18).

This approximation method is quite general, and may in principle be applied to other contexts (e.g., for other choices of the conditioning subsets $C_n \subseteq \mathbb{R}^n$). The message is that, whenever a local limit theorem in the Stone form is available, Theorem 1.1 provides a helpful tool in deriving the corresponding local limit theorem for the density.

3. Proof of Theorem 1.1

Let us first discuss the strategy of the proof. The starting point is Feller's observation (2.9), which shows that f_2 can be bounded from above by a non-increasing function, namely, the integrated tail of f. When f has finite mean (that is, assumption (2) holds with $\varepsilon = 1$), it follows that f_2 is d.R.i., but when the mean is infinite this bound is not enough.

The natural idea is then to bootstrap the estimate (2.9), deducing an estimate on f_4 from the bound on f_2, and so on, hoping that convolutions are regularizing enough so that for some $n \in \mathbb{N}$ the bound obtained for f_{2^n} yields the direct Riemann integrability. This turns out to be the case, though in a highly non straightforward way.

For convenience, we organize the proof in four steps.

3.1. **Some preliminary results.** Let us give a name to the (translated) upper and lower Riemann sums of a function $g : \mathbb{R} \to \mathbb{R}$: for $\delta \in (0, \infty)$ and $x \in \mathbb{R}$ we set

$$S_\delta^g(x) := \sum_{m \in \mathbb{Z}} \delta \left(\sup_{z \in [m\delta, (m+1)\delta)} g(z-x) \right), \quad s_\delta^g(x) := \sum_{m \in \mathbb{Z}} \delta \left(\inf_{z \in [m\delta, (m+1)\delta)} g(z-x) \right). \quad (3.1)$$

Note that both $S_\delta^g(x)$ and $s_\delta^g(x)$ are δ-periodic functions of x. Moreover, when g is non-negative, the following inequality holds, as we prove in §A.3:

$$S_\delta^g(x) \le \left(1 + 2\frac{\delta}{\delta'}\right) S_{\delta'}^g(x'), \qquad \forall x, x' \in \mathbb{R}, \ \forall \delta, \delta' > 0. \quad (3.2)$$

This shows in particular that the finiteness of $S_\delta^g(x)$ does not depend on δ, x.

Recall that, by equation (1.1), a non-negative function g is d.R.i. if and only if $S_\delta^g(0)$ and $s_\delta^g(0)$ converge to the same finite limit as $\delta \downarrow 0$. It is an easy exercise to prove the following lemma, which provides a useful reformulation of the d.R.i. condition.

Lemma 3.1. *A function $g : \mathbb{R} \to \mathbb{R}$ is d.R.i. if and only if it is Riemann integrable on every compact interval $[a, b] \subseteq \mathbb{R}$, and if in addition the upper Riemann sum $S_\delta^{|g|}(x)$ of $|g|$ is finite for some (hence all) $x \in \mathbb{R}$ and $\delta > 0$. In particular, every non-negative, continuous function with a finite upper Riemann sum is d.R.i..*

The following standard result will also be useful.

Lemma 3.2. *If $g \in L^1$ and $h \in L^1 \cap L^\infty$, then $g * f$ is bounded, continuous and vanishes at infinity (that is $\lim_{|x| \to +\infty} (g * f)(x) = 0$).*

Proof. If $g(x) = \sum_{i=1}^n c_i \mathbf{1}_{(a_i, b_i]}(x)$ is a step function, the theorem holds by direct verification, since $(g * h)(x) = \sum_{i=1}^n c_i \int_{[x-b_i, x-a_i)} h(z)\, \mathrm{d}z$ and $h \in L^1$. For every $g \in L^1$, take a sequence of step functions g_n such that $\|g - g_n\|_1 \to 0$. Since $|(g*h)(x) - (g_n * h)(x)| \le \|h\|_\infty \|g - g_n\|_1$ for every $x \in \mathbb{R}$, $g * h$ is the uniform limit of $g_n * h$ and the conclusion follows. ☐

3.2. **Setup.** If $f \in L^1(\mathbb{R}, Leb)$ satisfies assumption (1) of Theorem 1.1, it follows by Lemma 3.2 that that $f_k = f^{*k}$ is bounded and continuous (and vanishes at infinity) for every $k \ge k_0 + 1$. By Lemma 3.1, to prove Theorem 1.1 it suffices to show that an upper Riemann sum of $x \mapsto (1 + |x|^\varepsilon)|f_k(x)|$, say with mesh 1, is finite, that is

$$\sum_{m \in \mathbb{Z}} \left(\sup_{z \in [m, (m+1))} (1 + |z|^\varepsilon)\, |f_k(z)| \right) < \infty, \quad (3.3)$$

for all k large enough. Actually, if this relation holds for $k = \overline{k}$, one easily shows that it holds for every $k \ge \overline{k}$, cf. §A.4. Therefore it suffices to prove (3.3) for some $k \in \mathbb{N}$.

Since $|f_k| \le |f|_k$, that is $|f^{*k}| \le |f|^{*k}$, without loss of generality we may assume that the function f is non-negative (it suffices to replace f by $|f|$). Moreover, excluding the trivial case when $f = 0$ almost everywhere, in which there is nothing to prove, we may also impose the normalization $\int_\mathbb{R} f(x)\, \mathrm{d}x = 1$ (it suffices to multiply f by a constant). In this way, f may be viewed as a probability density. As a consequence, also f_k is a probability density: $f_k \ge 0$ and $\int_\mathbb{R} f_k(x)\, \mathrm{d}x = 1$, for all $k \in \mathbb{N}$. Let us set for convenience (recall assumption (2))

$$C := \int_\mathbb{R} |x|^\varepsilon f(x)\, \mathrm{d}x < \infty.$$

Since $|x_1 + \ldots + x_k|^\varepsilon \le (k \max_{1\le i\le k} |x_i|)^\varepsilon \le k^\varepsilon(|x_1|^\varepsilon + \ldots + |x_k|^\varepsilon)$, for every $k \in \mathbb{N}$

$$\int_{\mathbb{R}} |x|^\varepsilon f_k(x)\,\mathrm{d}x \;=\; \int_{\mathbb{R}^k} |x_1 + \ldots + x_k|^\varepsilon f(x_1)\cdots f(x_k)\,\mathrm{d}x_1\cdots\mathrm{d}x_k \;\le\; k^\varepsilon(kC) \;<\; \infty\,. \tag{3.4}$$

It follows in particular that f_k satisfies the hypothesis of Theorem 1.1, for every $k \in \mathbb{N}$. Therefore, we may assume that $f \in L^\infty$ (it suffices to replace f by f_{k_0}).

Summarizing, without any loss of generality, henceforth *we assume that f is a bounded probability density*, that is $f : \mathbb{R} \to \mathbb{R}$ is a measurable function such that

$$f(x) \ge 0 \quad \forall x \in \mathbb{R}\,, \qquad \int_{\mathbb{R}} f(x)\,\mathrm{d}x = 1\,, \qquad \sup_{x\in\mathbb{R}} f(x) \;<\; \infty\,, \tag{3.5}$$

and our goal is to show that (3.3) holds true for some $k \in \mathbb{N}$.

3.3. **A sequence of upper bounds.** By Markov's inequality, for all $t \ge 0$ we can write

$$g_1(t) \;:=\; \int_{|x|\ge t} f(x)\,\mathrm{d}x \;\le\; \min\left\{1, \frac{1}{t^\varepsilon}\int_{|x|\ge t} |x|^\varepsilon f(x)\,\mathrm{d}x\right\} \;\le\; \min\left\{1, \frac{C}{t^\varepsilon}\right\}. \tag{3.6}$$

Analogously, for every $k \in \mathbb{N}$, since $\mathbf{1}_{\{|x_1+\ldots+x_k|\ge t\}} \le \sum_{i=1}^k \mathbf{1}_{\{|x_i|\ge t/k\}}$, for all $t \ge 0$ we obtain

$$\begin{aligned} g_k(t) \;&:=\; \int_{|x|\ge t} f_k(x)\,\mathrm{d}x \;=\; \int_{\mathbb{R}^k} f(x_1)\cdots f(x_k)\,\mathbf{1}_{\{|x_1+\ldots+x_k|\ge t\}}\,\mathrm{d}x_1\cdots\mathrm{d}x_k \\ &\le\; \min\left\{1, k\,g_1\!\left(\frac{t}{k}\right)\right\} \;\le\; \min\left\{1, \frac{k^{1+\varepsilon}C}{t^\varepsilon}\right\}. \end{aligned} \tag{3.7}$$

For consistency and for later convenience, for $t \le 0$ we set $g_k(t) := g_k(0) = 1$, for all $k \in \mathbb{N}$. It follows from (3.7) that for all $k \in \mathbb{N}$

$$\int_{\mathbb{R}} g_k\!\left(\frac{|w|}{3} - 1\right)^p \mathrm{d}w \;=\; 6\,g_k(0) + 2\cdot 3\int_0^\infty g_k(t)^p\,\mathrm{d}t \;<\; \infty\,, \qquad \forall p \;>\; \frac{1}{\varepsilon}\,. \tag{3.8}$$

For every $n \in \mathbb{N}$, we define a map $\Phi_n : L^\infty \to L^\infty$ by

$$(\Phi_n(h))(x) \;:=\; 2\int_{|z|>(|x|-3)/2} \mathrm{d}z\, f_n(z)\,h(x-z)\,, \qquad \forall x \in \mathbb{R}\,. \tag{3.9}$$

The reason for such a definition is explained by the following crucial lemma, which will enable us to bound the left hand side of (3.3) by the integral of a suitable function.

Lemma 3.3. *Let $h \in L^\infty(\mathbb{R}, Leb)$, with $h \ge 0$, and $\ell \in \mathbb{N}$ be such that*

$$\sup_{x\in[a,a+1)} f_\ell(x) \le \int_{a-1}^{a+2} h(w)\,\mathrm{d}w\,, \qquad \forall a \in \mathbb{R}\,. \tag{3.10}$$

Then

$$\sup_{x\in[a,a+1)} f_{2\ell}(x) \le \int_{a-1}^{a+2} \Phi_\ell(h)(w)\,\mathrm{d}w\,, \qquad \forall a \in \mathbb{R}\,. \tag{3.11}$$

Proof. We start giving a couple of slight generalizations of (3.10). First, by a straightforward translation argument,

$$\sup_{x\in[a,a+1)} f_\ell(x-z) \;\le\; \int_{a-1}^{a+2} h(w-z)\,\mathrm{d}w\,, \qquad \forall a, z \in \mathbb{R}\,. \tag{3.12}$$

Next we claim that, for every $z \in \mathbb{R}$ and $t > 0$,

$$\sup_{x\in[a,a+1)} \Big(f_\ell(x-z) \mathbb{1}_{\{|x|<t\}} \Big) \le \int_{a-1}^{a+2} h(w-z) \mathbb{1}_{\{|w|<t+3\}} \, \mathrm{d}w \,, \qquad \forall a, z \in \mathbb{R} \,,\ \forall t > 0 \,. \tag{3.13}$$

In fact, if $t < |a| - 1$ then the left hand side of (3.13) is zero and there is nothing to prove. On the other hand, if $t \ge |a| - 1$ the right hand sides of (3.13) and (3.12) coincide, because $\mathbb{1}_{\{|w|\le t+3\}} = 1$ for all w in the domain of integration $(a-1, a+2)$, hence (3.13) follows from (3.12) simply because $f_\ell(x-z) \mathbb{1}_{\{|x|\le t\}} \le f_\ell(x-z)$.

Since $f_{2\ell} = f_\ell * f_\ell$, for all $x \ge 0$ we can write

$$f_{2\ell}(x) = \int_{\mathbb{R}} f_\ell(z) f_\ell(x-z) \, \mathrm{d}z = 2 \int_{z > x/2} f_\ell(z) f_\ell(x-z) \, \mathrm{d}z \,,$$

having exploited the symmetry $z \leftrightarrow x - z$. For $x \le 0$ we can write an analogous formula, with $\{z > x/2\}$ replaced by $\{z < -x/2\}$. We can combine these relations in the following single inequality:

$$f_{2\ell}(x) \le 2 \int_{|z|>|x|/2} f_\ell(z) f_\ell(x-z) \, \mathrm{d}z \,, \qquad \forall x \in \mathbb{R} \,, \tag{3.14}$$

hence

$$\sup_{x\in[a,a+1)} f_{2\ell}(x) \le 2 \int_{\mathbb{R}} \mathrm{d}z \, f_\ell(z) \sup_{x\in[a,a+1)} \Big(f_\ell(x-z) \mathbb{1}_{\{|x|<2|z|\}} \Big) \,, \qquad \forall a, x \in \mathbb{R},\ \forall t > 0 \,.$$

We now apply (3.13), getting

$$\begin{aligned} \sup_{x\in[a,a+1)} f_{2\ell}(x) &\le 2 \int_{\mathbb{R}} \mathrm{d}z \, f_\ell(z) \int_{a-1}^{a+2} \mathrm{d}w \, h(w-z) \mathbb{1}_{\{|w|<2|z|+3\}} \\ &= \int_{a-1}^{a+2} \mathrm{d}w \left(2 \int_{\mathbb{R}} \mathrm{d}z \, f_\ell(z) \, h(w-z) \mathbb{1}_{\{|w|<2|z|+3\}} \right) , \end{aligned}$$

which, recalling (3.9), is exactly (3.11). □

Applying iteratively Lemma 3.3 we now obtain a sequence of upper bounds for $f_{2^n}(\cdot)$. Let us start with $n = 1$, i.e., with $f_{2^1}(\cdot) = f_2(\cdot)$, showing that (3.10) holds for a suitable choice of $h(\cdot)$. Recalling (3.14) and (3.6), for all $x \in \mathbb{R}$ we can write

$$f_2(x) \;\le\; 2 \int_{|z|>|x|/2} \mathrm{d}z \, f(z) \, f(x-z) \;\le\; 2 \, \|f\|_\infty \int_{|z|>|x|/2} \mathrm{d}z \, f(z) \;=\; 2 \, \|f\|_\infty \, g_1(|x|/2) \,.$$

The function $x \mapsto g_1(|x|/2)$ is non-decreasing for $x \le 0$ and non-increasing for $x \ge 0$, hence for every $a \ge 1$ we have

$$\sup_{x\in[a,a+1)} f_2(x) \le 2 \, \|f\|_\infty \, g_1(a/2) \le \int_{a-1}^{a} 2 \, \|f\|_\infty \, g_1(|w|/2) \, \mathrm{d}w \,,$$

and analogously for $a \le -2$

$$\sup_{x\in[a,a+1)} f_2(x) \le 2 \, \|f\|_\infty \, g_1(|a+1|/2) \le \int_{a+1}^{a+2} 2 \, \|f\|_\infty \, g_1(|w|/2) \, \mathrm{d}w \,.$$

Altogether, for every $a \in \mathbb{R} \setminus [-2, 1]$

$$\sup_{x\in[a,a+1)} f_2(x) \le \int_{a-1}^{a+2} 2 \, \|f\|_\infty \, g_1(|w|/2) \, \mathrm{d}w \,. \tag{3.15}$$

Let us show that an analogous estimate holds also for $a \in [-2, 1]$. Note that the right hand side of (3.15) is a continuous function of a. Furthermore, it is strictly positive for $a \in [-2, 1]$, because $g_1(|w|/2)$ is strictly positive in a neighborhood of $w = 0$ — it is continuous and it equals one in zero, cf. (3.6) — and 0 is in the domain of integration $[a-1, a+2]$ for every $a \in [-2, 1]$. Therefore, the infimum of the right hand side of (3.15) over the compact interval $a \in [-2, 1]$, call it B, is strictly positive. As the left hand side of (3.15) is bounded from above by $\|f_2\|_\infty \le \|f\|_\infty$, it follows that relation (3.15) holds for $a \in [-2, 1]$ provided we multiply the right hand side by the constant $\|f\|_\infty / B$. Summarizing, for all $a \in \mathbb{R}$

$$\sup_{x \in [a, a+1)} f_2(x) \le D \int_{a-1}^{a+2} \overline{g}_1(|w|/2)\, \mathrm{d}w\,, \qquad \text{where} \quad D := 2\,\|f\|_\infty \max\left\{1, \frac{\|f\|_\infty}{B}\right\}. \tag{3.16}$$

We have thus shown that (3.10) holds true for f_2, with $h(\cdot) = D\,\overline{g}_1(|\cdot|/2)$. Applying iteratively Lemma 3.3, for every $n \in \mathbb{N}$ we obtain

$$\sup_{x \in [a, a+1)} f_{2^n}(x) \le \int_{a-1}^{a+2} \overline{h}_n(x)\, \mathrm{d}x\,, \qquad \forall a \in \mathbb{R}\,, \tag{3.17}$$

where recalling (3.16) we set

$$\begin{aligned} \overline{h}_1(x) &:= D\, g_1(|x|/2)\,, \\ \overline{h}_n(x) &:= \big(\Phi_{2^{n-1}}(\overline{h}_{n-1})\big)(x) \;=\; \big((\Phi_{2^{n-1}} \circ \Phi_{2^{n-2}} \circ \ldots \circ \Phi_2)(\overline{h}_1)\big)(x)\,, \qquad \forall n \ge 2\,. \end{aligned} \tag{3.18}$$

3.4. **Conclusion.** Let us observe that, for every $\varepsilon > 0$,

$$c_\varepsilon := \sup_{a \in \mathbb{R}} \left(\frac{\sup_{z \in [a-1, a+2]}(1 + |z|^\varepsilon)}{\inf_{z \in [a-1, a+2]}(1 + |z|^\varepsilon)} \right) < \infty\,,$$

and we can write $(1 + |x|^\varepsilon) \le c_\varepsilon (1 + |x'|^\varepsilon)$ for all $x, x' \in \mathbb{R}$ with $|x - x'| \le 3$. Applying (3.17), it follows that we can estimate the left hand side of (3.3) for $k = 2^n$ as follows:

$$\begin{aligned} \sum_{m \in \mathbb{Z}} \sup_{z \in [m, (m+1))} \big((1 + |z|^\varepsilon)\, f_{2^n}(z)\big) &\le \sum_{m \in \mathbb{Z}} c_\varepsilon\, (1 + |m|^\varepsilon) \sup_{z \in [m, (m+1))} f_{2^n}(z) \\ &\le \sum_{m \in \mathbb{Z}} c_\varepsilon\, (1 + |m|^\varepsilon) \int_{(m-1)}^{(m+2)} \overline{h}_n(w)\, \mathrm{d}w \;\le\; \sum_{m \in \mathbb{Z}} c_\varepsilon^2 \int_{(m-1)}^{(m+2)} (1 + |w|^\varepsilon)\, \overline{h}_n(w)\, \mathrm{d}w \\ &= 3\, c_\varepsilon^2 \int_{\mathbb{R}} (1 + |w|^\varepsilon)\, \overline{h}_n(w)\, \mathrm{d}w\,. \end{aligned}$$

Therefore to prove (3.3) it suffices to show that there exists $n \in \mathbb{N}$ large enough such that

$$\int_{\mathbb{R}} (1 + |w|^\varepsilon)\, \overline{h}_n(w)\, \mathrm{d}w \;<\; \infty\,, \tag{3.19}$$

where $\overline{h}_n(\cdot)$ is defined in (3.18). To this purpose, we show that the maps Φ_n are regularizing.

Lemma 3.4. *If $h \in L^\infty \cap L^q$, for some $q \in (1, \infty)$, then $\Phi_n(h) \in L^p$ for every $p \in ((1-\varepsilon)q, \infty) \cap [1, \infty)$ and for every $n \in \mathbb{N}$.*

Proof. Recalling the definition (3.9) of the operator Φ_n, it follows by Jensen's inequality that for every $h \in L^\infty$ and $x \in \mathbb{R}$

$$|\Phi_n(h)(x)|^p \;\le\; 2^p \int_{|z| > (|x| - 3)/2} \mathrm{d}z\, f_n(z)\, |h(x - z)|^p\,, \qquad \forall p \ge 1\,, \tag{3.20}$$

hence

$$\|\Phi_n(h)\|_p^p = \int_{\mathbb{R}} \mathrm{d}x\, |\Phi_n(h)(x)|^p \le 2^p \int_{\mathbb{R}} \mathrm{d}z\, f_n(z) \int_{|x|<2|z|+3} \mathrm{d}x\, |h(x-z)|^p .$$

Under the change of variables $x \to w := x - z$, the domain $\{|x| < 2|z| + 3\}$ becomes $\{-(2|z|+3) - z < w < (2|z|+3) - z\} \subseteq \{-3(|z|+1) < w < 3(|z|+1)\}$. Therefore enlarging the domain of integration and recalling that $g_n(t) := \int_{|z|>t} \mathrm{d}z\, f_n(z)$ we obtain

$$\begin{aligned} \|\Phi_n(h)\|_p^p &\le 2^p \int_{\mathbb{R}} \mathrm{d}z\, f_n(z) \int_{|w|<3(|z|+1)} \mathrm{d}w\, |h(w)|^p \\ &= 2^p \int_{\mathbb{R}} \mathrm{d}w\, |h(w)|^p \int_{|z|>|w|/3-1} \mathrm{d}z\, f_n(z) = 2^p \int_{\mathbb{R}} \mathrm{d}w\, |h(w)|^p\, g_n(|w|/3-1) , \end{aligned}$$

where we recall that $g_n(t) := g_n(0) = 1$ for $t < 0$. For every $\gamma \in (0,1)$, by Hölder's inequality we then obtain

$$\|\Phi_n(h)\|_p^p \le 2^p \left(\int_{\mathbb{R}} \mathrm{d}w\, |h(w)|^{p/(1-\gamma)} \right)^{1-\gamma} \left(\int_{\mathbb{R}} \mathrm{d}w\, g_n(|w|/3-1)^{1/\gamma} \right)^{\gamma} . \tag{3.21}$$

Looking back at (3.8), we see that the second integral in the right hand side is finite for all $\gamma \in (0,\varepsilon)$. Now observe that if $h \in L^\infty \cap L^q$ then $h \in L^{q'}$ for all $q' \ge q$, therefore the first integral in the right hand side is finite whenever $p/(1-\gamma) \ge q$. Summarizing, we have shown that $\|\Phi_n(h)\|_p < \infty$, that is $\Phi_n(h) \in L^p$, for every $p \in [1,\infty)$ (recall relation (3.20)) such that $p/(1-\gamma) \ge q$ for some $\gamma \in (0,\varepsilon)$, i.e., for every $p \in [1,\infty) \cap ((1-\varepsilon)q, \infty)$. □

We are almost done. Recall that we need to show that (3.19) holds true for n large enough, where $\overline{h}_n(\cdot)$ is defined in (3.18). By (3.6), we know that $\|\overline{h}_1\|_q < \infty$ for $q = 2/\varepsilon$. Applying iteratively Lemma 3.4, it follows that $\|\overline{h}_n\|_p < \infty$ for all $p \in ((1-\varepsilon)^{n-1}q, \infty) \cap [1,\infty)$. By choosing n large enough we may assume henceforth that $\|\overline{h}_{n-1}\|_1 = \int_{\mathbb{R}} \overline{h}_{n-1}(w)\, \mathrm{d}w < \infty$.

By definition we have $\overline{h}_n = \Phi_{2^{n-1}}(\overline{h}_{n-1})$, therefore recalling (3.9) we can write

$$\int_{\mathbb{R}} (1 + |x|^\varepsilon)\, \overline{h}_n(x)\, \mathrm{d}x = 2 \int_{\mathbb{R}} \mathrm{d}x \int_{|z|>(|x|-3)/2} \mathrm{d}z\, (1 + |x|^\varepsilon)\, f_{2^{n-1}}(z)\, \overline{h}_{n-1}(x-z) .$$

In the domain $\{|z| > (|x|-3)/2\}$ we have $|x|^\varepsilon \le (2|z|+3)^\varepsilon \le 2^\varepsilon(2^\varepsilon|z|^\varepsilon + 3^\varepsilon)$, because $(a+b)^\varepsilon \le (2\max\{a,b\})^\varepsilon \le 2^\varepsilon(a^\varepsilon + b^\varepsilon)$ for all $a, b, \varepsilon \ge 0$. Therefore

$$\begin{aligned} \int_{\mathbb{R}} (1+|x|)^\varepsilon\, \overline{h}_n(x)\, \mathrm{d}x &\le 2 \int_{\mathbb{R}} \mathrm{d}z\, (1 + 6^\varepsilon + 4^\varepsilon|z|^\varepsilon)\, f_{2^{n-1}}(z) \bigg(\int_{|x|<2|z|+3} \mathrm{d}x\, \overline{h}_{n-1}(x-z) \bigg) \\ &\le 2\, \|\overline{h}_{n-1}\|_1 \int_{\mathbb{R}} \mathrm{d}z\, (1 + 6^\varepsilon + 4^\varepsilon|z|^\varepsilon)\, f_{2^{n-1}}(z) < \infty , \end{aligned}$$

thanks to (3.4). The proof of (3.3), and hence of Theorem 1.1, is complete.

Appendix A. Some technical proofs

A.1. **Proof of Lemma 1.2.** Observe that $\widehat{f} \in L^\infty$ for every $f \in L^1$. We recall Theorem 3 in [8, §XV.3] and its Corollary, concerning a function $g \in L^1$ and its Fourier transform $\widehat{g}$:

$$\text{if } \widehat{g} \in L^1, \text{ then } g \in L^\infty \text{ (and is continuous);} \quad \text{if } g \in L^\infty \text{ and if } \widehat{g} \ge 0, \text{ then } \widehat{g} \in L^1. \tag{A.1}$$

Assume that $\widehat{f} \in L^p$ for some $p \in [1,\infty)$. Since $\widehat{f} \in L^\infty$, it follows that $\widehat{f} \in L^q$ for every $q \in [p,\infty]$. Since Fourier transform turns convolutions into products, we have $\widehat{f_n} = (\widehat{f})^n$ for every $n \in \mathbb{N}$, therefore for all $n \geq p$ we have $\widehat{f_n} \in L^1$ and, by (A.1), $f_n \in L^\infty$. Therefore f satisfies assumption (1) of Theorem 1.1.

Now assume that $f_{k_0} \in L^\infty$ for some $k_0 \in \mathbb{N}$. If we define $g(x) := f_{k_0}(-x)$, we have $\widehat{g} = (\widehat{f_{k_0}})^* = ((\widehat{f})^{k_0})^*$, where we denote by a^* the complex conjugate of $a \in \mathbb{C}$. It follows that $\widehat{f_{k_0} * g} = \widehat{f_{k_0}} * \widehat{g} = |\widehat{f}|^{2k_0} \geq 0$. Note that $f_{k_0} * g \in L^\infty$, because both f_{k_0} and g are in $L^1 \cap L^\infty$, therefore, by (A.1), $\widehat{f_{k_0} * g} = |\widehat{f}|^{2k_0} \in L^1$, that is $\widehat{f} \in L^{2k_0}$.

A.2. **Proof of Lemma 1.3.** Let us denote by $\| \cdot \|_p$ the norm in L^p, i.e.,

$$\|h\|_p := \left(\int_{\mathbb{R}} |h(w)|^p \, \mathrm{d}w \right)^{1/p} \quad \text{if } p \in [1, \infty) , \qquad \|h\|_\infty := \operatorname{ess\,sup}_{x \in \mathbb{R}} |h(x)| .$$

Recalling (1.2), by Hölder's inequality

$$\begin{aligned} |f_2(x)| &\leq \left(\int_{\mathbb{R}} |f(x-y)| \, |f(y)|^p \, \mathrm{d}y \right)^{\frac{1}{p}} \|f\|_1^{\frac{p-1}{p}} \\ &\leq \left(\int_{\mathbb{R}} |f(x-y)|^p \, |f(y)|^p \, \mathrm{d}y \right)^{\frac{1}{p^2}} \|f\|_1^{\frac{p-1}{p}} \, \|f\|_p^{\frac{p-1}{p}} , \end{aligned}$$

therefore $f_2 \in L^1 \cap L^{p^2}$. By iteration, $f_{2^k} \in L^1 \cap L^{p^{2^k}}$ for every $k \in \mathbb{N}$. If k is such that $p^{2^k} \geq 2$, it follows that $f_{2^k} \in L^2$. Since Fourier transform is an isometry on L^2, it follows that $\widehat{f_{2^k}} = (\widehat{f})^{2^k} \in L^2$, that is $\widehat{f} \in L^{2^{k+1}}$. The conclusion follows by Lemma 1.2.

A.3. **Proof of relation** (3.2)**.** We recall that $g \geq 0$ by assumption. Arguing as in [14, §1.8], for fixed $x, x' \in \mathbb{R}$ and $\delta, \delta' > 0$, let A denote the set of $(m, m') \in \mathbb{Z}^2$ such that

$$[m'\delta' - x', (m'+1)\delta' - x') \cap [m\delta - x, (m+1)\delta - x) \neq \emptyset .$$

For every $m' \in \mathbb{Z}$, there are at most $(\delta'/\delta) + 2$ values of $m \in \mathbb{Z}$ for which $(m, m') \in A$, hence

$$\begin{aligned} \sum_{m \in \mathbb{Z}} \sup_{z \in [m\delta, (m+1)\delta)} g(z - x) &\leq \sum_{m \in \mathbb{Z}} \max_{m' \in \mathbb{Z} \colon (m,m') \in A} \sup_{z \in [m'\delta', (m'+1)\delta')} g(z - x') \\ &\leq \sum_{(m,m') \in A} \sup_{z \in [m'\delta', (m'+1)\delta')} g(z - x') \leq \left(\frac{\delta'}{\delta} + 2 \right) \sum_{m' \in \mathbb{Z}} \sup_{z \in [m'\delta', (m'+1)\delta')} g(z - x') . \end{aligned}$$

Now denote by B the set of $(m, m') \in \mathbb{Z}^2$ such that

$$[m\delta - x, (m+1)\delta - x) \subseteq [m'\delta' - x', (m'+1)\delta' - x') .$$

Plainly, for every m there is *at most* one value of m' such that $(m, m') \in B$, while for every m' there are *at least* $(\delta'/\delta) - 2$ values of $m \in \mathbb{Z}$ for which $(m, m') \in B$. Therefore

$$\begin{aligned} \sum_{m \in \mathbb{Z}} \inf_{z \in [m\delta, (m+1)\delta)} g(z - x) &\geq \sum_{m \in \mathbb{Z}} \sum_{m' \in \mathbb{Z}} \mathbb{1}_{\{(m,m') \in B\}} \inf_{z \in [m\delta, (m+1)\delta)} g(z - x) \\ &\geq \sum_{m' \in \mathbb{Z}} \inf_{z \in [m'\delta', (m'+1)\delta')} g(z - x') \left(\sum_{m' \in \mathbb{Z}} \mathbb{1}_{\{(m,m') \in B\}} \right) \\ &\geq \left(\frac{\delta'}{\delta} - 2 \right) \sum_{m' \in \mathbb{Z}} \inf_{z \in [m'\delta', (m'+1)\delta')} g(z - x') . \end{aligned}$$

The previous relations shown that, for all $x, x' \in \mathbb{R}$ and $\delta, \delta' > 0$,

$$S^g_\delta(x) \le \left(1 + 2\frac{\delta}{\delta'}\right) S^g_{\delta'}(x') , \qquad s^g_\delta(x) \ge \left(1 - 2\frac{\delta}{\delta'}\right) s^g_{\delta'}(x') , \tag{A.2}$$

which implies in particular (3.2).

A.4. **Bootstrapping relation** (3.3)**.** Let us set $g_k(x) := (1 + |x|^\varepsilon)|f_k(x)|$, so that the left hand side of (3.3) can be expressed as $S^{g_k}_1(0)$ (recall (3.1)). Since $(a + b)^\varepsilon \le 2^\varepsilon(a^\varepsilon + b^\varepsilon)$ for $a, b, \varepsilon \ge 0$, recalling (1.2) we can write

$$\begin{aligned} S^{g_{k+1}}_1(0) &\le 2^\varepsilon \int_{\mathbb{R}} |f(y)| \sum_{m\in\mathbb{Z}} \left(\sup_{z\in[m,(m+1))} (1 + |z - y|^\varepsilon + |y|^\varepsilon) \, |f_k(z - y)| \right) \mathrm{d}y \\ &= 2^\varepsilon \left\{ \left(\int_{\mathbb{R}} |f(y)| \, S^{g_k}_1(z - y) \, \mathrm{d}y \right) + \left(\int_{\mathbb{R}} |y|^\varepsilon \, |f(y)| \, S^{|f_k|}_1(z - y) \, \mathrm{d}y \right) \right\} \\ &\le 3 \cdot 2^\varepsilon \left\{ \left(\int_{\mathbb{R}} |f(y)| \, \mathrm{d}y \right) + \left(\int_{\mathbb{R}} |y|^\varepsilon \, |f(y)| \, \mathrm{d}y \right) \right\} S^{g_k}_1(0) , \end{aligned}$$

where the last inequality follows from (3.2) with $x = z - y$, $x' = 0$ and $\delta = \delta' = 1$. The term in brackets is finite, by the assumptions of Theorem 1.1, hence $S^{g_{k+1}}_1(0) < \infty$ if $S^{g_k}_1(0) < \infty$. This shows that if (3.3) holds for $k = \overline{k}$, then it holds for all $k \ge \overline{k}$.

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