SINGULARITY AND REGULARITY OF THE CRITICAL 2D STOCHASTIC HEAT FLOW

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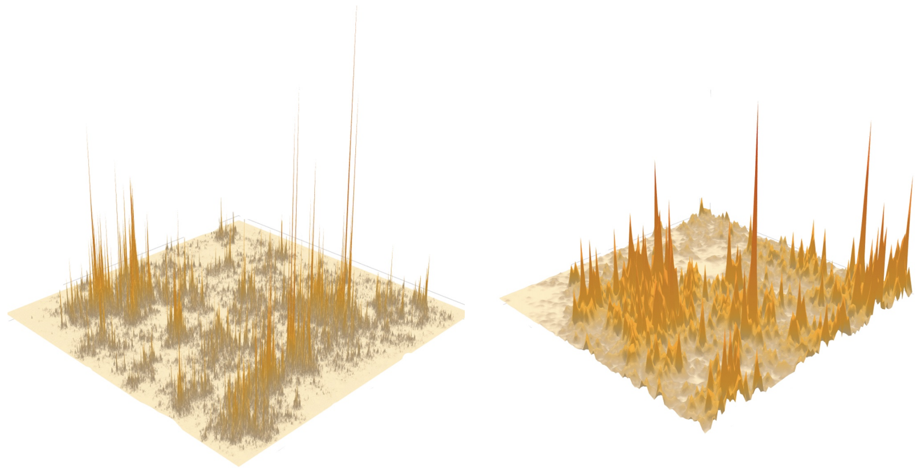
ABSTRACT. The Critical 2D Stochastic Heat Flow (SHF) provides a natural candidate solution to the ill-posed 2D Stochastic Heat Equation with multiplicative space-time white noise. In this paper, we initiate the investigation of the spatial properties of the SHF. We prove that, as a random measure on $\mathbb{R}^2$, it is a.s. singular w.r.t. the Lebesgue measure. This is obtained by probing a "*quasi-critical*" regime and showing the asymptotic log-normality of the mass assigned to vanishing balls, as the disorder strength is sent to zero at a suitable rate, accompanied by similar results for critical 2D directed polymers. We also describe the regularity of the SHF, showing that it is a.s. Hölder $\mathcal{C}^{-\varepsilon}$ for any $\varepsilon > 0$, implying the absence of atoms, and we establish local convergence to zero in the long time limit.



The picture on the left is a simulation of the Critical 2D SHF and illustrates its singularity. The picture on the right is a simulation in the quasi-critical regime, slightly below the critical window, which is smoother and will be used to approximate the Critical 2D SHF.

1. Introduction

The Critical 2D Stochastic Heat Flow (SHF) was constructed in [CSZ23a] as a family of measure-valued processes $\mathscr{Z}_t^{\vartheta}(\mathrm{d}x)$ with disorder strength parameter $\vartheta \in \mathbb{R}$, which give non-trivial solutions to the ill-defined two-dimensional Stochastic Heat Equation (SHE)

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \beta\, u(t,x)\,\xi(t,x)\,, \qquad t > 0,\ x \in \mathbb{R}^2, \qquad \text{(SHE)}$$

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where $\xi(t,x)$ denotes space-time white noise (more recently, an axiomatic characterisation of the SHF as a continuous measure-valued process was given in [Tsa24]). Dimension 2 is critical for the SHE as it is the dimension where the singularity of the noise matches the smoothing effect of the Laplacian and thus cannot be treated perturbatively. A comprehensive theory of singular Stochastic PDEs (SPDEs) below their critical dimension (known as "subcritical SPDEs") exists thanks to the breakthrough theories of regularity structure [Hai14], paracontrolled distributions [GIP15], renormalisation group theory [Kup16, Duc22], energy solutions [GJ14] and the huge volume of work they have inspired. The endeavour of treating critical singular SPDEs is only now starting to emerge [CSZ24, CT24], and the Critical 2D SHF is the first example describing a non-trivial and non-Gaussian solution to a critical equation at its phase transition point.

The Critical 2D SHF is an interesting object with a rich structure (see the recent review [CSZ24]). However, its fine properties have not yet been explored. The purpose of this paper is to initiate the study of its spatial characteristics. Consider the Critical 2D SHF $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$ started from the Lebesgue measure $\mathscr{Z}_0^\vartheta(\mathrm{d}x) = \mathrm{d}x$. We will prove the following results: For every $\vartheta \in \mathbb{R}$ and $t > 0$, almost surely,

- $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$ is singular with respect to the Lebesgue measure (Theorem 1.1);
- $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$ barely fails to be a function in the sense that it is in the negative Hölder spaces $C^{-\varepsilon}$ for every $\varepsilon > 0$ (Theorem 1.5), and hence contains no atoms.

Moreover, we show that

- $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$ converges in law to the 0 measure as $t \to \infty$, in the sense that the mass assigned to any finite ball converges to 0 in probability (Theorem 1.4).

The almost surely singularity of $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$ is a consequence of the result that:

- The mass density $\frac{1}{|B(x,\delta)|}\mathscr{Z}_t^\vartheta(B(x,\delta))$ on a ball $B(x,\delta)$ of shrinking radius $\delta \downarrow 0$ converges to a log-normal limit, if the disorder strength parameter $\vartheta = \vartheta(\delta) \to -\infty$ at a suitable rate (Theorem 1.2).

This is obtained by proving an analogous result (Theorem 1.8) for the averaged partition function of the directed polymer model in the so-called *quasi-critical regime*, which was introduced in [CCR23] as an interpolation between the sub-critical and critical regimes of the 2D directed polymer model. We remark that the directed polymer model is a very interesting and important disordered system on its own [Com17, Zyg24], and the Critical 2D SHF was first constructed in [CSZ23a] as the unique limit of 2D directed polymer partition functions in the critical regime.

The proof of Theorem 1.8 constitutes the bulk of this paper and is accomplished via an approximate *multiplicative, multi-scale decomposition of the polymer partition function*, see (4.7). Similar decompositions have also been applied in the sub-critical regime [CD24, CNZ25]. The novelty of our contribution is that we push such a decomposition to the quasi-critical regime, up to the onset of criticality, setting the foundations for understanding fine properties of the SHF. Along the way, we derive a *general hypercontractive bound* on the higher moments of the averaged polymer partition function in terms of its second moment (Theorem 1.11), valid in all regimes up to criticality.

For the rest of the introduction, we will first recall the construction and basic properties of the Critical 2D Stochastic Heat Flow. Our main results for the Critical 2D SHF and the directed polymer model will then be stated in Section 1.2 and 1.3 respectively. In

Section 1.4, we will formulate the hypercontractive moment bound mentioned above. Lastly in Section 1.5, we explain how limiting properties of the Critical 2D SHF as the disorder parameter $\vartheta \downarrow -\infty$ can always be analysed by studying the directed polymer partition functions in the quasi-critical regime.

1.1. Briefing on the Critical 2D SHF. To make sense of the two-dimensional stochastic heat equation (SHE), we need to first perform a regularisation on small spatial scales (ultraviolet cutoff) and then take a suitable limit. The regularisation can be accomplished in different ways. One way is to consider the mollified SHE

$$\partial_t u^\varepsilon = \frac{1}{2}\Delta u^\varepsilon + \beta_\varepsilon \, \xi^\varepsilon(t,x) u^\varepsilon, \tag{1.1}$$

where $\varepsilon > 0$ is the spatial scale of regularisation, $j(\cdot)$ is a smooth probability kernel on $\mathbb{R}^2$ and $j^\varepsilon(x) = \varepsilon^{-2} j(x/\varepsilon)$ is its scaled version, while $\xi^\varepsilon := j^\varepsilon * \xi$ is the spatial mollification of the white noise ξ.

Another way is to discretize space and time. Namely, the white noise ξ is replaced by a family of i.i.d. random variables $\omega = (\omega(n,x))_{n\in\mathbb{N}, x\in\mathbb{Z}^2}$ with law $\mathbb{P}$ and expectation $\mathbb{E}$, and

$$\mathbb{E}[\omega] = 0\,, \qquad \mathbb{E}[\omega^2] = 1\,, \qquad \exists\, \beta_0 > 0: \quad \lambda(\beta) := \log \mathbb{E}[\mathrm{e}^{\beta\omega}] < \infty \quad \forall \beta \in [0,\beta_0]\,. \tag{1.2}$$

Replacing derivatives in (SHE) by suitable difference operators, the solution can be expressed in terms of the (point-to-point) partition functions of the *directed polymer model*:

$$Z^{\beta_N}_{M,N}(y,x) = \mathrm{E}\Big[\mathrm{e}^{\sum_{n=M+1}^{N-1}\{\beta_N \omega(n,S_n) - \lambda(\beta_N)\}} \, \mathbb{1}_{\{S_N = x\}} \,\Big|\, S_M = y\Big]\,, \tag{1.3}$$

where E is the expectation with respect to the 2D simple symmetric random walk $S = (S_n)_{n\geqslant 0}$. More precisely, the diffusively rescaled plane-to-point partition functions (with $\mathbb{Z}_{\mathrm{even}} := \{2n : n \in \mathbb{Z}\}$ and $\mathbb{Z}^2_{\mathrm{even}} := \{(x,y) \in \mathbb{Z}^2 : x + y \in \mathbb{Z}_{\mathrm{even}}\}$)

$$u^{(N)}(t,x) := \sum_{y\in\mathbb{Z}^2_{\mathrm{even}}} Z^{\beta_N}_{0,Nt}(y,\sqrt{N}x), \qquad (t,x) \in \frac{1}{N}\mathbb{Z}_{\mathrm{even}} \times \frac{1}{\sqrt{N}}\mathbb{Z}^2_{\mathrm{even}}, \tag{1.4}$$

is the analogue of $u^\varepsilon(t,x)$ and solves a version of (SHE), discretised on spatial scale $1/\sqrt{N}$ and time scale $1/N$, with initial condition $u^{(N)}(0,\cdot) \equiv 1$.

It was first shown in [CSZ17] that on the intermediate disorder scale

$$\beta_N = \hat\beta \sqrt{\frac{\pi}{\log N}},$$

the directed polymer partition functions $u^{(N)}(t,x)$ undergo a phase transition (with critical value $\hat\beta_c = 1$) in two different senses:

- For each $(t,x) \in [0,1) \times \mathbb{R}^2$, $u^{(N)}(t,x)$ converges to a log-normal limit if $\hat\beta < 1$ and converges to 0 if $\hat\beta \geqslant 1$;
- The centered and rescaled field $\beta_N^{-1}(u^{(N)}(t,x) - 1)$ converges for $\hat\beta < 1$ to a Gaussian limit that solves the additive SHE (or Edwards-Wilkinson equation)

  $$\partial_t v = \frac{1}{2}\Delta v + \sqrt{\frac{\hat\beta^2}{1-\hat\beta^2}}\,\xi,$$

  where the noise strength diverges as $\hat\beta \uparrow 1$.

The same results were also proved in [CSZ17] for the solution u^ε of the mollified SHE (1.1) on the intermediate disorder scale $\beta_\varepsilon = \hat\beta\sqrt{\frac{2\pi}{\log 1/\varepsilon}}$. Therefore in the subcritical regime $\hat\beta < 1$, the 2D SHE is essentially the Edwards-Wilkinson equation.

It is the critical regime $\hat\beta = 1$ that leads to a non-Gaussian limit, called the *Critical 2D Stochastic Heat Flow (SHF)*. It turns out there is a whole critical window around $\hat\beta = 1$, determined by the relation

$$e^{\lambda(2\beta_N)-2\lambda(\beta_N)} - 1 = \frac{\pi}{\log N}\Big(1 + \frac{\vartheta + o(1)}{\log N}\Big) \quad \text{for some } \vartheta \in \mathbb{R}. \tag{1.5}$$

For a more explicit expression of β_N in terms of ϑ, see [CSZ23a, (3.12)]. For the mollifed SHE (1.1), the corresponding critical window is given by

$$\beta_\varepsilon^2 = \frac{2\pi}{\log\frac{1}{\varepsilon}}\Big(1 + \frac{\varrho + o(1)}{\log\frac{1}{\varepsilon}}\Big), \tag{1.6}$$

where $\varrho = \pi\vartheta + C$ (see [CSZ19b, (1.38)] for the precise value of C).

The main result of [CSZ23a] is that:

- If β_N is chosen to satisfy (1.5) for some $\vartheta \in \mathbb{R}$ and $u^{(N)}(t,\cdot)$ is regarded as a process of random measures on $\mathbb{R}^2$, then $(u^{(N)}(t,\cdot))_{t\geqslant 0}$ converges in finite dimensional distribution to a unique (in law) measure-valued process $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$, which was named the *Critical 2D Stochastic Heat Flow* in [CSZ23a].

Prior to [CSZ23a], the tightness of the sequence of random measures $u^{(N)}(t,\cdot)$ follows trivially from first moment bounds, while moment asymptotics of $u^{(N)}(t,\cdot)$ were studied in [BC98, CSZ19b, GQT21], which determined all positive integer moments of any subsequential weak limit of $u^{(N)}(t,\cdot)$. However, these moments diverge too fast to uniquely determine the limit (a lower bound of order $\exp(ck^2)$ for the k-th moment was given in [CSZ23b]). The uniqueness was finally achieved in [CSZ23a] by showing that the laws of $u^{(N)}(t,\cdot)$ form a Cauchy sequence, and hence must converge to a unique limit. The proof was based on coarse graining, coupled with a Lindeberg replacement principle.

Recently, Tsai [Tsa24] gave an axiomatic characterization of the critical 2D SHF and showed that there is a version that is almost surely continuous in time. This greatly facilitates the proof of convergence to the SHF. In particular, this axiomatic characterization was used in [Tsa24] to show that the solution u^ε of the mollified SHE (1.1) in the critical window (1.6) also converges to the SHF. In Tsai's characterisation, the SHF is the unique (in law) continuous measure-valued process that satisfies: (i) Independent "increments" property; (ii) An almost sure Chapman-Kolmogorov property (first defined and verified for the SHF by Clark and Mian [CM24]); (iii) matching first four moments with the SHF. The proof was also based on a Lindeberg replacement principle.

We also recall from [CSZ23a, Theorem 1.2] some basic properties of the Critical 2D SHF (for simplicity, we only consider constant initial configuration $\mathscr{Z}_0^\vartheta(\mathrm{d}x) = \mathrm{d}x$):

- (Scaling Covariance) For all $\mathsf{a} > 0$, we have

  $$(\mathscr{Z}_{\mathsf{a}t}^\vartheta(\mathrm{d}(\sqrt{\mathsf{a}}x)))_{0\leqslant t<\infty} \overset{\mathrm{dist}}{=} (\mathsf{a}\,\mathscr{Z}_t^{\vartheta+\log\mathsf{a}}(\mathrm{d}x))_{0\leqslant t<\infty}\,. \tag{1.7}$$

  Thus zooming out diffusively ($\mathsf{a}\uparrow\infty$) increases the disorder strength ϑ, while zooming in ($\mathsf{a}\downarrow 0$) decreases the disorder strength (cf. the pictures on the front page, where the picture on the right is a result of zooming into the picture on the left).

- (First and Second Moments) We have
$$\mathrm{E}[\mathscr{Z}_t^\vartheta(\mathrm{d}x)] = \mathrm{d}x\,,$$
$$\mathrm{E}[\mathscr{Z}_t^\vartheta(\mathrm{d}x)\mathscr{Z}_t^\vartheta(\mathrm{d}y)] = K_t^\vartheta(x,y)\,\mathrm{d}x\,\mathrm{d}y\,,$$
where $K_t^\vartheta(x,y) \sim C \log \frac{1}{|x-y|}$ as $|x-y| \to 0$. It was first computed in [BC98] before the realisation that this lies in the critical window of a phase transition [CSZ17].

For more properties of the Critical 2D SHF, see [CSZ23a] and the lecture notes [CSZ24].

1.2. New properties of the Critical 2D SHF. In this paper we investigate the spatial regularity of the SHF. We focus on its one-time marginal $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$, which is a *locally finite random measure on* $\mathbb{R}^2$ with $\mathscr{Z}_0^\vartheta(\mathrm{d}x) = \mathrm{d}x$ and $\mathbb{E}[\mathscr{Z}_t^\vartheta(\mathrm{d}x)] = \mathrm{d}x$ for all $t \geqslant 0$. Our main result is that, for each $t > 0$, $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$ is almost surely *singular* with respect to the Lebesgue measure, and hence not a function.

Let $\mathcal{U}_{B(x,\delta)}(\cdot)$ denote the uniform density on the Euclidean ball in $\mathbb{R}^2$:

$$\mathcal{U}_{B(x,\delta)}(\cdot) := \frac{1}{\pi\delta^2}\,\mathbb{1}_{B(x,\delta)}(\cdot) \qquad \text{where } \; B(x,\delta) := \big\{y \in \mathbb{R}^2 : \; |y-x| < \delta\big\}\,. \tag{1.8}$$

We will mostly focus on the SHF $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$ *averaged over balls*, that is

$$\mathscr{Z}_t^\vartheta(\mathcal{U}_{B(x,\delta)}) := \frac{\mathscr{Z}_t^\vartheta(B(x,\delta))}{\pi\delta^2}\,. \tag{1.9}$$

We can now state our first main result.

Theorem 1.1 (Singularity of SHF). *Fix any $t > 0$ and $\vartheta \in \mathbb{R}$. Almost surely, the SHF $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$ is singular with respect to Lebesgue measure on $\mathbb{R}^2$. In fact, the following holds:*

$$\textit{almost surely,} \qquad \lim_{\delta\downarrow 0}\, \mathscr{Z}_t^\vartheta\big(\mathcal{U}_{B(x,\delta)}\big) = 0 \quad \textit{for Lebesgue a.e. } x \in \mathbb{R}^2\,. \tag{1.10}$$

The singularity of the SHF with respect to Lebesgue can be deduced from property (1.10) via general arguments (see Proposition 3.2). In order to prove (1.10), we show that *in the "weak disorder limit" $\vartheta \to -\infty$, the SHF averaged on balls $\mathscr{Z}_t^\vartheta\big(\mathcal{U}_{B(x,\delta)}\big)$ is asymptotically log-normal for radius $\delta = \delta_\vartheta^\varrho \downarrow 0$ vanishing as any power of a suitable scale δ_ϑ.*

Theorem 1.2 (Log-normality of SHF in the weak disorder limit). *Let us define*

$$\delta_\vartheta := \mathrm{e}^{\frac{1}{2}\vartheta} = \mathrm{e}^{-\frac{1}{2}|\vartheta|} \longrightarrow 0 \qquad \textit{as } \vartheta \to -\infty\,. \tag{1.11}$$

Given any $t > 0$ and $x \in \mathbb{R}^2$, the following convergence in distribution holds:

$$\forall \varrho \in (0,\infty): \qquad \mathscr{Z}_t^\vartheta\big(\mathcal{U}_{B(x,\delta_\vartheta^\varrho)}\big) \xrightarrow[\vartheta\to-\infty]{d} \mathrm{e}^{\mathcal{N}(0,\sigma^2)-\frac{1}{2}\sigma^2} \qquad \textit{with } \sigma^2 = \log(1+\varrho)\,. \tag{1.12}$$

Remark 1.3. *We stress that the log-normality* (1.12) *emerges as $\vartheta \to -\infty$. For fixed $\vartheta \in \mathbb{R}$, the SHF averaged on balls $\mathscr{Z}_t^\vartheta\big(\mathcal{U}_{B(x,\delta)}\big)$ vanishes as $\delta \to 0$, as shown by* (1.10).

On a different note, the SHF $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$ is not the exponential of a (generalised) Gaussian field, i.e. it is not a Gaussian Multiplicative Chaos, see [CSZ23b].

In the proof of Theorem 1.1 we deduce (1.10) from (1.12) by exploiting *the monotonicity of fractional moments of $\mathscr{Z}_t^\vartheta(B)$ with respect to ϑ* (see Lemma 3.1).

Using the scaling covariance property (1.7), we also show that the SHF *locally vanishes* as the time horizon tends to infinity.

Theorem 1.4 (Long-time behavior of SHF). *Fix any* $\vartheta \in \mathbb{R}$. *Then,*

$$\textit{for any bounded set } A \subset \mathbb{R}^2: \qquad \mathscr{Z}_t^{\vartheta}(A) \xrightarrow[t\to\infty]{d} 0\,. \tag{1.13}$$

We finally investigate the regularity of the SHF $\mathscr{Z}_t^{\vartheta}(\mathrm{d}x)$ as a measure on $\mathbb{R}^2$, showing that it has negative Hölder regularity $\mathcal{C}^{-\varepsilon}$ for arbitrary small $\varepsilon > 0$ (the definition of negative Hölder spaces is recalled in Subsection 3.4). Since positive Hölder spaces $\mathcal{C}^{\varepsilon}$ consist of functions, this shows that, in a sense, the SHF $\mathscr{Z}_t^{\vartheta}(\mathrm{d}x)$ *barely fails to be a function.*

Theorem 1.5 (Regularity of the SHF). *Fix any* $t > 0$ *and* $\vartheta \in \mathbb{R}$. *Almost surely, the SHF* $\mathscr{Z}_t^{\vartheta}(\mathrm{d}x)$ *belongs to* $\mathcal{C}^{0-} := \bigcap_{\varepsilon>0} \mathcal{C}^{-\varepsilon}$ *and, hence, contains no atoms.*

The recent work of Nakashima [Nak25], Section 7, indicates that the fine regularity of the SHF should be captured by suitably defined log-Hölder spaces, which captures the logarithmic heights of the peaks. The interesting task of determining the precise logarithmic regularity of the SHF would require a detailed understanding of the structure of its peaks and it should be the subject of future works.

The results above are proved in Section 3. The proof of Theorems 1.2 and 1.5 are based on the approximation of the SHF via *partition functions of directed polymers*, which was used in the original construction of the SHF in [CSZ23a] and will be recalled next.

1.3. Results for directed polymers. To define the directed polymer model, let $S = (S_n)_{n\geqslant 0}$ be the simple symmetric random walk on $\mathbb{Z}^2$ with law P and expectation E. We denote its transition kernel by

$$q_n(z) := \mathrm{P}(S_n = z \,|\, S_0 = 0) \qquad \text{for } \; n \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}\,, \; z \in \mathbb{Z}^2\,. \tag{1.14}$$

We define the *expected replica overlap* $R_N = \mathrm{E}\big[\sum_{n=1}^N \mathbb{1}_{\{S_n = S_n'\}}\big]$ where S' is an independent copy of S with $S_0' = S_0 = 0$. By the local central limit theorem (2.3)

$$R_N = \sum_{n=1}^N \sum_{x\in\mathbb{Z}^2} q_n(x)^2 = \sum_{n=1}^N q_{2n}(0) = \frac{\log N}{\pi} + O(1) \qquad \text{as } N \to \infty \tag{1.15}$$

(see also [CSZ19a, Proposition 3.2] for a refined asymptotic behavior).

The *environment (disorder)* is given by a family $(\omega(n,z))_{n\in\mathbb{N}, z\in\mathbb{Z}^2}$ of i.i.d. random variables satisfying the assumptions in (1.2). Note that $\lambda(\beta) := \log \mathbb{E}[\mathrm{e}^{\beta\omega}] \sim \frac{1}{2}\beta^2$ as $\beta \to 0$. We introduce the quantity

$$\sigma_\beta^2 := \mathbb{V}\mathrm{ar}\big[\mathrm{e}^{\beta\omega - \lambda(\beta)}\big] = \mathrm{e}^{\lambda(2\beta) - 2\lambda(\beta)} - 1 \underset{\beta\downarrow 0}{\sim} \beta^2\,. \tag{1.16}$$

Given $\varphi, \psi : \mathbb{Z}^2 \to \mathbb{R}$, polymer length $N \in \mathbb{N}$, and inverse temperature (or disorder strength) $\beta \geqslant 0$, we define the averaged directed polymer partition function as follows:

$$Z_N^{\beta}(\varphi,\psi) := \sum_{z\in\mathbb{Z}^2} \varphi(z)\, Z_N^{\beta}(z,z')\, \psi(z') \quad \text{with} \quad Z_N^{\beta}(z,z') := \mathrm{E}\Big[\mathrm{e}^{\mathcal{H}^{\beta}_{(0,N]}(S)}\, \mathbb{1}_{S_N = z'} \,\Big|\, S_0 = z\Big]\,, \tag{1.17}$$

where

$$\mathcal{H}_I^\beta(S) := \sum_{n \in I \cap \mathbb{Z}} \big\{ \beta\, \omega(n, S_n) - \lambda(\beta) \big\} \qquad \text{for } I \subset \mathbb{R}\,. \tag{1.18}$$

When $\psi \equiv 1$, we will simplify notation and write $Z_N^\beta(\varphi) := Z_N^\beta(\varphi, 1)$.

Remark 1.6. *To comply with the periodicity of the simple random walk, we usually consider φ supported on the even sub-lattice*

$$\mathbb{Z}^2_{\rm even} := \{(x,y) \in \mathbb{Z}^2 \colon x + y \ \textit{is even}\}\,.$$

As explained in Section 1.1, the Critical 2D SHF $\mathscr{Z}_t^\vartheta(\cdot)$ is the *scaling limit of the diffusively rescaled partition functions* $Z_{tN}^{\beta_N}(\cdot\,\sqrt{N})$ regarded as a random measure on $\mathbb{R}^2$, if the disorder strength β_N is chosen to be in the following critical window:

$$\sigma^2_{\beta_N^{\rm crit}} = \frac{1}{R_N}\left(1 + \frac{\vartheta}{\log N}\right) \qquad \text{for some } \vartheta \in \mathbb{R}\,. \tag{1.19}$$

More precisely, denoting by $[\![x]\!]$ the point in $\mathbb{Z}^2_{\rm even}$ closest to $x \in \mathbb{R}^2$, the following convergence in distribution was proved in [CSZ23a, Theorem 1.1]: for any $t > 0$

$$Z_{tN}^{\beta_N^{\rm crit}}\big([\![x\sqrt{N}]\!]\big)\, \mathrm{d}x \ \xrightarrow[N\to\infty]{d} \ \mathscr{Z}_t^\vartheta(\mathrm{d}x), \tag{1.20}$$

which are regarded as random variables taking values in the space of locally finite measures on $\mathbb{R}^2$ equipped with the vague topology, i.e., the one generated by the integrals $\int \varphi\, \mathrm{d}\mu$ for continuous and compactly supported test functions $\varphi : \mathbb{R}^2 \to \mathbb{R}$.

We strengthen this result to a convergence in distribution of random variables taking values in negative Hölder spaces $\mathcal{C}^{-\varepsilon}$ for any $\varepsilon > 0$.

Theorem 1.7 (Improved convergence to the SHF). *Fix $\vartheta \in \mathbb{R}$ and consider $\beta_N^{\rm crit}$ in the critical regime* (1.19)*. For any $t > 0$, the following convergence in distribution holds:*

$$\forall \varepsilon > 0: \qquad Z_{tN}^{\beta_N^{\rm crit}}\big([\![x\sqrt{N}]\!]\big)\, \mathrm{d}x \ \xrightarrow[N\to\infty]{d} \ \mathscr{Z}_t^\vartheta(\mathrm{d}x) \quad \textit{in } \mathcal{C}^{-\varepsilon}\,. \tag{1.21}$$

This result directly implies Theorem 1.5 on the regularity of the SHF. The proof is given in Section 3 by exploiting moment bounds from [CSZ23a] (see Proposition 3.7).

We next look back at the log-normality of the SHF averaged on vanishing balls as $\vartheta \to -\infty$, see Theorem 1.2. We obtain this result via discrete approximations, namely we deduce it from an analogue result for directed polymer partition functions, which we state next.

In order to compare the SHF as $\vartheta \to -\infty$ with directed polymers, we need to tune the disorder strength β in a *quasi-critical regime* recently investigated in [CCR23], where we replace ϑ in (1.19) by a sequence $\vartheta_N = -|\vartheta_N| \to -\infty$ at an arbitrarily slow rate:

$$\sigma^2_{\beta_N^{\text{quasi-crit}}} := \frac{1}{R_N}\left(1 - \frac{|\vartheta_N|}{\log N}\right) \qquad \text{where } \ 1 \ll |\vartheta_N| \ll \log N\,. \tag{1.22}$$

We call this regime *quasi-critical* because it interpolates between the *critical regime* (1.19), corresponding to $|\vartheta_N| = O(1)$, and the *sub-critical regime* [CSZ17, CSZ20, CC22, CD24], corresponding to $|\vartheta_N| \approx \log N$, see (1.26) below.

Let us consider directed polymer partition functions $Z_N^\beta(\varphi)$ with initial conditions φ that are uniformly distributed on discrete balls, denoted by $\mathcal{U}_{B(z,R)}$ (same as their continuum counterparts (1.8), with some abuse of notation):

$$\mathcal{U}_{B(z,R)}(\cdot) := \frac{\mathbb{1}_{B(z,R)\cap\mathbb{Z}^2_{\rm even}}(\cdot)}{\big|B(z,R)\cap\mathbb{Z}^2_{\rm even}\big|} \,. \tag{1.23}$$

We can now state our log-normality result for directed polymers.

Theorem 1.8 (Quasi-critical log-normality). *Consider $\beta_N^{\text{quasi-crit}}$ in the quasi-critical regime* (1.22) *for a given sequence $1 \ll |\vartheta_N| \ll \log N$. Define the scale δ_N by*

$$\delta_N := \mathrm{e}^{-\frac{1}{2}|\vartheta_N|} \longrightarrow 0 \qquad \text{as } N\to\infty \,. \tag{1.24}$$

For any $t > 0$ and $x \in \mathbb{R}^2$, the following convergence in distribution holds:

$$\forall \varrho \in (0,\infty): \qquad Z_{tN}^{\beta_N^{\text{quasi-crit}}}\big(\mathcal{U}_{B(x\sqrt{N},\delta_N^\varrho\sqrt{N})}\big) \xrightarrow[N\to\infty]{d} \mathrm{e}^{\mathcal{N}(0,\sigma^2)-\frac{1}{2}\sigma^2} \quad \text{with } \sigma^2 = \log(1+\varrho) \tag{1.25}$$

and, furthermore, all positive moments converge.

Log-normality was first proved in [CSZ17, Theorem 2.8] for $Z_N^\beta(x) := Z_N^\beta(\mathbb{1}_{\{x\}})$, i.e. the partition function started at a single point x (also called *point-to-plane* partition function), when $\beta = \beta_N^{\text{sub-crit}}$ is chosen in the *sub-critical regime*:

$$\sigma^2_{\beta_N^{\text{sub-crit}}} \sim \frac{\hat\beta^2}{R_N} \qquad \text{for some } \hat\beta \in (0,1) \qquad \big(\text{i.e. } \big(\beta_N^{\text{sub-crit}}\big)^2 \sim \tfrac{\pi\hat\beta^2}{\log N}\big) \,. \tag{1.26}$$

Our proof of Theorem 1.8 also covers this regime and allows for averaging over balls of *arbitrary sub-diffusive polynomial radius* $N^{\gamma/2+o(1)}$ as $N\to\infty$, for any $0 \leqslant \gamma < 1$. The few changes required are described in Remarks 4.6 and 4.8 (see also Remark 2.6).

Theorem 1.9 (Sub-critical log-normality). *Consider $\beta_N^{\text{sub-crit}}$ in the sub-critical regime* (1.26) *for some $\hat\beta \in (0,1)$. For any $t > 0$, $x \in \mathbb{R}^2$, one has the convergence in distribution*

$$\forall \gamma \in [0,1): \qquad Z_{tN}^{\beta_N^{\text{sub-crit}}}\big(\mathcal{U}_{B(x\sqrt{N},\sqrt{N}^{\gamma+o(1)})}\big) \xrightarrow[N\to\infty]{d} \mathrm{e}^{\mathcal{N}(0,\sigma^2)-\frac{1}{2}\sigma^2} \quad \text{with } \sigma^2 = \log\tfrac{1-\gamma\hat\beta^2}{1-\hat\beta^2} \tag{1.27}$$

and, furthermore, all positive moments converge.

Alternative proofs of the log-normality of the point-to-plane partition function in the sub-critical regime were given in [CC22] and, more recently, in [CD24], simplifying the original approach in [CSZ17, Theorem 2.8]. A key ingredient in all of these proofs is the identification of suitable *exponential time scales* which yield an approximate factorisation of the partition function.

Remarkably, *a similar structure also emerges in the quasi-critical regime* (1.22) when the partition function is averaged on scales $\delta_N^\varrho\sqrt{N}$, for any power ϱ, with δ_N as in (1.24). This key fact is at the core of our proof of Theorem 1.8 (see Section 4 for more details).

Remark 1.10 (Quasi-critical vs. sub-critical regime). *Comparing* (1.25) *with* (1.27) *for $\gamma = 0$, we can draw an analogy between the following two quantities:*

- *the quasi-critical partition function $Z_N^{\text{quasi-crit}} := Z_N^{\beta_N^{\text{quasi-crit}}}\big(\mathcal{U}_{B(0,\delta_N^\varrho\sqrt{N})}\big)$ of size N, averaged on the ball of radius $\delta_N^\varrho\sqrt{N}$ centred at 0;*

- *the sub-critical point-to-plane partition function* $Z_L^{\text{sub-crit}} := Z_L^{\beta_L^{\text{sub-crit}}}(0)$ *of size* L*, with disorder strength* $\hat\beta^2 = \frac{\varrho}{1+\varrho}$*, that is* $\frac{1}{1-\hat\beta^2} = 1+\varrho$ *(to match* σ^2 *in* (1.25) *and* (1.27)*).*

More precisely, if we divide space into squares of side length $\delta_N^\varrho \sqrt{N}$ *and time into intervals of size* $(\delta_N^\varrho)^2 N$*, we can view the quasi-critical model* $Z_N^{\text{quasi-crit}}$ *as effectively a sub-critical model* $Z_L^{\text{sub-crit}}$ *with rescaled time horizon* $L \approx 1/(\delta_N^\varrho)^2$ *and effective disorder strength* $\hat\beta^2 = \frac{\varrho}{1+\varrho}$*.*

This analogy is made quantitative by our strategy of proof for Theorem 1.8, described in Section 4. This suggests that, at a conceptual level, other results that hold in the sub-critical regime could be transferred to the quasi-critical regime via this correspondence.

We stress, however, that the quasi-critical regime (1.22) *presents a fundamental technical challenge: unlike in the sub-critical regime, the main contribution to the polynomial chaos expansion of the partition function now comes from* chaos of unbounded order *(see the proofs of Proposition 2.3 and Theorem 2.5). As a consequence, many fundamental tools break down (e.g., hypercontractivity) and novel arguments are required.*

1.4. Moment bounds. A key tool in our analysis are *moment bounds on the partition function* $Z_L^\beta(\varphi)$, see (1.17) (we denote the system size by L in place of N for later convenience). Such bounds, based on a functional operator approach, have been exploited in several contexts, see [GQT21, CSZ23a, LZ23, CCR23, CZ23, CZ24, CN25]. We provide here a *universal bound* of independent interest, which applies to all regimes of β mentioned so far (sub-critical, quasi-critical and critical) and to general initial conditions $\varphi(\cdot)$ supported on sub-diffusive or diffusive scales $O(\sqrt{L})$.

We focus on initial conditions which are *probability mass functions* on $\mathbb{Z}^2$,[†] i.e.

$$\varphi(\cdot) \geqslant 0\,, \qquad \sum_{x\in\mathbb{Z}^2} \varphi(x) = 1\,,$$

with finite *mean-squared displacement from its center of mass*:

$$\mathbb{D}[\varphi] := \sum_{z\in\mathbb{Z}^2} |z-m_\varphi|^2\, \varphi(z) < \infty \qquad \text{with} \qquad m_\varphi := \sum_{z\in\mathbb{Z}^2} z\,\varphi(z)\,. \tag{1.28}$$

We require two natural bounds on φ.

- *Exponential localisation on at most diffusive scale:* for some $\hat t > 0$, $\mathsf{c}_1 < \infty$
$$\exists z_0 \in \mathbb{R}^2: \qquad \sum_{z\in\mathbb{Z}^2} \varphi(z)\, \mathrm{e}^{2\,\hat t\, \frac{|z-z_0|}{\sqrt{L}}} \leqslant \mathsf{c}_1 \tag{1.29}$$
(the factor 2 in the exponent is for later convenience). This allows $\varphi(\cdot)$ to be localised on a diffusive or sub-diffusive scale, as it implies $\sqrt{\mathbb{D}[\varphi]} = O(\sqrt{L})$.
- *Local uniformity:* for some $\mathsf{c}_2 < \infty$
$$\|\varphi\|_{\ell^2}^2 = \sum_{z\in\mathbb{Z}^2} \varphi(z)^2 = \mathrm{E}[\varphi(Z)] \leqslant \frac{\mathsf{c}_2}{\mathbb{D}[\varphi]}\,, \tag{1.30}$$
where Z is a random point in $\mathbb{Z}^2$ with law φ. Since $\|\varphi\|_{\ell^2}^2 \leqslant \|\varphi\|_{\ell^\infty} \sum_{z\in\mathbb{Z}^2} \varphi(z) = \|\varphi\|_{\ell^\infty}$, a sufficient condition is
$$\|\varphi\|_{\ell^\infty} \leqslant \frac{\mathsf{c}_2}{\mathbb{D}[\varphi]}\,, \tag{1.31}$$

[†]Since $\varphi \mapsto Z_L^\beta(\varphi)$ is linear, any $\varphi \geqslant 0$ with $\sum_{x\in\mathbb{Z}^2} \varphi(x) < \infty$ can be normalised to a probability mass function.

which means that *the peaks of φ are comparable to those of a uniform distribution* (note that φ puts most of its mass in a ball of radius $\sqrt{\mathbb{D}[\varphi]}$, by Chebyshev).

We do not restrict β to any particular regime, but we consider partition functions with *uniformly bounded variance*: for some $\mathsf{c}_3 < \infty$

$$\mathbb{V}\mathrm{ar}[Z_L^\beta(\varphi)] \leqslant \mathsf{c}_3 \,. \tag{1.32}$$

We will show that, together with (1.29), this implies that β lies within or below the critical regime as $L \to \infty$, see Lemma 5.4.

We are ready to state our general moment bound.

Theorem 1.11 (General moment bound). *Given $h \in \mathbb{N}$ and $\hat{t}$, c_1, c_2, $\mathsf{c}_3 \in (0, \infty)$, there are constants $L_h, \mathfrak{C}_h < \infty$ (depending also on $\hat{t}, \mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3$) such that*

$$\big|\mathbb{E}\big[\big(Z_L^\beta(\varphi) - \mathbb{E}[Z_L^\beta(\varphi)]\big)^h\big]\big| \leqslant \mathfrak{C}_h \, \mathbb{V}\mathrm{ar}[Z_L^\beta(\varphi)]^{\frac{h}{2}} \tag{1.33}$$

uniformly for $\beta \geqslant 0$, $L \geqslant L_h$ and probability mass functions φ satisfying (1.29), (1.30), (1.32). *The bound* (1.33) *still holds if, on the LHS, we replace $Z_L^\beta(\varphi)$ with its restriction to any subset of random walk paths in its definition* (1.17).

We prove Theorem 1.11 in Section 5 in a strengthened form, see Theorem 5.2, where we relax the assumption (1.30) and we consider partition functions $Z_L^\beta(\varphi, \psi)$ averaged at both endpoints (i.e. we allow for a "final condition" ψ, besides the initial condition φ).

Remark 1.12 (Beyond diffusive scales). *We prove Theorem 1.11 under fairly general assumptions: the bounded variance condition* (1.32) *is necessary, as we explain in Remark 1.14, and the local uniformity assumption* (1.30) *is mild (and we further relax it in Section 5).*

Only the exponential localisation condition on at most diffusive scale (1.29) *imposes some real restriction. For instance, for initial conditions $\varphi(\cdot)$ localised at scale $\sqrt{N}$, one can consider system sizes $L = \varepsilon N$ and prove a moment bound like* (1.33) *uniformly in $\varepsilon > 0$, see e.g.* [CCR23, Proposition 2.3] *in the quasi-critical regime, but this goes beyond the scope of Theorem 1.11 because assumption* (1.29) *is not satisfied uniformly in $\varepsilon > 0$.*

We believe that our proof of Theorem 1.11 could be extended in order to relax the localisation condition (1.29)*, but we refrain from doing so in the present paper.*

Remark 1.13 (Hypercontractivity). *The bound* (1.33) *shows that, under the assumptions of Theorem 1.11, a form of* hypercontractivity *holds for the diffusively averaged partition function: moments of order $h > 2$ are controlled by the $\frac{h}{2}$-power of the second moment.*

We point out that hypercontractivity is a general property of Wiener chaos and polynomial chaos when the main contribution comes from chaos of bounded order [Jan97, MOO10]. *This is the case for the directed polymer partition function only in the sub-critical regime* (1.26)*, because in the quasi-critical and critical regimes* (1.22) *and* (1.19) the main contribution comes from chaos of unbounded order. *The fact that the partition function still satisfies a form of hypercontractivity in these latter regimes, by Therorem 1.11, is highly non-trivial.*

Remark 1.14 (Intermittency). *The bounded variance assumption* (1.32) *is crucial for Theorem 1.11. In fact it is* necessary *for the moment bound* (1.33) *to hold as it is a general fact that, for a sequence of nonnegative random variables X_n with mean* 1 *and diverging* *variance, we have*

$$\forall h > 2: \quad \mathbb{E}\big[X_N^h\big] \gg \mathbb{E}\big(X_N^2\big)^{h/2}.$$

This can be seen by defining the size-biased law $\mathrm{d}\tilde{\mathbb{P}}_N := X_N \, \mathrm{d}\mathbb{P}$ *and using Jensen's inequality:*

$$\mathbb{E}\big[X_N^h\big] = \tilde{\mathbb{E}}\big[X_N^{h-1}\big] \geqslant \tilde{\mathbb{E}}\big[X_N\big]^{h-1} = \mathbb{E}\big[X_N^2\big]^{h-1} \gg \mathbb{E}\big[X_N^2\big]^{\frac{h}{2}},$$

since $h > 2$ *and the second moment* $\mathbb{E}[X_N^2]$ *diverges as* $N \to \infty$.

In the case $X_N = Z_{L_N}^{\beta_N}(\varphi_N)$ *with the support of* φ_N *shrinking to* 0 *fast enough such that the variance diverges as* $N \to \infty$, *we actually expect a stronger intermittency of the form*

$$\forall h \geqslant 3: \qquad \mathbb{E}\big[Z_{L_N}^{\beta_N}(\varphi_N)^h\big] \geqslant \mathbb{E}[Z_{L_N}^{\beta_N}(\varphi_N)^2]^{\binom{h}{2}}.$$

Such results have been proved in the continuum setting of the Critical 2D SHF [CSZ23b, LZ24].

1.5. Extensions and related results. Theorem 1.2, which deals with the regime $\vartheta \to -\infty$, is proved by approximating the SHF by the directed polymer partition functions in the *quasi-critical regime* (1.22) (see Section 3). This strategy is actually very general and leads us to formulate the following "meta-theorem".

Metatheorem 1.15. *Consider any "reasonable" statement about the SHF* $\mathscr{Z}_t^\vartheta(\mathrm{d}x)$ *in the regime* $\vartheta \to -\infty$. *Such a statement holds if one can prove the corresponding statement for the rescaled directed polymer partition functions* $Z_{tN}^{\beta}([\![x\sqrt{N}]\!])\,\mathrm{d}x$ *with* $\beta = \beta_N^{\text{quasi-crit}}$ *in the quasi-critical regime* (1.22), *for any sequence* $|\vartheta_N| \to \infty$ *slowly enough.*

By "reasonable" statement we mean some property of locally finite measures on $\mathbb{R}^2$ which is *continuous with respect to the vague topology or the topology of* $\mathcal{C}^{-\varepsilon}$. Indeed, the basic idea behind Claim 1.15 is that the convergence in distribution (1.20) or (1.21), which we know to hold *in the critical regime*, can also be applied to $\vartheta = \vartheta_N \to -\infty$ slowly enough, allowing us to effectively transfer the statement from directed polymers to the SHF. To lighten the exposition, we refrain from formulating a more precise result: we rather refer to the proof of Theorem 1.2 in Section 3 for a concrete application of this idea.

For example, the quasi-critical regime (1.22) was recently investigated in [CCR23] for *diffusive initial conditions*, such as $\mathcal{U}_{B(0,\delta\sqrt{N})}$ for fixed $\delta > 0$. It was shown that the averaged partition function concentrates around its mean:

$$Z_{tN}^{\beta_N^{\text{quasi-crit}}}\big(\mathcal{U}_{B(0,\delta\sqrt{N})}\big) \xrightarrow[N\to\infty]{d} 1 \equiv \mathbb{E}\Big[Z_{tN}^{\beta_N^{\text{quasi-crit}}}\big(\mathcal{U}_{B(0,\delta\sqrt{N})}\big)\Big],$$

its variance vanishes at rate $|\vartheta_N|^{-1}$ [CCR23, Proposition 2.1], and *Gaussian fluctuations* emerge at the corresponding scale [CCR23, Theorem 1.1]:

$$\sqrt{|\vartheta_N|}\,\Big\{Z_{tN}^{\beta_N^{\text{quasi-crit}}}\big(\mathcal{U}_{B(0,\delta\sqrt{N})}\big) - 1\Big\} \xrightarrow[N\to\infty]{d} \mathcal{N}(0, a_\delta^2) \qquad \text{with} \quad 0 < a_\delta^2 < \infty. \tag{1.34}$$

From this one can deduce a corresponding results for the SHF in the weak disorder limit $\vartheta \to -\infty$, see [CCR23, Theorem 1.2], in the spirit of the metatheorem just stated:

$$\sqrt{|\vartheta|}\big(\mathscr{Z}_t^\vartheta(\mathcal{U}_{B(x,\delta)}) - 1\big) \xrightarrow[\vartheta\to-\infty]{d} \mathcal{N}(0, a_\delta^2). \tag{1.35}$$

Remark 1.16. *We expect* $\mathscr{Z}_t^{\vartheta}(\mathcal{U}_{B(x,\delta)})$ *to have Gaussian fluctuations, as* $\vartheta \to -\infty$*, on all spatial scales* $\delta = O(1)$ *satisfying* $\delta = \mathrm{e}^{-o(|\vartheta|)}$*, i.e. much larger than the scales* δ_ϑ^ϱ *on which log-normality arises in Theorem 1.2. Similarly, we expect the averaged partition function in* (1.34) *to have a Gaussian limit when averaged on spatial scales* $\delta\sqrt{N}$ *with* $\delta_N = O(1)$ *satisfying* $\delta = \mathrm{e}^{-o(|\vartheta_N|)}$*, i.e. much larger than the scale* δ_N^ϱ *appearing in Theorem 1.8.*

We note that in the subcritical regime (1.26)*, Gaussian fluctuations for spatial averages on such mesoscopic spatial scales have been established for the solution of the 2D KPZ equation in* [Tao24] *(the subcritical 2D SHE and polymer partition functions are expected to have the same fluctuations on the mesoscopic scale).*

2. Basic tools

In this section we collect some basic definitions and tools that we use in the proof. In particular, we present some key second moment computations on the partition function in the quasi-critical regime (1.22). For simplicity, we will abbreviate $\beta_N^{\text{quasi-crit}}$ by β_N.

2.1. Partition functions. Given $A < B \in 2\mathbb{Z}$ and a function $\varphi : \mathbb{Z}^2_{\text{even}} \to \mathbb{R}$, we denote by $Z_{(A,B]}(\varphi)$ the directed polymer partition function on the time interval $(A, B]$ with initial condition φ at time A:

$$Z^\beta_{(A,B]}(\varphi) := \mathrm{E}\Big[\mathrm{e}^{\mathcal{H}^\beta_{(A,B]}} \,\Big|\, S_A \sim \varphi\Big] := \sum_{x\in\mathbb{Z}^2_{\text{even}}} \varphi(x)\,\mathrm{E}[\mathrm{e}^{\mathcal{H}^\beta_{(A,B]}} | S_A = x] \tag{2.1}$$

where $\mathcal{H}^\beta_I$ is defined in (1.18). Note that $\mathbb{E}[\mathrm{e}^{\mathcal{H}^{\beta,\omega}_I(S)}] = 1$, and hence

$$\mathbb{E}[Z^\beta_{(A,B]}(\varphi)] = \sum_{x\in\mathbb{Z}^2_{\text{even}}} \varphi(x)\,.$$

In the special case $(A, B] = (0, L]$ and $\varphi = \mathbb{1}_{\{x\}}$, we write for short

$$Z^\beta_L(\varphi) := Z^\beta_{(0,L]}(\varphi) \qquad \text{and} \qquad Z^\beta_L(x) := Z^\beta_L(\mathbb{1}_{\{x\}})\,. \tag{2.2}$$

2.2. Random walk. Recall the random walk transition kernel $q_n(\cdot)$ from (1.14). We give two versions of the local central limit theorem for the simple symmetric random walk on $\mathbb{Z}^2$, see [LL10, Theorems 2.3.5 and 2.3.11]: uniformly for $z \in \mathbb{Z}^2$ and $n \in 2\mathbb{N}$, as $n \to \infty$

$$\begin{aligned} q_n(z) &= \Big(g_{\frac{n}{2}}(z) + O\big(\tfrac{1}{n^2}\big)\Big)\, 2\cdot\mathbb{1}_{\mathbb{Z}^2_{\text{even}}}(z) \\ &= g_{\frac{n}{2}}(z)\,\mathrm{e}^{O(\frac{1}{n})+O(\frac{|z|^4}{n^3})}\; 2\cdot\mathbb{1}_{\mathbb{Z}^2_{\text{even}}}(z)\,\mathbb{1}_{q_n(z)>0} \end{aligned} \qquad \text{where} \quad g_t(x) := \frac{\mathrm{e}^{-\frac{|x|^2}{2t}}}{2\pi t}\,, \tag{2.3}$$

the factor $2\cdot\mathbb{1}_{\mathbb{Z}^2_{\text{even}}}(z)$ is due to periodicity, while the time argument $\frac{n}{2}$ in the heat kernel comes from the random walk covariance $\mathrm{E}[S_n^{(i)}S_n^{(j)}] = \frac{n}{2}\mathbb{1}_{i=j}$ for $i,j \in \{1,2\}$. In particular, $q_{2n}(0) \sim \frac{1}{\pi}\cdot\frac{1}{n}$ as $n \to \infty$. For later use, we fix $0 < \mathfrak{a}_- < \mathfrak{a}_+ < \infty$ such that

$$\frac{\mathfrak{a}_-}{n} \leqslant q_{2n}(0) \leqslant \frac{\mathfrak{a}_+}{n} \qquad \forall n \in \mathbb{N}\,. \tag{2.4}$$

We generalize the expected replica overlap R_L from (1.15) by defining, for $z \in \mathbb{Z}^2_{\text{even}}$,

$$R_L(z) := \sum_{n=1}^{L} q_{2n}(z)\,, \tag{2.5}$$

which is nothing but the random walk Green's function (on a bounded time interval). We also introduce the corresponding quadratic form:

$$R_L(\varphi,\varphi) := \sum_{z,w\in\mathbb{Z}^2_{\mathrm{even}}} \varphi(z)\, R_L(z-w)\, \varphi(w)\,. \tag{2.6}$$

By (2.4), we can bound

$$R_L(z) - R_{\lfloor\frac12 L\rfloor}(z) = \sum_{n=\lfloor\frac12 L\rfloor+1}^{L} q_{2n}(z) \leqslant \sum_{n=\lfloor\frac12 L\rfloor+1}^{L} q_{2n}(0) \leqslant \mathfrak{a}_+$$

uniformly over $z\in\mathbb{Z}^2$. Therefore,

$$\text{for any probability mass function } \varphi: \qquad R_L(\varphi,\varphi) - R_{\lfloor\frac12 L\rfloor}(\varphi,\varphi) \leqslant \mathfrak{a}_+\,. \tag{2.7}$$

The continuum analogue of $R_L(\cdot)$ is the Green's function $\mathcal{G}(x) = \mathcal{G}(|x|)$ given by

$$\mathcal{G}(x) \;:=\; \int_0^1 g_t(x)\,\mathrm{d}t \;=\; \int_0^1 \frac{\mathrm{e}^{-\frac{|x|^2}{2t}}}{2\pi\, t}\,\mathrm{d}t \;=\; \frac{1}{2\pi}\int_{|x|^2}^{\infty} \frac{\mathrm{e}^{-\frac{r}{2}}}{r}\,\mathrm{d}r\,. \tag{2.8}$$

The following result compares $R_L(\cdot)$ and $\mathcal{G}(\cdot)$. The proof is given in Appendix A.

Lemma 2.1 (Green's function). *Uniformly for $L\in\mathbb{N}$ and $z\in\mathbb{Z}^2$ we can write*

$$R_L(z) \;=\; 2\,\mathcal{G}\Big(\tfrac{|z|+1}{\sqrt{L}}\Big)\,\mathbb{1}_{z\in\mathbb{Z}^2_{\mathrm{even}}} + O(1) \;=\; \tfrac{1}{\pi}\log\Big(1+\tfrac{L}{1+|z|^2}\Big)\,\mathbb{1}_{z\in\mathbb{Z}^2_{\mathrm{even}}} + O(1)\,. \tag{2.9}$$

Moreover, for any $t\in(0,\infty)$, there is $c_t>0$ such that

$$\begin{array}{c}\text{uniformly for } L\in\mathbb{N},\ z\in\mathbb{Z}^2 \\ \text{with } |z|\leqslant t\sqrt{L}\wedge L:\end{array} \qquad R_L(z) \geqslant c_t\,\log\Big(1+\tfrac{L}{1+|z|^2}\Big)\,\mathbb{1}_{z\in\mathbb{Z}^2_{\mathrm{even}}}\,. \tag{2.10}$$

(The restriction $|z|\leqslant L$ in (2.10) ensures that $R_L(z)\geqslant q_{2L}(z)>0$ for $z\in\mathbb{Z}^2_{\mathrm{even}}$.)

2.3. Polynomial chaos expansion. Let us introduce random variables

$$\xi^{\beta}_{n,x} := \frac{\mathrm{e}^{\beta\omega(n,x)-\lambda(\beta)}}{\sigma_\beta}\,,$$

which are i.i.d. with zero mean and unit variance, thanks to the definition (1.16) of σ_β. We can represent the point-to-plane partition function $Z^\beta_L(x)$ as a *polynomial chaos expansion*:

$$Z^\beta_L(x) = 1 + \sum_{k=1}^{L} \sigma_\beta^k \sum_{\substack{0=n_0<n_1<\ldots<n_k\leqslant L\\ x_0=x,\ x_1,\ldots,x_k\in\mathbb{Z}^2}} \prod_{i=1}^{k} q_{n_i-n_{i-1}}(x_i-x_{i-1})\,\xi^\beta_{n_i,x_i}\,. \tag{2.11}$$

see e.g. [CSZ20, eq. (2.17)]. This follows from the definition (1.17)-(1.18) by writing

$$\mathrm{e}^{\mathcal{H}^\beta_{(0,L]}} = \prod_{n=1}^{L}\prod_{x\in\mathbb{Z}^2} \mathrm{e}^{\{\beta\omega(n,x)-\lambda(\beta)\}\mathbb{1}_{\{S_n=x\}}} = \prod_{n=1}^{L}\prod_{x\in\mathbb{Z}^2}\Big\{1+\sigma_\beta\,\xi^\beta_{n,x}\,\mathbb{1}_{\{S_n=x\}}\Big\}$$

and then expanding the product.

Remark 2.2 (Switching off some disorder). *We will later consider partition functions where disorder is "switched off" in a time interval* $(A, B] \subseteq (0, L]$*, meaning that the Hamiltonian* $\mathcal{H}^{\beta}_{(0,L]}$ *is replaced by* $\mathcal{H}^{\beta}_{(0,A]} + \mathcal{H}^{\beta}_{(B,L]}$*, see* (1.18)*. This amounts to setting* $\xi_{n,x} = 0$ *for all* $n \in (A, B]$*, which is equivalent to restricting the polynomial chaos* (2.11) *to sequences of times* $n_1, \ldots, n_k$ *which avoid the interval* $(A, B]$*.*

2.4. Second moment of point-to-plane partition function.

Recalling (1.16), we define a weighted renewal function $U_\beta(\cdot)$ by setting $U_\beta(0) := 1$ and for $n \geqslant 1$

$$\begin{aligned} U_\beta(n) &:= \sum_{k \geqslant 1} (\sigma_\beta^2)^k \sum_{\substack{0 =: n_0 < n_1 < \cdots < n_k = n \\ x_0 := 0,\ x_1, \ldots, x_k \in \mathbb{Z}^2}} \prod_{i=1}^{k} q_{n_i - n_{i-1}}(x_i - x_{i-1})^2 \\ &= \sum_{k \geqslant 1} (\sigma_\beta^2)^k \sum_{0 =: n_0 < n_1 < \cdots < n_k = n} \prod_{i=1}^{k} q_{2(n_i - n_{i-1})}(0)\,, \end{aligned} \tag{2.12}$$

where the last equality holds by the fact that $\sum_{z \in \mathbb{Z}^2} q_n(z)^2 = \sum_{z \in \mathbb{Z}^2} q_n(z)\, q_n(-z) = q_{2n}(0)$.

The quantity $U_\beta(\cdot)$ arises in the second moment of the *point-to-plane partition function*:

$$\mathbb{E}\big[Z_L^\beta(0)^2\big] = \overline{U}_\beta(L) := \sum_{n=0}^{L} U_\beta(n)\,, \tag{2.13}$$

which follows by the polynomial chaos expansion (2.11) noting that terms indexed by distinct space-time sequences $(n_1, x_1), \ldots, (n_k, x_k)$ are orthogonal in L^2.

The second moment $\mathbb{E}\big[Z_L^\beta(0)^2\big]$ is uniformly bounded in $L \leqslant N$ and $N \in \mathbb{N}$ when $\beta = \beta_N$ lies in the sub-critical regime (1.26) [CSZ19a]. The next result considers the quasi-critical regime (1.22) and identifies how the second moment diverges as a function of ϑ_N in (1.22). The proof, based on renewal theory, is deferred to Appendix A.

Proposition 2.3 (Second moment of point-to-plane partition function). *For* $\beta = \beta_N$ *in the quasi-critical regime* (1.22)*, uniformly over* $L \in 2\mathbb{N}$ *with* $L \leqslant N$*, we have*

$$\mathbb{E}\big[Z_L^{\beta_N}(0)^2\big] = \overline{U}_{\beta_N}(L) \sim \frac{1}{1 - \frac{R_L}{R_N}\big(1 - \frac{|\vartheta_N|}{\log N}\big)} \qquad \text{as } N \to \infty\,. \tag{2.14}$$

2.5. Variance of averaged partition functions.

We finally compute the variance of the *averaged partition function* $Z_L^\beta(\varphi) = \sum_{x \in \mathbb{Z}^2_{\mathrm{even}}} \varphi(x)\, Z_L^\beta(x)$. We write

$$\mathbb{V}\mathrm{ar}\big[Z_L^\beta(\varphi)\big] = \sum_{x, x' \in \mathbb{Z}^2_{\mathrm{even}}} \varphi(x)\, \varphi(x')\ \mathbb{C}\mathrm{ov}\big[Z_L^\beta(x), Z_L^\beta(x')\big]\,.$$

Plugging in the polynomial chaos expansion (2.11) and renaming $n_1 = m$ and $n_k = n$, by (2.12), we can write

$$\begin{aligned} \mathbb{C}\mathrm{ov}\big[Z_L^\beta(x), Z_L^\beta(x')\big] &= \sum_{0 < m \leqslant n \leqslant L} \sum_{x_1 \in \mathbb{Z}^2} q_m(x_1 - x)\, q_m(x_1 - x')\, \sigma_\beta^2\, U_\beta(n - m) \\ &= \sum_{0 < m \leqslant L} q_{2m}(x - x')\ \sigma_\beta^2\ \overline{U}_\beta(L - m) \\ &= \sum_{0 < m \leqslant L} q_{2m}(x - x')\ \sigma_\beta^2\ \mathbb{E}\big[Z_{L-m}^\beta(0)^2\big]\,, \end{aligned}$$

where we applied (2.13). In summary, introducing the shorthand

$$q_{2m}(\varphi,\varphi) := \sum_{x,x' \in \mathbb{Z}^2_{\mathrm{even}}} \varphi(x)\,\varphi(x')\,q_{2m}(x-x')\,,$$

we have thus obtained the key formula

$$\mathbb{V}\mathrm{ar}\big[Z^{\beta}_{L}(\varphi)\big] \;=\; \sum_{0<m\leqslant L} q_{2m}(\varphi,\varphi)\;\sigma_\beta^2\;\mathbb{E}\big[Z^{\beta}_{L-m}(0)^2\big]\,. \tag{2.15}$$

Since $L \mapsto \mathbb{E}\big[Z^{\beta}_{L}(0)^2\big] = \overline{U}_\beta(L)$ is increasing, recalling the definition (2.6) of $R_L(\varphi,\varphi)$, we obtain the bounds

$$R_{\lfloor \frac{1}{2}L\rfloor}(\varphi,\varphi)\;\sigma_\beta^2\;\mathbb{E}\big[Z^{\beta}_{\lfloor \frac{1}{2}L\rfloor}(0)^2\big] \;\leqslant\; \mathbb{V}\mathrm{ar}\big[Z^{\beta}_{L}(\varphi)\big] \;\leqslant\; R_L(\varphi,\varphi)\;\sigma_\beta^2\;\mathbb{E}\big[Z^{\beta}_{L}(0)^2\big]\,. \tag{2.16}$$

Remark 2.4. *We can also rewrite* (2.15) *more explicitly as*

$$\mathbb{V}\mathrm{ar}\big[Z^{\beta}_{L}(\varphi)\big] \;=\; \sum_{k\geqslant 1}(\sigma_\beta^2)^k \sum_{0<n_1<\cdots<n_k\leqslant L} q_{2n_1}(\varphi,\varphi)\prod_{i=2}^{k} q_{2(n_i-n_{i-1})}(0)\,. \tag{2.17}$$

We now compute the asymptotic behavior of $\mathbb{V}\mathrm{ar}\big[Z^{\beta}_{L}(\varphi)\big]$ for $\beta=\beta_N$ in the quasi-critical regime (1.22), allowing for general system size $L=L_N$ and initial condition $\varphi=\varphi_N$ (this will be essential for the proof of our results).

In Theorem 1.8, we consider the partition function $Z^{\beta_N}_{L_N}(\varphi_N)$ of size $L_N=N$ and initial conditions φ_N averaged on balls of radius $\delta_N^w\sqrt{N}$ for $w\in(0,\infty)$ (recall δ_N from (1.24)). Our next results computes the variance of $Z^{\beta_N}_{L_N}(\varphi_N)$, showing that *it is bounded away from zero and infinity* for general initial conditions φ_N that are "spread out" on the scale $\delta_N^w\sqrt{N}$ and for general system sizes $L_N = N\,(\delta_N^2)^{\ell+o(1)}$ with $\ell<w$.

Theorem 2.5 (Variance of averaged partition functions). *Let β_N be in the quasi-critical regime* (1.22) *for a given sequence $1 \ll |\vartheta_N| \ll \log N$. Recall δ_N from* (1.24).

Let us fix two exponents $0\leqslant \ell\leqslant w<\infty$. For $N\in\mathbb{N}$ we consider:

- *two sequences $L_N\in 2\mathbb{N}$ (system size), $W_N\geqslant 0$ (scaling factor) such that, as $N\to\infty$,*

$$L_N = N\,(\delta_N^2)^{\ell+o(1)} = N\,\mathrm{e}^{-\ell\,|\vartheta_N|+o(|\vartheta_N|)} \qquad \text{with } \ell\geqslant 0\,, \tag{2.18}$$

$$W_N = N\,(\delta_N^2)^{w+o(1)} = N\,\mathrm{e}^{-w\,|\vartheta_N|+o(|\vartheta_N|)} \qquad \text{with } w\geqslant \ell\,; \tag{2.19}$$

- *probability mass functions $\varphi_N : \mathbb{Z}^2_{\mathrm{even}} \to [0,\infty)$ "spread out" on scale $\sqrt{W_N}$ in the following sense (recall* (2.9)*):*

$$R_{L_N}(\varphi_N,\varphi_N) \;=\; \frac{1}{\pi}\log\frac{L_N}{W_N} + o(|\vartheta_N|) \;=\; \frac{w-\ell}{\pi}\,|\vartheta_N| + o(|\vartheta_N|)\,. \tag{2.20}$$

Then

$$\lim_{N\to\infty}\mathbb{V}\mathrm{ar}\big[Z^{\beta_N}_{L_N}(\varphi_N)\big] = \frac{w-\ell}{1+\ell}\,. \tag{2.21}$$

Moreover, the convergence (2.21) *holds uniformly over system sizes $(L_N)_{N\in\mathbb{N}}$, scaling factors $(W_N)_{N\in\mathbb{N}}$ and initial conditions $(\varphi_N)_{N\in\mathbb{N}}$ for which* (2.18), (2.19), (2.20) *hold uniformly.*

Remark 2.6 (Sub-critical regime). *Theorem 2.5 can also be applied to the sub-critical regime* (1.26)*: it suffices to take $|\vartheta_N| \sim (1-\hat\beta^2)\log N$ with $\hat\beta^2\in(0,1)$ (cf.* (1.22) *and* (1.26)*),* *but the final result* (2.21) *must be updated as follows:*

$$\lim_{N\to\infty} \mathbb{V}\mathrm{ar}\big[Z_{L_N}^{\beta_N}(\varphi_N)\big] = \frac{(w-\ell)\,\hat\beta^2}{1+\ell\,\hat\beta^2}\,. \tag{2.22}$$

The minor changes required in the proof are described in Remark A.2.

The proof of Theorem 2.5 is deferred to Appendix A. Condition (2.20) means intuitively that, sampling two points x, y independently from φ_N, their distance $|x-y|$ is roughly order $\sqrt{W_N}$. This is made precise by the next results, also proved in Appendix A.

Proposition 2.7 (Equivalent condition for (2.20)**).** *Let L_N, W_N be as in* (2.18), (2.19). *Condition* (2.20) *for the probability mass functions φ_N is equivalent to*

$$\sum_{x,y\in\mathbb{Z}^2_{\mathrm{even}}} \varphi_N(x)\,\varphi_N(y)\,\log\Big(1+\tfrac{L_N}{1+|x-y|^2}\Big) = \log\frac{L_N}{W_N} + o(|\vartheta_N|) \qquad \text{as } N\to\infty\,. \tag{2.23}$$

Proposition 2.8 (Sufficient condition for (2.20)**).** *Let L_N, W_N be as in* (2.18), (2.19). *For probability mass functions φ_N to satisfy* (2.20) *(or, equivalently,* (2.23)*) it suffices that they are "mostly supported on a ball of radius $\sqrt{W_N}$ with atoms of size $O(1/W_N)$" in the following sense: there exist $z_N\in\mathbb{Z}^2$ and $0\leqslant t_N = o(\vartheta_N)$ such that*

$$\sum_{|x-z_N|\leqslant\sqrt{W_N\,\mathrm{e}^{t_N}}} \varphi_N(x) = 1-o(1) \qquad \text{and} \qquad \sup_{x\in\mathbb{Z}^2}\varphi_N(x)\leqslant\frac{\mathrm{e}^{t_N}}{W_N}\,, \tag{2.24}$$

In particular, by (2.24), condition (2.20) is satisfied when φ_N is the uniform distribution on a ball or when φ_N is the random walk transition kernel, see (1.23) and (1.14):

$$\varphi_N = \mathcal{U}_{B\left(0,\sqrt{W_N\,\mathrm{e}^{t_N}}\right)} \quad \text{and} \quad \varphi_N = q_{W_N\,\mathrm{e}^{t_N}} \quad \text{with } t_N = o(|\vartheta_N|) \text{ satisfy (2.20)}\ . \tag{2.25}$$

We finally compute the variance of the partition function $Z_{Nt}^{\beta_N}(\varphi_N)$ when φ_N is the uniform distribution in the ball $B(0,\delta_N^{\varrho}\sqrt{N})$, as in Theorem 1.8.

Corollary 2.9. *Let β_N be in the quasi-critical regime* (1.22) *and recall δ_N from* (1.24). *For any $t>0$, $x\in\mathbb{R}^2$ we have*

$$\forall\varrho\in(0,\infty): \qquad \lim_{N\to\infty}\mathbb{V}\mathrm{ar}\big[Z_{Nt}^{\beta_N}(\mathcal{U}_{B(0,\delta_N^{\varrho}\sqrt{N})})\big] = \varrho\,. \tag{2.26}$$

Proof. The initial condition $\varphi_N = \mathcal{U}_{B(0,\delta_N^{\varrho}\sqrt{N})}$ fulfills (2.24) with $W_N := N\delta_N^{2\varrho}$ (recall (1.23)). Since W_N satisfies (2.19) with $w=\varrho$, while $L_N := Nt$ satisfies (2.18) with $\ell=0$, the assumptions of Theorem 2.5 are verified and we obtain (2.26) from (2.21). □

Remark 2.10. *Since* $\mathbb{V}\mathrm{ar}[\mathrm{e}^{\mathcal{N}(0,\sigma^2)-\frac12\sigma^2}] = \mathrm{e}^{\sigma^2}-1$, *relation* (2.26) *is consistent with* (1.25).

3. Proofs for the SHF

In this section, we prove our main results for the SHF. More precisely, we prove

- Theorem 1.1 (singularity) in Subsection 3.1;
- Theorem 1.2 (log-normality) in Subsection 3.2;
- Theorem 1.4 (long-time behavior) in Subsection 3.3;

- Theorem 1.5 (regularity) in Subsection 3.4, alongside the corresponding Theorem 1.7 for directed polymers.

We first state a basic monotonicity result. For integrable $\varphi : \mathbb{R}^2 \to \mathbb{R}$, we will write

$$\mathscr{Z}_t^\vartheta(\varphi) := \int_{\mathbb{R}^2} \varphi(x)\, \mathscr{Z}_t^\vartheta(\mathrm{d}x),$$

which is well-defined and has mean $\int_{\mathbb{R}^2} \varphi(x)\,\mathrm{d}x$.

Lemma 3.1 (Convex monotonicity for the SHF). *Fix $t > 0$ and a positive integrable function $\varphi : \mathbb{R}^2 \to \mathbb{R}^+$. The law of $\mathscr{Z}_t^\vartheta(\varphi)$ is* increasing in ϑ w.r.t. the convex order*, i.e., for any convex function $\Psi : \mathbb{R} \to \mathbb{R}$, we have*

$$\mathbb{E}\big[\Psi\big(\mathscr{Z}_t^{\vartheta'}(\varphi)\big)\big] \leqslant \mathbb{E}\big[\Psi\big(\mathscr{Z}_t^{\vartheta}(\varphi)\big)\big] \qquad \text{for } \vartheta' < \vartheta\,, \tag{3.1}$$

and the reverse inequality holds for concave Ψ.

In particular, fractional moments of the SHF are decreasing in ϑ:

$$\forall \alpha \in (0,1): \qquad \mathbb{E}\big[\mathscr{Z}_t^{\vartheta'}(\varphi)^\alpha\big] \geqslant \mathbb{E}\big[\mathscr{Z}_t^{\vartheta}(\varphi)^\alpha\big] \qquad \text{for } \vartheta' < \vartheta\,. \tag{3.2}$$

Proof. It is enough to prove (3.1) when $\Psi(x) = O(x)$ grows at most linearly as $x \to \infty$, since the general case follows by monotone convergence. We may also assume that φ is continuous and compactly supported, because such functions are dense in L^1.

By the weak convergence (1.20), it is enough to prove (3.1) when the SHF $\mathscr{Z}_t^\vartheta$ is replaced by the rescaled directed polymer partition function $Z_{tN}^{\beta_N^{\mathrm{crit}}}$, because the limit $N \to \infty$ is justified by uniform integrability (via boundedness in L^2). More generally, we claim that

$$\mathbb{E}\big[\Psi\big(Z_N^{\beta'}(\varphi)\big)\big] \leqslant \mathbb{E}\big[\Psi\big(Z_N^{\beta}(\varphi)\big)\big] \qquad \text{for } \beta' < \beta\,, \tag{3.3}$$

for any $N \in \mathbb{N}$ and for any positive integrable $\varphi : \mathbb{Z}^2_{\mathrm{even}} \to \mathbb{R}^+$.

Relation (3.3) is known to hold by the FKG inequality, see e.g. the proofs of [Com17, Proposition 3.1] or [CSZ17, Theorem 2.8] where the arguments are carried out for fractional moments, but they hold in general. ☐

3.1. Singularity of the SHF. We now prove Theorem 1.1 on the singularity of the SHF. Let us first recall some general facts about measures on the Euclidean space.

By the Lebesgue Decomposition Theorem [Fol99, Theorem 3.8], any σ-finite measure ν on $\mathbb{R}^2$ is the sum $\nu = \nu^{\mathrm{ac}} + \nu^{\mathrm{sing}}$ of an *absolutely continuous part* $\nu^{\mathrm{ac}}(\mathrm{d}x) = f(x)\,\mathrm{d}x$ and a *singular part* $\nu^{\mathrm{sing}}(\mathrm{d}x) \perp \mathrm{d}x$ which assigns all its mass to a set of zero Lebesgue measure.

If furthermore ν is *locally finite*, then the density $f(x)$ of ν^{ac} can be computed as follows: denoting by $B(x,\delta) := \{y \in \mathbb{R}^2 \colon |y-x| < \delta\}$ the Euclidean ball in $\mathbb{R}^2$, we have

$$f(x) = \lim_{\delta \downarrow 0} \frac{\nu(B(x,\delta))}{\pi\,\delta^2} \qquad \text{for Lebesgue a.e. } x \in \mathbb{R}^2\,, \tag{3.4}$$

see [Fol99, Theorem 3.22] (any locally finite measure is regular by [Fol99, Theorem 7.8]).

In particular, we summarise the following general result.

Proposition 3.2 (Singularity of measures). *Given a locally finite measure ν on $\mathbb{R}^2$, the limit in* (3.4) *exists for Lebesgue a.e. $x \in \mathbb{R}^2$ and recovers the density $f(x)$ of the absolutely continuous part of ν. In particular, ν is singular with respect to the Lebesgue measure if and only if the limit in* (3.4) *vanishes for Lebesgue a.e. $x \in \mathbb{R}^2$.*

Proof of Theorem 1.1. Applying Proposition 3.2 to the SHF $\nu(\mathrm{d}x) = \mathscr{Z}_t^{\vartheta}(\mathrm{d}x)$, we see that if (1.10) holds, then almost surely the SHF is singular with respect to Lebesgue.

It remains to prove (1.10), which we deduce from (1.12). We denote by $(\Omega, \mathcal{A}, \mathbb{P})$ the probability space on which the SHF is defined and we indicate explicitly the dependence on $\omega \in \Omega$ by $\mathscr{Z}_t^{\vartheta,\omega}(\mathrm{d}x)$. Recalling (1.9), we rephrase (1.10) as

$$\text{for a.e. } \omega \in \Omega: \qquad L(\omega, x) := \liminf_{n\to\infty} \ \mathscr{Z}_t^{\vartheta,\omega}\big(\mathcal{U}_{B(x,\delta_n)}\big) = 0 \quad \text{for Lebesgue a.e. } x \in \mathbb{R}^2 \,, \tag{3.5}$$

where we have fixed (arbitrarily) $\delta_n := \frac{1}{n}$ and we have replaced lim by lim inf, in order to obtain a measurable function $L(\omega, x) \in [0, \infty]$ defined for all $\omega \in \Omega$ and $x \in \mathbb{R}^2$. We stress that the limit in (1.10) *exists as* $\delta \downarrow 0$ *for Lebesgue a.e.* $x \in \mathbb{R}^2$, see Proposition 3.2, hence it must coincide with $L(\omega, x)$ for a.e. $\omega \in \Omega$ and for Lebesgue a.e. $x \in \mathbb{R}^2$.

To complete the proof, we need to show that, for a.e. $\omega \in \Omega$, we have $L(\omega, x) = 0$ for Lebesgue a.e. $x \in \mathbb{R}^2$, or equivalently $\mathbb{E}[\int_{\mathbb{R}^2} L(\omega, x)^\alpha \, \mathrm{d}x] = 0$ for any fixed $\alpha \in (0, 1)$ (recall that $L(\omega, x) \geqslant 0$). By Fubini's theorem, it is enough to show that for all $x \in \mathbb{R}^2$ we have $\mathbb{E}[L(\omega, x)^\alpha] = 0$. By Fatou's Lemma

$$\mathbb{E}[L(\omega, x)^\alpha] \leqslant \liminf_{n\to\infty} \ \mathbb{E}\big[\mathscr{Z}_t^{\vartheta,\omega}\big(\mathcal{U}_{B(x,\delta_n)}\big)^\alpha\big] \tag{3.6}$$

and it remains to estimate the RHS. To this end, we exploit the *monotonicity of fractional moments* (3.2) by replacing ϑ with $\vartheta'_\delta \downarrow -\infty$ and applying the *log-normality* (1.12). Let us fix a parameter $\varrho \in (0, \infty)$.

- We rewrite the log-normality (1.12) by renaming $\delta_\vartheta^\varrho = \mathrm{e}^{\frac{1}{2}\varrho\vartheta}$ as δ, i.e. expressing $\vartheta = -\frac{1}{\varrho}\log\frac{1}{\delta^2}$ as a function of δ, so that (1.12) becomes
$$\mathscr{Z}_t^{-\frac{1}{\varrho}\log\frac{1}{\delta^2}}\big(\mathcal{U}_{B(x,\delta)}\big) \ \xrightarrow[\delta\downarrow 0]{d} \ \mathrm{e}^{\mathcal{N}(0,\sigma^2)-\frac{1}{2}\sigma^2} \qquad \text{with} \quad \sigma^2 = \log(1+\varrho) \,. \tag{3.7}$$
We note that this weak convergence also implies convergence of fractional moments, because the LHS of (3.7) is bounded in L^1 (recall that $\mathbb{E}[\mathscr{Z}_t^{\vartheta}(\mathrm{d}x)] = \mathrm{d}x$).
- If we set $\vartheta'_\delta := -\frac{1}{\varrho}\log\frac{1}{\delta^2} \to -\infty$ as $\delta \downarrow 0$, then we can apply the monotonicity of fractional moments (3.2) with $\varphi = \mathcal{U}_{B(x,\delta)}$ to estimate the RHS of (3.6).

Overall, we obtain for any fixed $\alpha \in (0, 1)$

$$\forall \varrho \in (0,\infty): \qquad \mathbb{E}[L(\omega, x)^\alpha] \leqslant \mathbb{E}\Big[\Big(\mathrm{e}^{\mathcal{N}(0,\sigma^2)-\frac{1}{2}\sigma^2}\Big)^\alpha\Big] = \mathrm{e}^{\frac{1}{2}(\alpha^2-\alpha)\sigma^2} = \frac{1}{(1+\varrho)^{\frac{\alpha(1-\alpha)}{2}}} \,,$$

where in the last equality we plugged in the value of $\sigma = \log(1 + \varrho)$ from (3.7). Since $\alpha(1 - \alpha) > 0$ for $\alpha \in (0, 1)$, letting $\varrho \to \infty$ we finally obtain $\mathbb{E}[L(\omega, x)^\alpha] = 0$. $\square$

3.2. Log-normality of the SHF. We prove Theorem 1.2 on the log-normality of the SHF. More precisely we deduce it from the corresponding result for directed polymers, see Theorem 1.8 (which we prove in Section 4).

To this end, recalling the uniform distribution on continuum and discrete balls, see (1.8) and (1.23), from the convergence in distribution (1.20), we obtain

$$\forall x \in \mathbb{R}^2, \ \delta > 0: \qquad Z_{tN}^{\beta_N^{\mathrm{crit}}}\Big(\mathcal{U}_{B(x\sqrt{N},\delta\sqrt{N})}\Big) \ \xrightarrow[N\to\infty]{d} \ \mathscr{Z}_t^{\vartheta}\big(\mathcal{U}_{B(x,\delta)}\big) \,. \tag{3.8}$$

Strictly speaking we cannot plug $\mathcal{U}_{B(x,\delta)}(\cdot)$ as a test function into (1.20), because it is not continuous. However, for any $\varepsilon > 0$, we can approximate $\varphi_\varepsilon(\cdot) \leqslant \mathcal{U}_{B(x,\delta)}(\cdot) \leqslant \psi_\varepsilon(\cdot)$ with continuos functions $\varphi_\varepsilon, \psi_\varepsilon$ such that $0 \leqslant \psi_\varepsilon(\cdot) - \varphi_\varepsilon(\cdot) \leqslant \mathbb{1}_{A_\varepsilon}(\cdot)$, where we define the annulus $A_\varepsilon := B(x, \delta + \varepsilon) \backslash B(x, \delta - \varepsilon)$. Replacing $\mathcal{U}_{B(x,\delta)}(\cdot)$ by φ_ε or ψ_ε in (1.20), we commit an error in L^1 which is $O(\varepsilon)$, i.e. the Lebesgue measure of A_ε. This justifies (3.8).

Proof of Theorem 1.2. We fix $t > 0$, $x \in \mathbb{R}^2$, $\varrho \in (0, \infty)$ and define $\delta_\vartheta := \mathrm{e}^{\frac{1}{2}\vartheta}$ as in (1.11). It suffices to prove the convergence in distribution (1.12) when ϑ ranges in an arbitrary negative sequence $\vartheta_k = -|\vartheta_k| \to -\infty$, which we fix henceforth. Introducing the shorthands

$$Y_k := \mathscr{Z}_t^{\vartheta_k}\big(\mathcal{U}_{B(x,\delta^{\varrho}_{\vartheta_k})}\big) , \qquad Y := \mathrm{e}^{\mathcal{N}(0,\sigma^2) - \frac{1}{2}\sigma^2} \quad \text{with} \quad \sigma^2 = \log(1+\varrho) ,$$

we need to show that $Y_k \to Y$ in distribution. It suffices to fix any bounded and continuous function $\Phi : \mathbb{R} \to \mathbb{R}$ and to prove that

$$\lim_{k\to\infty} \mathbb{E}[\Phi(Y_k)] = \mathbb{E}[\Phi(Y)] . \tag{3.9}$$

The idea is to approximate the SHF with the directed polymer partition function. Recalling the convergence in distribution (3.8) in the critical regime (1.19), we abbreviate

$$W_{N,k} := Z_{tN}^{\beta_N^{\mathrm{crit}}(k)}\big(\mathcal{U}_{B(x\sqrt{N},\delta^{\varrho}_{\vartheta_k}\sqrt{N})}\big) \qquad \text{where} \quad \sigma^2_{\beta_N^{\mathrm{crit}}(k)} = \frac{1}{R_N}\left(1 + \frac{\vartheta_k}{\log N}\right) . \tag{3.10}$$

For fixed $k \in \mathbb{N}$, we have $W_{N,k} \to Y_k$ in distribution as $N \to \infty$. Therefore we can choose $N = N_k$ large enough so that $\mathbb{E}[\Phi(W_{N_k,k})]$ is close to $\mathbb{E}[\Phi(Y_k)]$. Let us define

$$\begin{aligned} N_1 &:= \min\big\{N \in \mathbb{N} \colon\ |\mathbb{E}[\Phi(W_{N,1})] - \mathbb{E}[\Phi(Y_1)]| \leqslant 1\big\} , \\ N_k &:= \min\big\{N > N_{k-1} \colon\ N \geqslant \mathrm{e}^{k|\vartheta_k|} \ \text{ and } \ |\mathbb{E}[\Phi(W_{N,k})] - \mathbb{E}[\Phi(Y_k)]| \leqslant \tfrac{1}{k}\big\} , \end{aligned}$$

so that we have by construction $N_1 < N_2 < \ldots$ and, for every $k \in \mathbb{N}$,

$$N_k \geqslant \mathrm{e}^{k|\vartheta_k|} , \qquad |\mathbb{E}[\Phi(W_{N_k,k})] - \mathbb{E}[\Phi(Y_k)]| \leqslant \tfrac{1}{k} . \tag{3.11}$$

By the triangle inequality, our goal (3.9) holds if we show that

$$\lim_{k\to\infty} \mathbb{E}[\Phi(W_{N_k,k})] = \mathbb{E}[\Phi(Y)] . \tag{3.12}$$

We claim that this holds by (1.25), because *the directed polymer partition functions* $W_{N_k,k}$ *from* (3.10) *satisfy the assumptions of Theorem 1.8.* Note indeed that:

- the sequence $\beta_N^{\mathrm{crit}}(k)$ for $N = N_k$ is in the *quasi-critical regime* (1.22), since by (3.10)
$$\sigma^2_{\beta_{N_k}^{\mathrm{crit}}(k)} = \frac{1}{R_{N_k}}\left(1 + \frac{\vartheta_k}{\log N_k}\right) = \frac{1}{R_{N_k}}\left(1 - \frac{|\vartheta_k|}{\log N_k}\right) \qquad \text{with} \quad 1 \ll |\vartheta_k| \ll \log N_k ,$$
where $|\vartheta_k| \ll \log N_k$ holds by the first inequality in (3.11);
- the initial condition $\mathcal{U}_{B(x\sqrt{N},\delta^{\varrho}_{\vartheta_k}\sqrt{N})}$ for $N = N_k$ satisfies the requirement that $\delta_{\vartheta_k} \downarrow 0$ at rate (1.24), because by definition $\delta_{\vartheta_k} := \mathrm{e}^{\frac{1}{2}\varrho\,\vartheta_k} = \mathrm{e}^{-\frac{1}{2}\varrho\,|\vartheta_k|}$ as in (1.11).

We can thus apply Theorem 1.8 to $(W_{N_k,k})_{k\in\mathbb{N}}$: relation (1.25) along the subsequence $N = N_k$ yields directly (3.12) and completes the proof. ☐

Remark 3.3. *The strategy in the proof of Theorem 1.2 is very general and it shows that the convergence in distribution* (1.20) *or* (1.21)*, which are proved for each* ϑ *in the critical regime* (1.19)*, can effectively be transferred to the quasi critical regime* (1.22)*, provided we take* $\vartheta_N \to -\infty$ *slow enough. This naturally leads to Metatheorem 1.15.*

3.3. Long-time behaviour of the SHF. We prove Theorem 1.4 on the long-time behaviour of the SHF. This is a corollary of Theorem 1.1 on the singularity of the SHF, together with the following scale-covariance property, proved in [CSZ23a, Theorem 1.2]:

$$\forall \mathsf{a} > 0: \qquad \mathscr{Z}^{\vartheta}_{\mathsf{a}t}\big(\mathcal{U}_{B(0,\sqrt{\mathsf{a}}\,R)}\big) \overset{d}{=} \mathscr{Z}^{\vartheta+\log \mathsf{a}}_{t}\big(\mathcal{U}_{B(0,R)}\big)\,, \tag{3.13}$$

which holds for any $t, R \in (0,\infty)$ and $\vartheta \in \mathbb{R}$.

Proof of Theorem 1.4. To prove (1.13), it suffices to show that

$$\forall R < \infty: \qquad \mathscr{Z}^{\vartheta}_{t}\big(\mathcal{U}_{B(0,R)}\big) \xrightarrow[t\to\infty]{d} 0\,, \tag{3.14}$$

which follows if we show that for some fixed $\alpha \in (0,1)$ the fractional moment vanishes:

$$\forall R < \infty: \qquad \lim_{t\to\infty} \mathbb{E}\big[\mathscr{Z}^{\vartheta}_{t}\big(\mathcal{U}_{B(0,R)}\big)^{\alpha}\big] \;=\; 0\,. \tag{3.15}$$

Exploiting first the scaling relation (3.13) with ϑ replaced by $\vartheta - \log \mathsf{a}$ and $\mathsf{a} = t^{-1}$, and then the monotonicity of fractional moments (3.2), we obtain for all $t \geqslant 1$

$$\mathbb{E}\big[\mathscr{Z}^{\vartheta}_{t}\big(\mathcal{U}_{B(0,R)}\big)^{\alpha}\big] = \mathbb{E}\Big[\mathscr{Z}^{\vartheta+\log t}_{1}\Big(\mathcal{U}_{B(0,R/\sqrt{t})}\Big)^{\alpha}\Big] \leqslant \mathbb{E}\Big[\mathscr{Z}^{\vartheta}_{1}\Big(\mathcal{U}_{B(0,R/\sqrt{t})}\Big)^{\alpha}\Big]\,.$$

Applying (1.10) with $\delta = \frac{R}{\sqrt{t}}$, we see that $A_t := \mathscr{Z}^{\vartheta}_{1}\Big(\mathcal{U}_{B(0,R/\sqrt{t})}\Big) \to 0$ in distribution as $t \to \infty$. The random variables $(A_t)_{t\geqslant 1}$ are bounded in L^1, because $\mathbb{E}[A_t] = 1$, hence by uniform integrability we obtain $\mathbb{E}[A_t^{\alpha}] \to 0$ for any $\alpha \in (0,1)$, which completes the proof. ☐

3.4. Improved convergence and regularity of the SHF. Theorem 1.7 entails that, almost surely, $\mathscr{Z}^{\vartheta}_{t} \in \mathcal{C}^{-\varepsilon}$ for every $\varepsilon > 0$, hence Theorem 1.5 follows (delta measures δ_x in $\mathbb{R}^d$ are in $\mathcal{C}^{\gamma}$ only for $\gamma \leqslant -d$, hence $\mathscr{Z}^{\vartheta}_{t}$ is non-atomic).

The rest of this subsection is devoted to the proof of Theorem 1.7. We first recall the definition of negative Hölder spaces (see [FM17, Section 2] or [CZ20, Section 12] for more details). Let us introduce some notation in any dimension $d \in \mathbb{N}$.

- Let C_c^{∞} denote the family of smooth and compactly supported functions $\varphi : \mathbb{R}^d \to \mathbb{R}$.
- For $r \in \mathbb{N}_0 = \{0,1,2,\ldots\}$, let C_c^r denote the family of compactly supported functions of class C^r, for which we define

$$\|\varphi\|_{C^r} := \max_{r_1,\ldots,r_d \in \mathbb{N}_0:\, r_1+\ldots+r_d \leqslant r} \|\partial_1^{r_1}\cdots\partial_d^{r_d}\varphi\|_{\infty}\,.$$

- Let $\mathscr{B}^r$ denote the family of functions $\varphi \in C_c^{\infty}$ supported on $\overline{B(0,1)}$ with $\|\varphi\|_{C^r} \leqslant 1$.
- Given a function $\varphi : \mathbb{R}^d \to \mathbb{R}$, we denote by φ_x^{λ} its λ-scaled version centred at x:

$$\varphi_x^{\lambda}(\cdot) := \lambda^{-d}\,\varphi\big(\lambda^{-1}(\,\cdot - x)\big) \qquad \text{for } x \in \mathbb{R}^d,\ \lambda > 0\,.$$

Definition 3.4 (Negative Hölder spaces). *Given $\gamma < 0$, the negative Hölder space $\mathcal{C}^{\gamma}$ is the family of linear functionals $T : C_c^{\infty} \to \mathbb{R}$ with the following property: for any $K \in \mathbb{N}$, there is a constant $c_K < \infty$ such that*

$$|T(\varphi_x^{\lambda})| \leqslant c_K\,\lambda^{\gamma} \qquad \forall x \in B(0,K),\ \lambda \in (0,1],\ \varphi \in \mathscr{B}^r, \tag{3.16}$$

where $r = r(\gamma) := \lfloor -\gamma + 1 \rfloor$ (any integer $r > -\gamma$ would yield an equivalent definition).

Any distribution $T \in \mathcal{C}^\gamma$ can be canonically extended (by continuity) from C_c^∞ to C_c^r, for any integer $r > -\gamma$, hence we can consider $T(\varphi)$ for $\varphi \in C_c^r$. In order to prove that $T \in \mathcal{C}^\gamma$, it turns out that it is enough to check property (3.16) for a *finite family of* 2^d *well-chosen test functions* $\varphi \in C_c^r$: a so-called wavelet basis [FM17, Section 2], which we denote by

$$\phi \quad \text{and} \quad \{\psi^{(i)}\}_{1 \leqslant i < 2^d} \quad (\text{which satisfy } \textstyle\int_{\mathbb{R}^d} \psi^{(i)}(x)\, \mathrm{d}x = 0)\,. \tag{3.17}$$

(The details of such functions are immaterial for our goals.)

This yields a convenient criterion for a sequence $(\mathscr{T}_N^\omega)_{N\in\mathbb{N}}$ of random distributions to be tight in the Hölder space $\mathcal{C}^\gamma$ with $\gamma < 0$ (in the spirit of the classical Kolmogorov moment criterion for $\gamma > 0$). The following is a special case of [FM17, Theorem 2.30].

Theorem 3.5 (Tightness criterion for negative Hölder spaces). *Fix $\gamma < 0$ and an integer $r > -\gamma$. Let $\{\phi,\ \psi^{(i)} : 1 \leqslant i < 2^d\}$ be a C_c^r wavelet basis in $\mathbb{R}^d$, see* (3.17).

Let $(\mathscr{T}_N^\omega)_{N\in\mathbb{N}}$ be a sequence of random linear forms on C_c^r, that is, for every ω in a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and every $N \in \mathbb{N}$, we have a linear functional $\mathscr{T}_N^\omega : C_c^r \to \mathbb{R}$, such that $\omega \mapsto \mathscr{T}_N^\omega(\varphi)$ is a random variable for every $\varphi \in C_c^r$.

Assume that for some $p \in [1, \infty)$ and $C < \infty$ the following bounds hold:

$$\forall N \in \mathbb{N},\ x \in \mathbb{R}^d: \qquad \begin{cases} \mathbb{E}\Big[\big|\mathscr{T}_N^\omega(\phi(\,\cdot - x))\big|^p\Big]^{1/p} \leqslant C\,, \\ \mathbb{E}\Big[\big|\mathscr{T}_N^\omega\big((\psi^{(i)})_x^\lambda\big)\big|^p\Big]^{1/p} \leqslant C\,\lambda^\gamma \qquad \forall \lambda \in (0,1],\ 1 \leqslant i < 2^d\,. \end{cases} \tag{3.18}$$

Then $(\mathscr{T}_N^\omega)_{N\in\mathbb{N}}$ is a tight sequence of random variables taking values in the space $\mathcal{C}^{\gamma'}$ for any $\gamma' < \gamma - \frac{d}{p}$.

Remark 3.6 (Topology of Hölder spaces). *Given $\gamma < 0$ and any distribution $T \in \mathcal{C}^\gamma$, let us denote $[\![T]\!]_{K,\gamma} = c_K$ the best constant in the inequality* (3.16)*. Defining the distance*

$$d_{\mathcal{C}^\gamma}(T, T') := \sum_{K\in\mathbb{N}} 2^{-K}\, \frac{[\![T - T']\!]_{K,\gamma}}{1 + [\![T - T']\!]_{K,\gamma}}\,,$$

we have that $\mathcal{C}^\gamma$ is a complete metric space, but it is not separable*, see* [FM17].

To ensure separability, one can define $\mathcal{C}_0^\gamma$ as the closure of smooth compactly supported functions C_c^∞ under the distance $d_{\mathcal{C}^\gamma}$. One has the strict inclusion $\mathcal{C}_0^\gamma \subset \mathcal{C}^\gamma$, however for any $\tilde{\gamma} > \gamma$ one can sandwich $\mathcal{C}^{\tilde{\gamma}} \subset \mathcal{C}_0^\gamma \subset \mathcal{C}^\gamma$ (so the difference is "small" in a sense).

The results in [FM17] *are formulated for the separable spaces $\mathcal{C}_0^\gamma$ (called $\mathcal{C}^\gamma$ in that paper). However, the tightness criterion in Theorem 3.5 applies also to the usual spaces $\mathcal{C}^\gamma$, because $\mathcal{C}_0^\gamma$ is a closed subset of $\mathcal{C}^\gamma$, hence compact sets in $\mathcal{C}_0^\gamma$ are also compact in $\mathcal{C}^\gamma$.*

We now turn to the proof of Theorem 1.7, which is based on the following moment bounds. Recall the initial conditions $\mathcal{U}_{B(z,R)}(\cdot)$ from (1.23) and the convergence in distribution (3.8).

Proposition 3.7 (Moment bounds). *Fix $t > 0$ and $\vartheta \in \mathbb{R}$. Let β_N^{crit} be in the critical regime* (1.19)*. For any $h \in 2\mathbb{N}$, $\varepsilon > 0$, $\delta_0 < \infty$, there is a constant $\mathtt{C} = \mathtt{C}_{p,\varepsilon,\delta_0}^{t,\vartheta} < \infty$ such that*

$$\sup_{N\in\mathbb{N}} \mathbb{E}\Big[Z_{tN}^{\beta_N^{\mathrm{crit}}}\big(\mathcal{U}_{B(x\sqrt{N},\delta\sqrt{N})}\big)^h\Big]^{1/h} \leqslant \mathtt{C}\,\delta^{-\varepsilon} \qquad \forall x \in \mathbb{R}^2,\ \delta \in (0, \delta_0)\,, \tag{3.19}$$

and hence

$$\mathbb{E}\Big[\mathscr{Z}_t^\vartheta\big(\mathcal{U}_{B(x,\delta)}\big)^h\Big]^{1/h} \leqslant \mathtt{C}\,\delta^{-\varepsilon} \qquad \forall x \in \mathbb{R}^2,\ \delta \in (0, \delta_0)\,. \tag{3.20}$$

Proof. In view of (3.8), the bound (3.20) follows by (3.19) and Fatou's Lemma.

To prove the bound (3.19), we apply equation (6.1) in [CSZ23a, Theorem 6.1] for $\psi \equiv 1$ (we just exchange $N, \tilde{N}$): given $h \in \mathbb{N}$ and $1 < p < \infty$, there is a constant $\mathsf{C}' = \mathsf{C}'(\vartheta, h, p) < \infty$ such that, uniformly in large $N, \tilde{N} \in \mathbb{N}$ and integrable $\varphi : \mathbb{R}^2 \to \mathbb{R}$, we have

$$\mathbb{E}\Big[\Big(Z_{\tilde{N}}^{\beta_N^{\rm crit}}(\varphi) - 1\Big)^h\Big]^{\frac{1}{h}} \leqslant \bigg(\frac{\mathsf{C}'}{\log(1 + N/\tilde{N})}\bigg)^{\frac{1}{h}} \left\|\frac{\varphi}{w}\right\|_p \|w\|_q \,,$$

where q is the dual of p (i.e. $\frac{1}{p} + \frac{1}{q} = 1$) and $w(\cdot) := \mathrm{e}^{-|\cdot|}$ is a weight function. In particular, for $\tilde{N} = tN$ and $\varphi(\cdot) := \frac{1}{\pi\delta^2}\mathbb{1}_{B(x,\delta)}(\cdot)$, we obtain

$$\mathbb{E}\Big[\Big(Z_{tN}^{\beta_N^{\rm crit}}\big(\mathcal{U}_{B(x\sqrt{N},\delta\sqrt{N})}\big) - 1\Big)^h\Big]^{\frac{1}{h}} \leqslant \bigg(\frac{\mathsf{C}'}{\log(1 + \frac{1}{t})}\bigg)^{\frac{1}{h}} \bigg(\int\limits_{B(x,\delta)} \frac{\mathrm{e}^{p|y|}}{(\pi\delta^2)^p}\,\mathrm{d}y\bigg)^{\frac{1}{p}} \bigg(\int\limits_{\mathbb{R}^2} \mathrm{e}^{-q|y|}\,\mathrm{d}y\bigg)^{\frac{1}{q}}.$$

The first integral is bounded by $\mathrm{e}^{\delta_0}\,(\pi\delta^2)^{\frac{1}{p}-1}$ while the second integral equals $(2\pi\, q^{-2})^{1/q}$ which is uniformly bounded for $1 < q < \infty$. For a suitable constant C'', we then obtain

$$\mathbb{E}\Big[\Big(Z_{tN}^{\beta_N^{\rm crit}}\big(\mathcal{U}_{B(x\sqrt{N},\delta\sqrt{N})}\big) - 1\Big)^h\Big]^{\frac{1}{h}} \leqslant \mathsf{C}''\,\delta^{-2(1-\frac{1}{p})}\,,$$

and taking $p > 1$ sufficiently close to 1 we have $2(1 - \frac{1}{p}) \leqslant \varepsilon$. The bound (3.19) then follows because $\|Z\|_h \leqslant 1 + \|Z - 1\|_h$ by the triangle inequality. ☐

Proof of Theorem 1.7. Fix $t > 0$ and $\vartheta \in \mathbb{R}$. Our goal is to prove (1.21). We also fix $\varepsilon > 0$ and some integer $r > \varepsilon$. For $N \in \mathbb{N}$, we define the random linear form

$$\mathscr{T}_N^\omega(\varphi) := \int_{\mathbb{R}^2} \varphi(y)\, Z_{tN}^{\beta_N^{\rm crit},\omega}\big([\![\sqrt{N}y]\!]\big)\,\mathrm{d}y = \int_{\mathbb{R}^2} \tfrac{1}{N}\,\varphi\big(\tfrac{z}{\sqrt{N}}\big)\, Z_{tN}^{\beta_N^{\rm crit},\omega}\big([\![z]\!]\big)\,\mathrm{d}z\,,$$

where $[\![x]\!]$ is the point in $\mathbb{Z}^2_{\rm even}$ closest to $x \in \mathbb{R}^2$. If φ is supported on the ball $B(0, R)$, then $\frac{1}{N}\,\varphi_x^\lambda(\frac{\cdot}{\sqrt{N}}) = \frac{1}{\lambda^2 N}\,\varphi(\frac{\cdot - x\sqrt{N}}{\lambda\sqrt{N}})$ is supported in $B(x\sqrt{N}, R\lambda\sqrt{N})$. Recalling (1.23), we can then bound

$$|\mathscr{T}_N^\omega(\varphi_x^\lambda)| \leqslant \|\varphi\|_\infty\, \frac{\mathscr{T}_N^\omega(B(x\sqrt{N}, R\lambda\sqrt{N}))}{\lambda^2 N} \leqslant c\,\|\varphi\|_\infty\, R^2\, Z_{tN}^{\beta_N^{\rm crit},\omega}\big(\mathcal{U}_{B(x\sqrt{N},R\lambda\sqrt{N})}\big)\,,$$

where the constant c accounts for the discrepancy between the cardinality $\big|B(z, R) \cap \mathbb{Z}^2_{\rm even}\big|$ and the area πR^2. Applying the bound (3.19) for $\delta = R\lambda$ and $h = p \in 2\mathbb{N}$, we obtain

$$\forall x \in \mathbb{R}^2,\ \lambda \in (0,1]: \qquad \sup_{N\in\mathbb{N}} \mathbb{E}\Big[\big|\mathscr{T}_N^\omega(\varphi_x^\lambda)\big|^p\Big]^{1/p} \leqslant C\,\lambda^{-\varepsilon} \qquad \text{with } \ C := \mathsf{C}\,c\,\|\varphi\|_\infty\, R^{2-\varepsilon}\,.$$

In particular, choosing $\varphi = \phi$ or $\varphi = \psi^{(i)}$, $1 \leqslant i < 2^d$, we see that both bounds in (3.18) are satisfied for $\gamma = -\varepsilon$, hence $(\mathscr{T}_N^\omega)_{N\in\mathbb{N}}$ is tight in $\mathcal{C}^{\gamma'}$ for all $\gamma' < -\varepsilon - \frac{2}{p}$. Since $\varepsilon > 0$ and $p \in 2\mathbb{N}$ are arbitrary, we conclude that $(\mathscr{T}_N^\omega)_{N\in\mathbb{N}}$ is tight in $\mathcal{C}^{-\varepsilon}$ for any $\varepsilon > 0$.

By the direct half of Prohorov's theorem [Bil99, Theorem 5.1] (which holds for metric spaces), tightness implies relative compactness. It remains to show that, for any weakly converging subsequence $\mathscr{T}_{N_k}^\omega \to \mathscr{T}^\omega$, the limit $\mathscr{T}^\omega$ has the same law as the SHF $\mathscr{Z}_t^{\vartheta,\omega}$.

The law of any random element $\mathscr{T}^\omega$ of $\mathcal{C}^\gamma$ is determined by the laws of the random vectors $(\mathscr{T}^\omega(\varphi_1), \ldots, \mathscr{T}^\omega(\varphi_k))$ for $k \in \mathbb{N}$ and $\varphi_1, \ldots, \varphi_k \in C_c^\infty$. By the linearity of $\varphi \mapsto \mathscr{T}^\omega(\varphi)$ and the Cramer-Wold device, it is enough to focus on the law of $\mathscr{T}^\omega(\varphi)$ for a given $\varphi \in C_c^\infty$. It only remains to show that $\mathscr{T}^\omega(\varphi)$ has the same distribution as $\mathscr{Z}_t^{\vartheta,\omega}(\varphi)$: but this follows from the convergence in distribution (1.20) in the vague topology, which yields $\mathscr{T}_{N_k}^{\omega}(\varphi) \to \mathscr{Z}_t^{\vartheta,\omega}(\varphi)$ in distribution for any $\varphi \in C_c^\infty \subseteq C_c^0$. The proof is completed. □

4. Proof of Theorem 1.8

In this section we prove (1.25). By translation invariance, we only consider the case $x = 0$. We also set for simplicity $t = 1$ (the proof extends to any $t > 0$ with almost no change). Throughout this section, we fix $\varrho \in (0, \infty)$ and, to lighten notation, we abbreviate

$$Z_N^{\mathrm{av}} := Z_N^{\beta_N}\big(\mathcal{U}_{\delta_N^\varrho \sqrt{N}}\big) \qquad \text{where} \quad \mathcal{U}_{\delta_N^\varrho \sqrt{N}} := \mathcal{U}_{B(0,\delta_N^\varrho \sqrt{N})} \,. \tag{4.1}$$

where we recall that $\beta_N = \beta_N^{\text{quasi-crit}}$ is in the quasi-critical regime (1.22) for a given sequence $1 \ll |\vartheta_N| \ll \log N$, and δ_N is defined in (1.24). We then rephrase our goal (1.25) as follows:

$$Z_N^{\mathrm{av}} \xrightarrow[N\to\infty]{d} \mathrm{e}^{\mathcal{N}(0,\sigma^2) - \frac{1}{2}\sigma^2} \qquad \text{with } \sigma^2 = \log(1+\varrho) \,. \tag{4.2}$$

Once (4.2) is proved, we have the convergence of positive moments: for all $p \geqslant 0$

$$\lim_{N\to\infty} \mathbb{E}\big[(Z_N^{\mathrm{av}})^p\big] = \mathbb{E}\big[\big(\mathrm{e}^{\mathcal{N}(0,\sigma^2)-\frac{1}{2}\sigma^2}\big)^p\big] = \mathrm{e}^{\frac{p(p-1)}{2}\,\sigma^2} = (1+\varrho)^{\frac{p(p-1)}{2}} \,.$$

This follows by the weak convergence (4.2) once we show that positive moments are uniformly bounded, say $\sup_{N\in\mathbb{N}} \mathbb{E}[(Z_N^{\mathrm{av}})^{2h}] < \infty$ for all $h \in \mathbb{N}$ (so that $(Z_N^{\mathrm{av}})^p$ is uniformly integrable). We already know from Corollary 2.9 that $\mathbb{V}\mathrm{ar}[Z_N^{\mathrm{av}}]$ is bounded, hence any moment $\mathbb{E}[(Z_N^{\mathrm{av}})^{2h}]$ is bounded too by Theorem 1.11 (the assumptions (1.29), (1.31) and (1.32) of Theorem 1.11 are satisfied by $\varphi = \mathcal{U}_{(0,\delta_N^\varrho\sqrt{N})}$ with $\mathbb{D}[\varphi] \sim (\delta_N^\varrho)^2 N$).

Remark 4.1. *The log-normality* (4.2) *is reminiscent of the corresponding result for the point-to-plane partition function $Z_L^{\beta}(0)$ in the sub-critical regime $\beta = \beta_L^{\text{sub-crit}}$, see* (1.26) *with $\hat\beta \in (0,1)$, that we proved in* [CSZ17, Theorem 2.8]. *As we described in Remark 1.10, the heuristic behind* (4.2) *is that, after coarse-graining space on the scale $\delta_N^\varrho\sqrt{N}$ and time on the scale $\delta_N^{2\varrho}N$, the averaged partition function Z_N^{av} in the quasi-critical regime becomes comparable to a sub-critical point-to-plane partition function with disorder parameter $\hat\beta^2 = \frac{\varrho}{1+\varrho}$ and effective time horizon $L = 1/\delta_N^{2\varrho}$.*

The key to the analysis of $Z_L^{\beta_L^{\text{sub-crit}}}(0)$ in the sub-critical regime in [CSZ17] *was a* multiscale structure *with time scales L^α for $\alpha \in (0,1)$. We will see that the same multiscale structure emerges in our analysis of Z_N^{av} with corresponding time scales $(\delta_N^{2\varrho}N)\,L^\alpha$ with $L = 1/\delta_N^{2\varrho}$ and $\alpha \in (0,1)$, see* (4.5). *This justifies the heuristic comparison just described.*

We divide the proof of (4.2) into several steps.

Overall strategy. Let us fix a (large) integer $M \in \mathbb{N}$, which will be the number of time scales. For technical reasons, we will first approximate the original partition function Z_N^{av} by switching off the disorder in suitable time strips, which defines Z_N^{off} in (4.15) and provides some smoothing between consecutive time scales; we will then introduce almost diffusive restrictions on the polymer paths, which defines Z_N^{diff} in (4.24). We will show that Z_N^{off} and Z_N^{diff} (which also depend on M) are good approximations in the following sense:

$$\forall M \in \mathbb{N}: \qquad Z_N^{\mathrm{av}} - Z_N^{\mathrm{off}} \xrightarrow[N\to\infty]{L^2} 0\,, \qquad Z_N^{\mathrm{off}} - Z_N^{\mathrm{diff}} \xrightarrow[N\to\infty]{L^1} 0\,. \tag{4.3}$$

Therefore to prove our goal (4.2), we can just replace Z_N^{av} with Z_N^{diff}.

Remark 4.2. *A similar idea of switching-off the noise in suitably chosen strips to obtain a smoothing approximation was also used by Dunlap-Gu* [DG22] *in their treatment of nonlinear SHE in a subcritical regime.*

We introduce explicit time scales

$$0 =: N_0 < \tilde{N}_1 \ll N_1 \ll \tilde{N}_2 \ll N_2 \ll \ldots \ll \tilde{N}_M \ll N_M = N\,. \tag{4.4}$$

We define Z_N^{off} by switching off the noise in the time strips $(\tilde{N}_i, N_i]$, see (4.15), then we define Z_N^{diff} by restricting polymer paths at times $\tilde{N}_i$ and N_i to an almost diffusive ball, see (4.23) and (4.24). The scales N_i are defined as follows:

$$N_0 := 0\,, \qquad N_i := \Big[\!\!\Big[(N\delta_N^{2\varrho})\,(\tfrac{1}{\delta_N^{2\varrho}})^{\frac{i}{M}} \Big]\!\!\Big] = \Big[\!\!\Big[N\,(\delta_N^{2\varrho})^{1-\frac{i}{M}} \Big]\!\!\Big] \quad \text{for} \ \ i = 1, \ldots, M\,, \tag{4.5}$$

where $[\![a]\!] := 2\lfloor a/2 \rfloor$ is an even proxy for $a \in \mathbb{R}$. Note that $N_{i+1} \approx N_i\,(\frac{1}{\delta_N^{2\varrho}})^{\frac{1}{M}}$ for $i \geqslant 1$. The intermediate scales $\tilde{N}_i$ are then defined by

$$\tilde{N}_i := \left[\!\!\left[\frac{N_i}{(\log \frac{1}{\delta_N^{2\varrho}})^3} \right]\!\!\right] \qquad \text{for } i = 1, 2, \ldots, M\,. \tag{4.6}$$

By restricting random walk paths in the definition of Z_N^{diff} to the time interval $[0, N_i]$, we obtain a sequence of quantities $Z_{N,i}^{\mathrm{diff}}$ for $i = 1, \ldots, M$ with $Z_{N,M}^{\mathrm{diff}} = Z_N^{\mathrm{diff}}$, see (4.24). Then, by a telescopic product, we can write

$$Z_N^{\mathrm{diff}} = \prod_{i=1}^{M} \frac{Z_{N,i}^{\mathrm{diff}}}{Z_{N,i-1}^{\mathrm{diff}}} \qquad\qquad (\text{with } Z_{N,0}^{\mathrm{diff}} := 1)\,. \tag{4.7}$$

The choice of the scales N_i is made so that the *the ratios in the RHS have variance of the same order* $\frac{1}{M}$*, albeit with a varying prefactor* (see Theorem 4.5). The fact that these scales are well separated, see (4.4), ensures that *the ratios in the RHS are approximately independent.*

Remark 4.3 (Exponential time scales). *The choice of the scales* (4.5)*, which leads to the decomposition* (4.7) *into approximately independent factors, resembles what is observed in the sub-critical regime* (1.26) *in* [CSZ17] *and in the more recent papers* [CC22, CD24]*.*

Denoting by $\mathcal{F}_L := \sigma\{\omega(n,z) \colon n \leqslant L, z \in \mathbb{Z}^2\}$ the σ-algebra generated by disorder variables up to time L, we introduce the conditional expectation

$$m_{N,i} := \mathbb{E}\bigg[\frac{Z_{N,i}^{\mathrm{diff}}}{Z_{N,i-1}^{\mathrm{diff}}} \,\bigg|\, \mathcal{F}_{N_{i-1}} \bigg] \qquad \text{for } i = 1, \ldots, M\,, \tag{4.8}$$

which turns out to be close to 1 with high probability (see (4.30)). We define $\Delta_{N,i}$ as the normalised and centred version of the ratios in the RHS of (4.7):

$$\Delta_{N,i} := \frac{1}{m_{N,i}} \, \frac{Z_{N,i}^{\mathrm{diff}}}{Z_{N,i-1}^{\mathrm{diff}}} - 1\,. \tag{4.9}$$

This leads to the identity

$$\begin{aligned}\log Z_N^{\mathrm{diff}} &= \sum_{i=1}^{M} \log \frac{Z_{N,i}^{\mathrm{diff}}}{Z_{N,i-1}^{\mathrm{diff}}} = \sum_{i=1}^{M} \big\{ \log\big(1+\Delta_{N,i}\big) + \log m_{N,i} \big\} \\ &= \sum_{i=1}^{M} \big\{ \Delta_{N,i} - \tfrac{1}{2}\Delta_{N,i}^2 + r(\Delta_{N,i}) + \log m_{N,i} \big\} ,\end{aligned} \tag{4.10}$$

where $r(\cdot)$ is the remainder in the second order Taylor expansion of the logarithm:

$$r(x) := \log(1+x) - \big(x - \tfrac{x^2}{2}\big) . \tag{4.11}$$

To complete the proof of our goal (4.2), we are going to show that, for $M = M_N \to \infty$ slowly enough, the following three convergences in distribution hold:

$$\sum_{i=1}^{M_N} \Delta_{N,i} \xrightarrow[N\to\infty]{d} \mathcal{N}(0,\sigma^2) , \tag{4.12}$$

$$\sum_{i=1}^{M_N} \Delta_{N,i}^2 \xrightarrow[N\to\infty]{d} \sigma^2 , \tag{4.13}$$

$$\sum_{i=1}^{M_N} \Big(r(\Delta_{N,i}) + \log m_{N,i} \Big) \xrightarrow[N\to\infty]{d} 0 , \tag{4.14}$$

with σ^2 as in (4.2). Intuitively, these relations hold because *the random variables $\Delta_{N,i}$ are approximately independent*, due to the separation of time scales (4.5). The proof of (4.2) then follows by first choosing $M = M_N \to \infty$ slowly enough such that (4.3) still holds, and then applying the identity (4.10).

The rest of the proof is divided into the following steps:

- in Steps 1 and 2 we define Z_N^{off} and Z_N^{diff} and prove the two limits in (4.3);
- in Step 3 we give a convenient representation for the ratio $Z_{N,i}^{\mathrm{diff}}/Z_{N,i-1}^{\mathrm{diff}}$ as a partition function on the time interval $(N_{i-1}, N_i]$ with initial condition given by a *polymer distribution at time* N_{i-1}, and we show that the latter is close to the free random walk thanks to the fact that noise has been turned off in the time interval $(\tilde{N}_i, N_i]$.
- in Steps 4 and 5 we compute the variance of $\Delta_{N,i}$ and bound its higher moments;
- in Step 6 we prove (4.12) through the *martingale CLT*;
- in Step 7 we prove relations (4.13) and (4.14) by variance bounds.

4.1. Step 1: switching off the noise. The first approximation Z_N^{off} of the partition function Z_N^{av}, recall (4.1), is obtained by "switching off the noise" in the time strips $(\tilde{N}_i, N_i]$ for $1 \leqslant i \leqslant M$, see (4.5) and (4.6): this will ensure that the endpoint distribution of the polymer at time N_i is comparable to the random walk transition kernel. Recalling (2.1) and (1.18), we thus define

$$Z_N^{\mathrm{off}} := \mathrm{E}\Big[\mathrm{e}^{\mathcal{H}^{\beta_N}_{(0,N]\setminus \bigcup_{j=1}^{M} (\tilde{N}_j, N_j]}} \,\Big|\, S_0 \sim \mathcal{U}_{\delta_N^{\varrho}\sqrt{N}} \Big] . \tag{4.15}$$

In this step, we prove the first relation in (4.3):

$$\forall M \in \mathbb{N}: \qquad \lim_{N\to\infty} \mathbb{E}\big[\big(Z_N^{\mathrm{av}} - Z_N^{\mathrm{off}}\big)^2\big] = 0\,. \tag{4.16}$$

Let $\mathcal{G}$ be the σ-algebra generated by the disorder variables that have *not* been swithced off, namely $\mathcal{G} := \sigma\big(\omega(n,z)\colon n \in \bigcup_{i=1}^M (N_{i-1}, \tilde{N}_i],\, z \in \mathbb{Z}^2\big)$, then we can write

$$Z_N^{\mathrm{off}} = \mathbb{E}\Big[Z_N^{\mathrm{av}} \,\Big|\, \mathcal{G}\Big]\,,$$

that is Z_N^{off} is the orthogonal projection of Z_N^{av} onto the linear subspace of L^2 generated by $\mathcal{G}$-measurable random variables. It follows that

$$\mathbb{E}\big[\big(Z_N^{\mathrm{av}} - Z_N^{\mathrm{off}}\big)^2\big] = \mathbb{E}\big[(Z_N^{\mathrm{av}})^2\big] - \mathbb{E}\big[(Z_N^{\mathrm{off}})^2\big]\,,$$

hence to prove (4.16) we need to show that

$$\forall M \in \mathbb{N}: \qquad \lim_{N\to\infty} \Big\{\mathbb{E}\big[(Z_N^{\mathrm{av}})^2\big] - \mathbb{E}\big[(Z_N^{\mathrm{off}})^2\big]\Big\} = 0\,. \tag{4.17}$$

Proof of (4.17)**.** Let us define a variant $\tilde{Z}_{N,j}^{\mathrm{off}}$ of (4.15) where we only switch off disorder in a given interval $(\tilde{N}_j, N_j]$, namely

$$\text{for } \; j = 1,\ldots,M: \qquad \tilde{Z}_{N,j}^{\mathrm{off}} := \mathrm{E}\Big[\mathrm{e}^{\mathcal{H}^{\beta_N}_{(0,N]\setminus(\tilde{N}_j,N_j]}} \,\Big|\, S_0 \sim \mathcal{U}_{\delta_N^{\varrho}\sqrt{N}}\Big]\,. \tag{4.18}$$

We claim that we can bound the difference in (4.17) by

$$\mathbb{E}\big[(Z_N^{\mathrm{av}})^2\big] - \mathbb{E}\big[(Z_N^{\mathrm{off}})^2\big] = \mathbb{V}\mathrm{ar}\big[Z_N^{\mathrm{av}}\big] - \mathbb{V}\mathrm{ar}\big[Z_N^{\mathrm{off}}\big] \leqslant \sum_{j=1}^{M} \Big\{\mathbb{V}\mathrm{ar}\big[Z_N^{\mathrm{av}}\big] - \mathbb{V}\mathrm{ar}\big[\tilde{Z}_{N,j}^{\mathrm{off}}\big]\Big\}. \tag{4.19}$$

It then suffices to estimate separately each term in this sum.

In order to prove (4.19), recall the polynomial chaos expansion (2.11) for the point-to-plane partition function $Z_N^{\beta}(x)$, which yields a corresponding polynomial chaos expansion for the averaged partition function $Z_N^{\beta}(\varphi) = \sum_{x\in\mathbb{Z}^2_{\mathrm{even}}} \varphi(x)\, Z_N^{\beta}(x)$. The polynomial chaos expansion for Z_N^{off} from (4.15) is a *subset* of the polynomial chaos expansion for Z_N^{av}: it is obtained by restricting the sum to times $n_1,\ldots,n_k$ which *avoid all intervals* $(\tilde{N}_j, N_j]$ for $j = 1,\ldots,M$ (see Remark 2.2), hence its variance admits a formula like (2.17) with the same restriction on the sum. Then the difference $\mathbb{V}\mathrm{ar}\big[Z_N^{\mathrm{av}}\big] - \mathbb{V}\mathrm{ar}\big[Z_N^{\mathrm{off}}\big]$ is given again by formula (2.17) where the sequence of times $n_1,\ldots,n_k$ is now required to *intersect at least one of the intervals* $(\tilde{N}_j, N_j]$ *for* $j = 1,\ldots,M$. By a union bound, we obtain precisely (4.19).

Let us finally focus on a given term in the RHS of (4.19). Since $\mathbb{V}\mathrm{ar}\big[Z_N^{\mathrm{av}}\big] \to \varrho$ as $N \to \infty$, see (2.26), it is enough to show that

$$\forall j \in \{1,\ldots,M\}: \qquad \lim_{N\to\infty} \mathbb{V}\mathrm{ar}\big[\tilde{Z}_{N,j}^{\mathrm{off}}\big] = \varrho\,. \tag{4.20}$$

We recall that $\tilde{Z}_{N,j}^{\mathrm{off}}$ corresponds to switching off disorder between $A_N = \tilde{N}_j$ and $B_N = N_j$. It is instructive (and more transparent) to fix $0 < a \leqslant b < 1$ and consider general times

$$\begin{aligned} A_N &= N\,(\delta_N^{2\varrho})^{1-a+o(1)} = N\,\mathrm{e}^{-(1-a)\,\varrho\,|\vartheta_N| + o(|\vartheta_N|)}\,, \\ B_N &= N\,(\delta_N^{2\varrho})^{1-b+o(1)} = N\,\mathrm{e}^{-(1-b)\,\varrho\,|\vartheta_N| + o(|\vartheta_N|)}\,. \end{aligned}$$

Later we will specialize to $b = a = \frac{j-1}{M}$ due to the choice of N_j and $\tilde{N}_j$ in (4.5) and (4.6).

We compute $\mathbb{V}\mathrm{ar}\big[\tilde{Z}^{\mathrm{off}}_{N,j}\big]$ by formula (2.17) where the sum is restricted to times $n_1, \ldots, n_k$ that *do not intersect the interval* $(A_N, B_N]$. We split $\mathbb{V}\mathrm{ar}\big[\tilde{Z}^{\mathrm{off}}_{N,j}\big] = I_1 + I_2 + I_3$ as follows:

- part I_1: all times n_i take place before A_N;
- part I_2: all times n_i take place after B_N;
- part I_3: the first time n_1 takes place before A_N, the last time n_k takes place after B_N.

The first contribution I_1 is nothing but the variance of the averaged partition function of polymer length A_N, that is (recall (4.1))

$$I_1 = \mathbb{V}\mathrm{ar}\big[Z^{\beta_N}_{A_N}(\mathcal{U}_{\delta_N^{\varrho}\sqrt{N}})\big] \sim \frac{a\varrho}{1+(1-a)\varrho}\,, \tag{4.21}$$

where we applied (2.21) with $L_N = A_N$ for which $\ell = (1-a)\varrho$, see (2.18), and $W_N = N\delta_N^{2\varrho}$ for which $w = \varrho$, see (2.19) (and recall (2.25)).

The second contribution I_2, when all times n_i are after B_N, corresponds to switching off the noise in the whole interval $[0, B_N]$, hence we have a partition function on the interval $(B_N, N]$ (whose length is $L_N = N - B_N \sim N$ since $b < 1$, that is (2.18) holds with $\ell = 0$) with initial condition at time B_N given by $\mathcal{U}_{\delta_N^{\varrho}\sqrt{N}} * q_{B_N}$, i.e., the distribution of the random walk at time B_N with initial condition $\mathcal{U}_{\delta_N^{\varrho}\sqrt{N}}$ (which satisfies assumption (2.24) with $W_N = B_N$, hence (2.19) holds with $w = (1-b)\varrho$). We thus obtain by (2.21)

$$I_2 \sim (1-b)\varrho\,.$$

We finally consider the third contribution I_3, when there are times n_i both before A_N and after B_N: recalling (2.17), we can write

$$I_3 = \sum_{\substack{0<m\leqslant g\leqslant A_N \\ B_N<d\leqslant n\leqslant N}} q_{2m}\big(\mathcal{U}_{\delta_N^{\varrho}\sqrt{N}}, \mathcal{U}_{\delta_N^{\varrho}\sqrt{N}}\big)\, \sigma^2_{\beta_N}\, U_{\beta_N}(g-m)\, \sigma^2_{\beta_N}\, q_{2(d-g)}(0)\, U_{\beta_N}(n-d)\,,$$

where we recall that $U_{\beta_N}(\cdot)$ was defined in (2.12). Summing over n we obtain $\overline{U}_{\beta_N}(N-d)$, see (2.13), and restricting the sum to $d \leqslant \frac{1}{2}N$ we can bound $\overline{U}_{\beta_N}(N-d) \geqslant \overline{U}_{\beta_N}(\frac{1}{2}N)$. Summing $q_{2(d-g)}(0)$ over $B_N < d \leqslant \frac{1}{2}N$ then gives, recalling (1.15),

$$\sum_{B_N<d\leqslant\frac{1}{2}N} q_{2(d-g)}(0) \;=\; R_{\frac{1}{2}N-g} - R_{B_N-g} \;\sim\; \frac{1}{\pi}\log\frac{\frac{1}{2}N-g}{B_N-g} \;\geqslant\; \frac{1}{\pi}\log\frac{\frac{1}{2}N}{B_N}$$

because the minimum is attained at $g = 0$. Recalling that $\sigma^2_{\beta_N} \sim \frac{\pi}{\log N}$, we then obtain

$$\begin{aligned} I_3 &\geqslant \Bigg\{\sum_{0<m\leqslant g\leqslant A_N} q_{2m}\big(\mathcal{U}_{\delta_N^{\varrho}\sqrt{N}}, \mathcal{U}_{\delta_N^{\varrho}\sqrt{N}}\big)\, \sigma^2_{\beta_N}\, U_{\beta_N}(g-m)\Bigg\} \frac{\log\frac{\frac{1}{2}N}{B_N}}{\log N}\, \overline{U}_{\beta_N}(\tfrac{1}{2}N) \\ &= \mathbb{E}\big[Z^{\beta_N}_{A_N}(\mathcal{U}_{\delta_N^{\varrho}\sqrt{N}})^2\big] \frac{\log\frac{\frac{1}{2}N}{B_N}}{\log N}\, \overline{U}_{\beta_N}(\tfrac{1}{2}N)\,, \end{aligned}$$

where the equality follows by (2.13) and (2.15). Applying (2.14) we see that $\overline{U}_{\beta_N}(\frac{1}{2}N) \sim \frac{\log N}{|\vartheta_N|}$, while $\log\frac{\frac{1}{2}N}{B_N} \sim (1-b)\log\frac{1}{\delta_N^{2\varrho}} \sim (1-b)\varrho\,|\vartheta_N|$, see (1.24). Recalling (4.21) we then obtain

$$I_3 \geqslant (1+o(1))\, I_1 \cdot \frac{(1-b)\varrho\,|\vartheta_N|}{\log N} \cdot \frac{\log N}{|\vartheta_N|} \sim I_1\,(1-b)\varrho\,.$$

Overall, summing the three parts I_1, I_2 and I_3, we have shown that

$$\begin{aligned}\mathbb{V}\mathrm{ar}\big[\tilde Z^{\mathrm{off}}_{N,j}\big] &= I_1 + I_2 + I_3 \geqslant I_1\,(1+(1-b)\varrho) + I_2 + o(1) \sim \frac{1+(1-b)\varrho}{1+(1-a)\varrho}\, a\varrho + (1-b)\varrho \\ &= \varrho \,-\, (b-a)\frac{\varrho\,(1+\varrho)}{1+(1-a)\varrho} = \varrho \,-\, O(b-a)\,.\end{aligned}$$

This last expression vanishes for $a = b$, which completes the proof of (4.20). □

4.2. Step 2: almost diffusive approximation. In this step, we prove the second relation in (4.3):

$$\forall M \in \mathbb{N}: \qquad \lim_{N\to\infty} \mathbb{E}\big[|Z^{\mathrm{off}}_N - Z^{\mathrm{diff}}_N|\big] = 0\,, \tag{4.22}$$

Let $\mathcal{D}_m$ be the event that the random walk is “almost diffusive” at time m, in the following sense:

$$\mathcal{D}_m := \Big\{ |S_m| \leqslant \sqrt{m \log \tfrac{1}{\delta_N^2}} \Big\}\,. \tag{4.23}$$

We define Z^{diff}_N by restricting Z^{off}_N in (4.15) to the event $\bigcap_{j=0}^{M} \mathcal{D}_{\tilde N_j} \cap \mathcal{D}_{N_j}$. It is actually useful to define $Z^{\mathrm{diff}}_{N,i}$ for each scale N_i, see (4.5) (note that $N_i = N$ for $i = M$):

$$\begin{aligned}\text{for } i = 1,\ldots,M: \qquad Z^{\mathrm{diff}}_{N,i} &:= \mathrm{E}\bigg[\prod_{j=1}^{i} \mathrm{e}^{\mathcal{H}^{\beta_N}_{(N_{j-1},\tilde N_j]}}\, \mathbb{1}_{\mathcal{D}_{\tilde N_j}\cap \mathcal{D}_{N_j}} \,\bigg|\, S_0 \sim \mathcal{U}_{\delta_N^\varrho \sqrt{N}}\bigg] \\ \text{and we set} \quad Z^{\mathrm{diff}}_N &:= Z^{\mathrm{diff}}_{N,M}\,.\end{aligned} \tag{4.24}$$

Let us prove (4.22). Since $|Z^{\mathrm{off}}_N - Z^{\mathrm{diff}}_N| = Z^{\mathrm{off}}_N - Z^{\mathrm{diff}}_N$ and $\mathbb{E}[\mathrm{e}^{\mathcal{H}_{(a,b]}}] = 1$, we have

$$\begin{aligned}\mathbb{E}\big[|Z^{\mathrm{off}}_N - Z^{\mathrm{diff}}_N|\big] &= 1 - \mathrm{P}\bigg(\bigcap_{j=1}^{M} \mathcal{D}_{\tilde N_j} \cap \mathcal{D}_{N_j} \,\bigg|\, S_0 \sim \mathcal{U}_{\delta_N \sqrt{N}}\bigg) \\ &\leqslant \sum_{m \in \{\tilde N_1, N_1, \ldots, \tilde N_M, N_M\}} \mathrm{P}\Big(|S_m| > \sqrt{m \log \tfrac{1}{\delta_N^2}} \,\Big|\, S_0 \sim \mathcal{U}_{\delta_N^\varrho \sqrt{N}}\Big)\,.\end{aligned}$$

We recall that under $\mathrm{P}(\,\cdot\,|\, S_0 \sim \mathcal{U}_{\delta_N^\varrho\sqrt{N}})$ we have $|S_0| \leqslant \delta_N^\varrho \sqrt{N}$. Since $\tilde N_1 \gg N\delta_N^{2\varrho}$, see (4.5) and (4.6), for $m \geqslant \tilde N_1$ we can bound $\delta_N^\varrho\sqrt{N} \leqslant \sqrt{m}$, hence

$$\sqrt{m \log \tfrac{1}{\delta_N^2}} - \delta_N^\varrho \sqrt{N} \geqslant \sqrt{m}\Big(\sqrt{\log \tfrac{1}{\delta_N^2}} - 1\Big) \geqslant \frac{1}{2}\sqrt{m}\sqrt{\log \tfrac{1}{\delta_N^2}} \quad \text{for large } N\,.$$

Then (4.22) holds because for any $m \in \{\tilde N_1, N_1, \ldots, \tilde N_M, N_M\}$, we can use Gaussian tail estimates for the simple symmetric random walk to bound

$$\begin{aligned}\mathrm{P}\Big(|S_m| > \sqrt{m\log \tfrac{1}{\delta_N^2}} \,\Big|\, S_0 \sim \mathcal{U}_{\delta_N^\varrho\sqrt{N}}\Big) &\leqslant \mathrm{P}\Big(|S_m - S_0| > \tfrac{1}{2}\sqrt{m}\sqrt{\log \tfrac{1}{\delta_N^2}}\Big) \\ &\leqslant C\,\exp\Big(-\frac{1}{C}\,\Big|\sqrt{\log \tfrac{1}{\delta_N^2}}\Big|^2\Big) = C\,(\delta_N^2)^{\frac{1}{C}} \xrightarrow{N\to\infty} 0\,.\end{aligned} \tag{4.25}$$

This concludes the proof of (4.22).

4.3. Step 3: polymer distribution. In this step we give a convenient representation for the ratio $Z^{\mathrm{diff}}_{N,i}/Z^{\mathrm{diff}}_{N,i-1}$ in terms of a directed polymer. This will be exploited to estimate the moments of $\Delta_{N,i}$, see (4.9).

Let us introduce the polymer endpoint distribution $\mu_{N,i}(\cdot)$ at time N_i, corresponding to the partition function $Z^{\mathrm{diff}}_{N,i}$ in (4.24):

$$\text{for } i = 1, \ldots, M \text{ and } x \in \mathbb{Z}^2_{\mathrm{even}}: \qquad \mu_{N,i}(x) := \frac{Z^{\mathrm{diff}}_{N,i}\big(\mathcal{U}_{\delta_N^\varrho \sqrt{N}}, x\big)}{Z^{\mathrm{diff}}_{N,i}} . \tag{4.26}$$

where we define $Z^{\mathrm{diff}}_{N,i}\big(\mathcal{U}_{\delta_N^\varrho \sqrt{N}}, x\big)$ by restricting paths in the definition of $Z^{\mathrm{diff}}_{N,i}$ to $S_{N_i} = x$, that is

$$Z^{\mathrm{diff}}_{N,i}\big(\mathcal{U}_{\delta_N^\varrho \sqrt{N}}, x\big) := \mathrm{E}\Bigg[\Bigg(\prod_{j=1}^{i} \mathrm{e}^{\mathcal{H}^{\beta_N}_{(N_{j-1},\tilde{N}_j]}} \mathbb{1}_{\mathcal{D}_{\tilde{N}_j} \cap \mathcal{D}_{N_j}}\Bigg) \mathbb{1}_{\{S_{N_i}=x\}} \,\Bigg|\, S_0 \sim \mathcal{U}_{\delta_N^\varrho \sqrt{N}}\Bigg] . \tag{4.27}$$

By the Markov property, the following representation holds for $i = 2, \ldots, M$:

$$\begin{aligned} \frac{Z^{\mathrm{diff}}_{N,i}}{Z^{\mathrm{diff}}_{N,i-1}} &= \mathrm{E}\Big[\mathrm{e}^{\mathcal{H}^{\beta_N}_{(N_{i-1},\tilde{N}_i]}} \mathbb{1}_{\mathcal{D}_{\tilde{N}_i} \cap \mathcal{D}_{N_i}} \,\Big|\, S_{N_{i-1}} \sim \mu_{N,i-1}\Big] \\ &= \sum_{x \in \mathbb{Z}^2_{\mathrm{even}}} \mu_{N,i-1}(x)\, \mathrm{E}\Big[\mathrm{e}^{\mathcal{H}^{\beta_N}_{(N_{i-1},\tilde{N}_i]}} \mathbb{1}_{\mathcal{D}_{\tilde{N}_i} \cap \mathcal{D}_{N_i}} \,\Big|\, S_{N_{i-1}} = x\Big] . \end{aligned} \tag{4.28}$$

The same formula holds also for $i = 1$ provided we define

$$\mu_{N,0}(x) := \mathcal{U}_{\delta_N^\varrho \sqrt{N}}(x) .$$

Remark 4.4. *The switching off of the noise ensures that the "initial distribution" $\mu_{N,i-1}$ in* (4.28) *is sufficiently smooth, as we show below. This will be needed in the next steps to compute the variance and to estimate the moments of $\Delta_{N,i}$ from* (4.9).

Representation (4.28) is very useful. For instance, recalling (4.8), we can compute

$$m_{N,i-1} = \mathrm{P}\big(\mathcal{D}_{\tilde{N}_i} \cap \mathcal{D}_{N_i} \,\big|\, S_{N_{i-1}} \sim \mu_{N,i-1}\big) . \tag{4.29}$$

Note that for $S_{N_{i-1}} \sim \mu_{N,i-1}$, we have $|S_{N_{i-1}}| \leqslant \sqrt{N_{i-1} \log \frac{1}{\delta_N^2}}$ due to the restriction to the event $\mathcal{D}_{N_{i-1}}$, see (4.27) and (4.23). Therefore for any $i = 1, \ldots, M$, we can bound

$$\begin{aligned} m_{N,i-1} \geqslant 1 &- \mathrm{P}\Big(|S_{\tilde{N}_i} - S_{N_{i-1}}| > (\sqrt{\tilde{N}_i} - \sqrt{N_{i-1}})\sqrt{\log \tfrac{1}{\delta_N^2}}\Big) \\ &- \mathrm{P}\Big(|S_{N_i} - S_{N_{i-1}}| > (\sqrt{N_i} - \sqrt{N_{i-1}})\sqrt{\log \tfrac{1}{\delta_N^2}}\Big) . \end{aligned}$$

Since $N_{i-1} \ll \tilde{N}_i \ll N_i$, see (4.5) and (4.6), arguing as in (4.25), we have: for some $C < \infty$,

$$\text{for } i = 1, \ldots, M: \qquad 1 - C\,(\delta_N^2)^{\frac{1}{C}} \leqslant m_{N,i-1} \leqslant 1 . \tag{4.30}$$

We conclude this step by showing that the polymer distribution $\mu_{N,i}(x)$ *is close to the random walk transition kernel* $q_{N_i - \tilde{N}_i}(x)$, see (1.14). Intuitively, this holds because:

- we switched off disorder between times $\tilde{N}_i$ and N_i, see (4.27), therefore the polymer evolves between these times as simple random walk;

- we know that $|S_{\tilde N_i}| \leqslant \sqrt{\tilde N_i \log \frac{1}{\delta_N^2}}$ (due to the event $\mathcal{D}_{\tilde N_i}$), therefore $|S_{\tilde N_i}| \ll \sqrt{N_i}$ by the choice of $\tilde N_i$ in (4.6). To compute S_{N_i}, we can therefore pretend that $S_{\tilde N_i} \simeq 0$, namely we can approximate $S_{N_i} - S_{\tilde N_i} \simeq S_{N_i - \tilde N_i}$, which is distributed as $q_{N_i - \tilde N_i}(\cdot)$.

Let us now be precise: we set for short

$$\varepsilon_N := \big(\log \tfrac{1}{\delta_N^2}\big)^{-1/3} \xrightarrow[N\to\infty]{} 0\,, \tag{4.31}$$

and we prove that for any $M \in \mathbb{N}$ and $i = 1, \ldots, M$, we have

$$\begin{aligned} \mu_{N,i} \in \mathcal{M}_{N,i} := \Bigg\{ &\varphi(\cdot) \geqslant 0 \ \text{ supported in } \Big\{ |\cdot| \leqslant \sqrt{N_i \log \tfrac{1}{\delta_N^2}} \Big\} \\ &\qquad \text{with } \sum_x \varphi(x) = 1 \ \text{ and } \ \varphi(\cdot) \leqslant (1+\varepsilon_N)^2\, q_{N_i - \tilde N_i}(\cdot) \Bigg\}\,. \end{aligned} \tag{4.32}$$

(This relation requires $i \geqslant 1$, but we will not need it for $i = 0$.)

Proof of (4.32)**.** We only need to prove that $\mu_{N,i}(\cdot) \leqslant (1+\varepsilon_N)^2\, q_{N_i - \tilde N_i}(\cdot)$. We express $\mu_{N,i}(x)$ by summing over the polymer position at time $\tilde N_i < N_i$: denoting by $\phi_{\tilde N_i}(z)$ the corresponding distribution (defined as in (4.27)-(4.26) with $\{S_{N_i} = x\}$ replaced by $\{S_{\tilde N_i} = z\}$), and recalling the random walk transition kernel from (1.14), for $x \in \mathbb{Z}^2_{\text{even}}$ with $|x| \leqslant \sqrt{N_i \log \frac{1}{\delta_N^2}}$, we get

$$\mu_{N,i}(x) = \frac{\displaystyle\sum_{|z| \leqslant \sqrt{\tilde N_i \log \frac{1}{\delta_N^2}}} \phi_{\tilde N_i}(z)\, q_{N_i - \tilde N_i}(x - z)}{\displaystyle\sum_{|z| \leqslant \sqrt{\tilde N_i \log \frac{1}{\delta_N^2}}} \phi_{\tilde N_i}(z) \Bigg\{ \sum_{|x'| \leqslant \sqrt{N_i \log \frac{1}{\delta_N^2}}} q_{N_i - \tilde N_i}(x' - z) \Bigg\}}\,. \tag{4.33}$$

To obtain $\mu_{N,i}(\cdot) \leqslant (1+\varepsilon_N)^2\, q_{N_i - \tilde N_i}(\cdot)$, it suffices to prove the following bounds on the numerator and denominator in (4.33): for any $M \in \mathbb{N}$ and $i = 1, \ldots, M$, we have, for large N,

$$q_{N_i - \tilde N_i}(x - z) \leqslant (1+\varepsilon_N)\, q_{N_i - \tilde N_i}(x)\,, \tag{4.34}$$

$$\sum_{|x'| \leqslant \sqrt{N_i \log \frac{1}{\delta_N^2}}} q_{N_i - \tilde N_i}(x' - z) \geqslant (1+\varepsilon_N)^{-1}\,, \tag{4.35}$$

uniformly over $z, x \in \mathbb{Z}^2_{\text{even}}$ that satisfy (recall (4.5) and (4.6))

$$|z| \leqslant \sqrt{\tilde N_i \log \tfrac{1}{\delta_N^2}} = \sqrt{N_i \big(\log \tfrac{1}{\delta_N^2}\big)^{-2}}\,, \qquad |x| \leqslant \sqrt{N_i \log \tfrac{1}{\delta_N^2}}\,. \tag{4.36}$$

We first prove (4.34). We fix $M \in \mathbb{N}$ and $i \in \{1, \ldots, M\}$. By the local limit theorem (2.3), uniformly over for $x, z \in \mathbb{Z}^2_{\text{even}}$, we can write

$$\frac{q_{N_i - \tilde N_i}(x - z)}{q_{N_i - \tilde N_i}(x)} = \mathrm{e}^{\frac{|x|^2 - |x-z|^2}{N_i - \tilde N_i} + O\left(\frac{|x|^4 + |x-z|^4}{(N_i - \tilde N_i)^3}\right) + o(1)}\,.$$

For large N, we have $N_i - \tilde{N}_i \geqslant \frac{1}{2} N_i$ and $N_i \geqslant (\delta_N^2)^{-\frac{1}{M}} \geqslant (\log \frac{1}{\delta_N^2})^3$, see (4.5) and (4.6). Therefore by (4.36), we can bound

$$\frac{|x|^4 + |x-z|^4}{(N_i - \tilde{N}_i)^3} \leqslant C \, \frac{(\log \frac{1}{\delta_N^2})^2}{N_i} \leqslant \frac{C}{\log \frac{1}{\delta_N^2}} \,,$$

$$\left| \frac{|x|^2 - |x-z|^2}{N_i - \tilde{N}_i} \right| = \left| \frac{2\langle z, x\rangle - |z|^2}{N_i - \tilde{N}_i} \right| \leqslant 2 \, \frac{2|z|(|x| + |z|)}{N_i} \leqslant \frac{C}{\sqrt{\log \frac{1}{\delta_N^2}}} \,.$$

Both estimates are $o(\varepsilon_N)$ as $N \to \infty$, see (4.31), hence (4.34) follows. Finally, to prove (4.35), we note that the LHS equals

$$\begin{aligned} \mathrm{P}\Big(|z + S_{N_i - \tilde{N}_i}| \leqslant \sqrt{N_i \log \tfrac{1}{\delta_N^2}} \Big) &\geqslant \mathrm{P}\Big(|S_{N_i}| \leqslant \tfrac{1}{2} \sqrt{N_i \log \tfrac{1}{\delta_N^2}} \Big) \\ &\geqslant 1 - C \, \exp\big(- \tfrac{1}{C} \big| \sqrt{\log \tfrac{1}{\delta_N^2}} \big|^2 \big) \\ &= 1 - C \, (\delta_N^2)^C \leqslant (1 + \varepsilon_N)^{-1} \,, \end{aligned}$$

where the last inequality holds for N large enough, see (4.31). □

4.4. Step 4: variance computation. In this step, we compute the asymptotic variance of Δ_{N_i} from (4.9), which is needed to prove (4.12) and (4.13). We work under the conditional probability $\mathbb{P}(\cdot | \mathcal{F}_{N_{i-1}})$ and note from (4.8) and (4.9) that

$$\mathbb{E}[\Delta_{N_i} | \mathcal{F}_{N_{i-1}}] = 0 \,. \tag{4.37}$$

Therefore we focus on the second moment. In the next result, we exploit the control on the polymer distribution $\mu_{N,i-1}$ that we obtained in (4.32) in the previous step.

Theorem 4.5 (Second moment asymptotics). *For any $M \in \mathbb{N}$ and $i = 1, \ldots, M$, we have the a.s. convergence (uniformly over $\mathcal{F}_{N_{i-1}}$)*

$$\lim_{N \to \infty} \mathbb{E}[(\Delta_{N,i})^2 \,|\, \mathcal{F}_{N_{i-1}}] = \frac{1}{M} \, \frac{\varrho}{1 + (1 - \frac{i}{M}) \, \varrho} \,. \tag{4.38}$$

Proof. Recalling the definition (4.9) of $\Delta_{N,i}$ and the representation (4.28), we can write

$$\Delta_{N,i} = \frac{1}{m_{N,i}} \sum_{x \in \mathbb{Z}^2_{\mathrm{even}}} \mu_{N,i-1}(x) \, \mathrm{E}\Big[\Big(\mathrm{e}^{\mathcal{H}^{\beta_N}_{(N_{i-1}, \tilde{N}_i]}} - 1 \Big) \mathbb{1}_{\mathcal{D}_{\tilde{N}_i} \cap \mathcal{D}_{N_i}} \,\Big|\, S_{N_{i-1}} = x \Big] . \tag{4.39}$$

Removing the constraint $\mathbb{1}_{\mathcal{D}_{\tilde{N}_i} \cap \mathcal{D}_{N_i}}$, the RHS of (4.39) would simply become

$$\tilde{\Delta}_{N,i} := \frac{1}{m_{N,i}} \Big(Z^{\beta_N}_{L_N}(\varphi_N) - 1 \Big) \qquad \text{with} \qquad L_N = \tilde{N}_i - N_{i-1} \,, \quad \varphi_N = \mu_{N,i-1} \,. \tag{4.40}$$

We can now apply Theorem 2.5, because $L_N = \tilde{N}_i - N_{i-1}$ satisfies (2.18) with $\ell = (1 - \frac{i}{M}) \, \varrho$, (cf. (4.5) and (4.6)), while $\varphi_N = \mu_{N,i-1}$ satisfies (2.20) with $W_N = N_{i-1} - \tilde{N}_{i-1}$ thanks to (4.32), (2.24), and (2.25), and (2.19) holds with $w = (1 - \frac{i-1}{M}) \, \varrho$. Recalling from (4.30) that $m_{N,i} = 1 - o(1)$, relation (2.21) then yields

$$\lim_{N \to \infty} \mathbb{E}\big[(\tilde{\Delta}_{N,i})^2 \big| \mathcal{F}_{N_{i-1}} \big] = \lim_{N \to \infty} \mathbb{V}\mathrm{ar}\big[Z^{\beta_N}_{L_N}(\varphi_N) \big] = \frac{w - \ell}{1 + \ell} = \frac{\varrho / M}{1 + (1 - \frac{i}{M}) \, \varrho} \,, \tag{4.41}$$

which matches our goal (4.38). It remains to show that removing the constraint $\mathbb{1}_{\mathcal{D}_{\tilde N_i}\cap\mathcal{D}_{N_i}}$ from (4.39) is immaterial, that is, almost surely we have $\mathbb{E}\big[(\Delta_{N,i}-\tilde\Delta_{N,i})^2\,\big|\,\mathcal{F}_{N_{i-1}}\big]\to 0$. To this end, we note that

$$\tilde\Delta_{N,i}-\Delta_{N,i}=\frac{1}{m_{N,i}}\sum_{x\in\mathbb{Z}^2_{\rm even}}\mu_{N,i-1}(x)\,\mathrm{E}\Big[\Big(\mathrm{e}^{\mathcal{H}^{\beta_N}_{(N_{i-1},\tilde N_i]}}-1\Big)\,\mathbb{1}_{\mathcal{D}^c_{\tilde N_i}\cup\mathcal{D}^c_{N_i}}\,\Big|\,S_{N_{i-1}}=x\Big].$$

Since in the expectation above, disorder is restricted to the time interval $(N_{i-1},\tilde N_i]$ with $N_{i-1}\ll\tilde N_i\ll N_i$, we can show that the contribution from the event $\mathcal{D}_{\tilde N_i}\cap\mathcal{D}^c_{N_i}$, which implies $|S_{N_i}|>\sqrt{N_i\log\frac{1}{\delta_N^2}}$, is negligible via an analysis similar to that performed in the proof of (4.32), which uses the fact that the simple symmetric random walk has a negligible probability of having super-diffusive displacement on the time interval $[\tilde N_i,N_i]$.

So we will focus on showing that, conditional on $\mathcal{F}_{N,i}$, the L^2 norm of

$$\frac{1}{m_{N,i}}\sum_{x\in\mathbb{Z}^2_{\rm even}}\mu_{N,i-1}(x)\,\mathrm{E}\Big[\Big(\mathrm{e}^{\mathcal{H}^{\beta_N}_{(N_{i-1},\tilde N_i]}}-1\Big)\,\mathbb{1}_{\mathcal{D}^c_{\tilde N_i}}\,\Big|\,S_{N_{i-1}}=x\Big]\tag{4.42}$$

is negligible. First, recalling (4.30) we have that $m_{N,i-1}\geqslant 1-C\,(\delta_N^2)^{\frac{1}{C}}$ and so we can neglect this term. Secondly, by (4.32), we can bound $\mu_{N,i-1}(x)\leqslant Cq_{N_{i-1}-\tilde N_{i-1}}(x)$. Using the chaos expansion (2.11), the L^2 norm of the sum in (4.42) can be bounded by a multiple of

$$\begin{aligned}&\sum_{\substack{x,x',z,z'\in\mathbb{Z}^2_{\rm even}\\|y|,|y'|>(\tilde N_i\log\delta_N^{-2})^{1/2}}}\ \sum_{N_{i-1}\leqslant a<b\leqslant\tilde N_i} q_{N_{i-1}-\tilde N_{i-1}}(x)q_{N_{i-1}-\tilde N_{i-1}}(x')\,q_{a-N_{i-1}}(z-x)q_{a-N_{i-i}}(z-x')\\
&\qquad\qquad\times\sigma^2_{\beta_N}\,U_{\beta_N}(b-a,z'-z)\;q_{\tilde N_i-b}(y-z')q_{\tilde N_i-b}(y'-z')\\
&=\sum_{\substack{z,z'\in\mathbb{Z}^2_{\rm even}\\|y|,|y'|>(\tilde N_i\log\delta_N^{-2})^{1/2}}}\ \sum_{N_{i-1}\leqslant a<b\leqslant\tilde N_i} q_{a-\tilde N_{i-1}}(z)^2\,\sigma^2_{\beta_N}U_{\beta_N}(b-a,z'-z)\;q_{\tilde N_i-b}(y-z')q_{\tilde N_i-b}(y'-z'),\end{aligned}\tag{4.43}$$

where we used the Chapman-Kolmogorov equation to go from the first line to the second and we used the notation

$$U_\beta(n,z):=\sum_{k\geqslant1}(\sigma^2_\beta)^k\sum_{\substack{0=:n_0<n_1<\cdots<n_k=n\\x_0:=0,\ x_1,\ldots,x_{k-1}\in\mathbb{Z}^2,x_k=z}}\prod_{i=1}^k q_{n_i-n_{i-1}}(x_i-x_{i-1})^2,\tag{4.44}$$

that is, the renewal function from (2.12) with the end point pinned at z. To bound (4.43), we distinguish between two cases: either $|z'|\leqslant\frac12(\tilde N_i\log\delta_N^{-2})^{1/2}$, or z' satisfies the opposite inequality. In the first case, the decay from the random walk kernels will make the contribution to (4.43) negligible since $|y-z|,|y'-z|>\frac12(\tilde N_i\log\delta_N^{-2})^{1/2}$. In the second case, we can drop the constraints on y,y' in the sum to obtain that, the corresponding contribution to (4.43) is bounded (up to constants) by

$$\sum_{\substack{z\in\mathbb{Z}^2_{\rm even}\\|z'|>\frac12(\tilde N_i\log\delta_N^{-2})^{1/2}}}\ \sum_{N_{i-1}\leqslant a<b\leqslant\tilde N_i} q_{a-\tilde N_{i-1}}(z)^2\,\sigma^2_{\beta_N}U_{\beta_N}(b-a,z'-z).\tag{4.45}$$

In the sum above, we can identify (a, z) with (n_1, x_1) in (4.44) and use (4.44) to rewrite (4.45) and bound it by

$$\sum_{\substack{0<b'\leqslant \tilde N_i-\tilde N_{i-1}\\ |z'|>\frac12(\tilde N_i\log\delta_N^{-2})^{1/2}}} U_{\beta_N}(b',z') \quad\leqslant \sum_{\substack{0<b'\leqslant \tilde N_i\\ |z'|>\frac12(\tilde N_i\log\delta_N^{-2})^{1/2}}} U_{\beta_N}(b',z'), \tag{4.46}$$

where in the inequality, we enlarged the range of summation for b'. We can then use the next bound, which can be proved by following the same steps as in [CSZ23a, Lemma 3.5]:

$$\sum_{x\in\mathbb{Z}^2} U_{\beta_N}(n,x)e^{\lambda|x|} \leqslant ce^{c\lambda^2 n}U_{\beta_N}(n).$$

From this bound, the negligibility of (4.46) as $N\to\infty$ follows easily by a Markov type inequality with appropriate choices λ. □

Remark 4.6 (Sub-critical regime). *In the sub-critical regime* (1.26)*, we need to modify* (4.38) *to:*

$$\lim_{N\to\infty}\mathbb{E}[(\Delta_{N,i})^2\,|\,\mathcal{F}_{N_{i-1}}] = \frac{1}{M}\,\frac{\varrho\,\hat\beta^2}{1+(1-\frac{i}{M})\,\varrho\,\hat\beta^2}\,. \tag{4.47}$$

The proof is the same, except that in (4.41) *we need to apply* (2.22) *in place of* (2.21)*, see Remark 2.6.*

Step 5. (Higher moment bounds). In this step, we control higher moments of $\Delta_{N,i}$ defined in (4.9), proving the following bound (recall the second moment computation (4.38)).

Proposition 4.7 (High moment bounds). *For any $h\in\mathbb{N}$, there is a constant $\mathfrak{C}_h<\infty$ such that, for any $M\in\mathbb{N}$ and $i=1,\ldots,M$, we have the a.s. bound (uniformly over $\mathcal{F}_{N_{i-1}}$):*

$$\limsup_{N\to\infty}\big|\mathbb{E}[(\Delta_{N,i})^h\,|\,\mathcal{F}_{N_{i-1}}]\big| \leqslant \mathfrak{C}_h\left(\frac{1}{M}\,\frac{\varrho}{1+(1-\frac{i}{M})\varrho}\right)^{\frac h2}. \tag{4.48}$$

Note that the case $h=1$ is trivial by (4.37), while the case $h=2$ holds by (4.38), and hence we focus on $h\geqslant 3$, in which case the bound is a direct consequence of Theorem 1.11.

Proof. As in the proof Theorem 4.5, see (4.39), conditioned on $\mathcal{F}_{N_{i-1}}$, $\Delta_{N,i}$ can be written as a modified partition function where the random walk is restricted to $\mathcal{D}_{\tilde N_i}\cap\mathcal{D}_{N_i}$. In (4.39), we can ignore the mean $m_{N,i}=1-o(1)$ (see (4.30)). Thanks to Theorem 1.11 (whose assumptions we check in a moment), we can obtain a moment upper bound by removing the random walk restriction and applying (1.33) to get

$$\big|\mathbb{E}[(\Delta_{N,i})^h\,|\,\mathcal{F}_{N_{i-1}}]\big| \leqslant \mathfrak{C}_h\,\mathbb{V}\mathrm{ar}\big[Z^{\beta_N}_{L_N}(\varphi_N)\,\big|\,\mathcal{F}_{N_{i-1}}\big]^{\frac h2}\,.$$

Our goal (4.48) then follows from (4.38), since, as we showed in the proof of Theorem 4.5,

$$\lim_{N\to\infty}\mathbb{V}\mathrm{ar}\big[Z^{\beta_N}_{L_N}(\varphi_N)\,\big|\,\mathcal{F}_{N_{i-1}}\big] = \lim_{N\to\infty}\mathbb{E}[(\Delta_{N,i})^2\,|\,\mathcal{F}_{N_{i-1}}] = \frac{1}{M}\,\frac{\varrho}{1+(1-\frac{i}{M})\varrho}\,. \tag{4.49}$$

It only remains to check the assumptions of Theorem 1.11, namely that β_N from (1.22) and $L_N=\tilde N_i - N_{i-1}$, $\varphi_N=\mu_{N,i-1}$ from (4.40) fullfill conditions (1.29), (1.30) and (1.32).

- The bounded variance condition (1.32) clearly holds by (4.49).

- By (4.32), we have $\varphi_N = \mu_{N,i-1} \leqslant (1+\varepsilon_N)^2 \, q_{N_{i-1}-\tilde{N}_{i-1}} \leqslant 2\, q_{N_{i-1}-\tilde{N}_{i-1}}$, so φ_N is exponentially concentrated on the scale $\sqrt{N_{i-1}-\tilde{N}_{i-1}} \sim \sqrt{N_{i-1}}$ much smaller than $\sqrt{\tilde{N}_i} \sim \sqrt{L_N}$, thus condition (1.29) holds by the local limit theorem (2.3).
- Again, $\varphi_N \leqslant 2\, q_{N_{i-1}-\tilde{N}_{i-1}}$ yields $\|\varphi_N\|_{\ell^\infty} \sim \frac{C}{N_{i-1}-\tilde{N}_{i-1}} \sim \frac{C'}{\mathbb{D}[\varphi_N]}$, see (1.28) and (2.3), hence condition (1.30) is fullfilled.

The proof is complete. □

Remark 4.8 (Sub-critical regime). *In the sub-critical regime* (1.26)*, we need to modify* (4.48) *to:*

$$\limsup_{N\to\infty} \big| \mathbb{E}[(\Delta_{N,i})^h \,|\, \mathcal{F}_{N_{i-1}}] \big| \leqslant \mathfrak{C}_h \left(\frac{1}{M} \, \frac{\varrho\, \hat{\beta}^2}{1+(1-\frac{i}{M})\varrho\,\hat{\beta}^2} \right)^{\frac{h}{2}}, \tag{4.50}$$

in agreement with the variance (4.47)*.*

Step 6. Proof of (4.13) and (4.12). We will apply the Central Limit Theorem for arrays of martingale differences. In particular, we will make use of the following special version of [HH80, Theorem 3.5].

Theorem 4.9. *For each $n \geqslant 1$, let $(S_{n,i})_{1\leqslant i\leqslant M_n}$ be a mean zero square integrable martingale adapted to the filtration $(\mathcal{F}_{n,i})_{1\leqslant i\leqslant M_n}$. Let $\Delta_{n,i} := S_{n,i} - S_{n,i-1}$ be the associated martingale differences. Denote*

$$V_{n,i}^2 := \sum_{j=1}^{i} \mathbb{E}\big[\Delta_{n,j}^2 \,\big|\, \mathcal{F}_{n,j-1}\big] \quad \text{and} \quad U_{n,i}^2 := \sum_{j=1}^{i} \Delta_{n,j}^2.$$

Assume that

$$\mathbb{E}\big[|V_{n,M_n}^2 - \sigma^2|\big] \xrightarrow[n\to\infty]{} 0, \quad \text{and} \quad \max_{i\leqslant M_n} \mathbb{E}\big[\Delta_{n,i}^2 \,\big|\, \mathcal{F}_{n,i-1}\big] \xrightarrow[n\to\infty]{\mathbb{P}} 0. \tag{4.51}$$

Then the following three statements are equivalent:

$$\text{(i)} \quad \sum_{i\leqslant M_n} \mathbb{E}\big[\Delta_{n,i}^2 \, \mathbb{1}_{|\Delta_{n,i}|>\varepsilon}\big] \xrightarrow[n\to\infty]{} 0, \tag{4.52}$$

$$\text{(ii)} \quad \mathbb{E}\big[|U_{n,M_n}^2 - \sigma^2|\big] \xrightarrow[n\to\infty]{} 0, \tag{4.53}$$

$$\text{(iii)} \quad S_n := \sum_{i=1}^{M_n} \Delta_{n,i} \xrightarrow[n\to\infty]{d} \mathcal{N}(0,\sigma^2). \tag{4.54}$$

In our setting, the first condition in (4.51) follows from Theorem 4.5 (note the uniformity over $\mathcal{F}_{N_i}$ in the convergence therein) by choosing $M_N \to \infty$ slowly enough and then applying a Riemann sum approximation that shows $\sigma^2 = \int_0^\varrho \frac{\mathrm{d}t}{1+t} = \log(1+\varrho)$.

The second condition in (4.51) follows from the higher moment estimate (4.48) and a union bound as follows:

$$\begin{aligned}\mathbb{P}\Big(\max_{i\leqslant M_N}\mathbb{E}\big[\Delta_{N,i}^2\,\big|\,\mathcal{F}_{N,i-1}\big]>\varepsilon\Big)&\leqslant\sum_{i=1}^{M_N}\mathbb{P}\Big(\mathbb{E}\big[\Delta_{N,i}^2\,\big|\,\mathcal{F}_{N,i-1}\big]>\varepsilon\Big)\\&\leqslant\frac{1}{\varepsilon^2}\sum_{i=1}^{M_N}\mathbb{E}\big[\Delta_{N,i}^4\big]=O\Big(\frac{1}{M_N}\Big),\end{aligned}$$

where the last equality is justified by choosing $M_N\to\infty$ slowly enough. Indeed, by (4.48), for any fixed $M<\infty$, there is $N=\mathcal{N}_M$ large enough such that

$$\forall N\geqslant\mathcal{N}_M,\ \forall 1\leqslant i\leqslant M:\qquad \mathbb{E}[(\Delta_{N,i})^4\,|\,\mathcal{F}_{N_{i-1}}]\leqslant\mathfrak{C}_4\left(\frac{2\varrho}{M}\right)^2.$$

We can assume that $\lim_{M\to\infty}\mathcal{N}_M=\infty$. Hence we can choose a sequence $(M_N)_{N\to\infty}$ with $M_N\to\infty$ slowly enough such that $N\geqslant\mathcal{N}_{M_N}$, and hence

$$\sum_{i=1}^{M_N}\mathbb{E}[(\Delta_{N,i})^4\,|\,\mathcal{F}_{N_{i-1}}]\leqslant\mathfrak{C}_4\,M_N\left(\frac{2\varrho}{M_N}\right)^2=O\Big(\frac{1}{M_N}\Big).$$

Condition (4.52) follows in the same way via Chebyshev's inequality and the higher moment estimate (4.48). In turn, this implies (4.53) and (4.54), which are respectively our desired relation (4.13) and (4.12).

Step 7. Proof of (4.14)**.** To show the negligibility of the second term in (4.14), i.e., $\sum_{i=1}^{M_N}\log m_{N,i}$, we apply (4.30) to obtain

$$0\leqslant-\sum_{i=1}^{M_N}\log m_{N,i}\leqslant-\sum_{i=1}^{M_N}\log(1-C(\delta_N^2)^{\frac{1}{C}})\leqslant C'M_N(\delta_N)^{\frac{1}{C}},$$

which goes to 0 if $M_N\to\infty$ slowly enough such that $M_N\ll(\delta_N)^{-\frac{1}{C}}$.

For the first term in (4.14), recall that $r(x)=\log(1+x)-\big(x-\frac{x^2}{2}\big)$. Using the elementary estimate $|r(x)|\leqslant C|x|^3\wedge x^2$, we obtain

$$\begin{aligned}\Big|\sum_{i=1}^{M_N}\mathbb{E}\big[r(\Delta_{N,i})\big]\Big|&\leqslant C\sum_{i=1}^{M_N}\Big(\mathbb{E}\big[\,|\Delta_{N,i}|^2\,\mathbb{1}_{|\Delta_{N,i}|>\varepsilon}\,\big]+\mathbb{E}\big[\,|\Delta_{N,i}|^3\,\mathbb{1}_{|\Delta_{N,i}|\leqslant\varepsilon}\,\big]\Big)\\&\leqslant C\sum_{i=1}^{M_N}\Big(\mathbb{E}\big[\,\Delta_{N,i}^2\,\mathbb{1}_{|\Delta_{N,i}|>\varepsilon}\,\big]+\varepsilon\mathbb{E}\big[\,\Delta_{N,i}^2\,\big]\Big),\end{aligned}$$

which converges to 0 by (4.52) and (4.53) and by letting ε be arbitrarily small.

The proof of (4.2) is now complete. □

5. Higher moment bounds

We prove a strengthened version of the general moment bound in Theorem 1.11. We consider the averaged *point-to-point* partition function $Z_L(\varphi,\psi)$ defined in (1.17): for φ,ψ :

$\mathbb{Z}^2 \to \mathbb{R}$ and $L \in \mathbb{N}$,

$$Z_L^\beta(\varphi, \psi) := \sum_{z,w \in \mathbb{Z}^2} \varphi(z)\, Z_L^\beta(z,w)\, \psi(w) \qquad \text{with} \quad Z_L^\beta(z,w) := \mathrm{E}\Big[\mathrm{e}^{\mathcal{H}^{\beta,\omega}_{(0,L)}(S)}\, \mathbb{1}_{\{S_L = w\}} \Big| S_0 = z\Big]. \tag{5.1}$$

Let φ be a probability mass function on $\mathbb{Z}^2$ satisfying the localization condition (1.29), which we recall here for convenience: for some $\hat{t} > 0$, $\mathsf{c}_1 < \infty$

$$\exists z_0 \in \mathbb{R}^2: \qquad \sum_{z \in \mathbb{Z}^2} \varphi(z)\, \mathrm{e}^{2\,\hat{t}\, \frac{|z - z_0|}{\sqrt{L}}} \leqslant \mathsf{c}_1 \,. \tag{5.2}$$

Instead of (1.30), we impose the weaker condition that, for some $\mathsf{c}_2' < \infty$,

$$\frac{\log\big(L\, \|\varphi\|_{\ell^2}^2\big)}{R_L(\varphi,\varphi)} \leqslant \mathsf{c}_2' \,, \tag{5.3}$$

where $R_L(\varphi,\varphi)$ is defined in (2.6). The fact that (5.3) is indeed implied by (1.30) (when (5.2) holds) is shown in the next result, proved in Appendix B.1.

Lemma 5.1. *If a probability mass function φ fullfills* (5.2)*, there exists $\mathfrak{c} = \mathfrak{c}(\hat{t}, \mathsf{c}_1) > 0$ such that*

$$R_L(\varphi,\varphi) \geqslant R_{L/2}(\varphi,\varphi) \geqslant \mathfrak{c}\, \log\Big(1 + \tfrac{L/2}{1 + 4\,\mathbb{D}[\varphi]}\Big)\,, \tag{5.4}$$

and hence condition (1.30) *implies condition* (5.3)*.*

We also require condition (1.32), that is boundedness of the variance of the *point-to-plane partition function* $Z_L^\beta(\varphi) = Z_L^\beta(\varphi, 1)$, which we recall here for convenience:

$$\mathbb{V}\mathrm{ar}[Z_L^\beta(\varphi)] \leqslant \mathsf{c}_3 \,. \tag{5.5}$$

We stress that we impose no assumption on ψ in $Z_L^\beta(\varphi,\psi)$.

We can now state our strengthened moment bound, which generalises Theorem 1.11.

Theorem 5.2 (Strengthened general moment bound). *Given $h \in \mathbb{N}$ and $\hat{t}$, c_1, c_2', $\mathsf{c}_3 \in (0,\infty)$, there exist constants $L_h, \mathfrak{C}_h < \infty$ (depending also on $\hat{t}, \mathsf{c}_1, \mathsf{c}_2', \mathsf{c}_3$) such that*

$$\Big|\mathbb{E}\big[\big(Z_L^\beta(\varphi,\psi) - \mathbb{E}[Z_L^\beta(\varphi,\psi)]\big)^h\big]\Big| \leqslant \mathfrak{C}_h\, \mathbb{V}\mathrm{ar}[Z_L^\beta(\varphi)]^{\frac{h}{2}} \left\| \psi(\cdot)\, \mathrm{e}^{-\frac{\hat{t}}{2} \frac{|\cdot - z_0|}{\sqrt{L}}} \right\|_{\ell^\infty}^h \tag{5.6}$$

uniformly for $\beta \in [0, \beta_0]$, $L \geqslant L_h$, for probability mass functions φ and $z_0 \in \mathbb{Z}^2$ satisfying (5.2)*,* (5.3) *and* (5.5)*, and for arbitrary function ψ. Furthermore:*

- *z_0 in* (5.6) *(from* (5.2)*) can be replaced by the mean m_φ of φ (see* (1.28)*);*
- *the bound* (5.6) *still holds if, on the LHS, we replace $Z_L^\beta(\varphi,\psi)$ by its restriction to any subset of random walk paths in its definition* (5.1)*.*

The rest of this section is devoted to proving Theorem 5.2.

5.1. Preliminary lemmas. We collect here some technical lemmas that will be useful in the proof. We first show that, for any probability mass function φ satisfying condition (5.2), we have a uniform lower bound on $R_{L/2}(\varphi,\varphi)$. The proof is given in Appendix B.2.

Lemma 5.3. *Given $\hat{t}, \mathsf{c}_1 \in (0,\infty)$, there exists $\eta > 0$ (depending on $\hat{t}, \mathsf{c}_1$) such that, for any probability mass function φ satisfying* (5.2) *with constants $\hat{t}$ and c_1, we have*

$$R_{L/2}(\varphi,\varphi) \geqslant \eta \,. \tag{5.7}$$

We next show that the assumptions of Theorem 5.2 force β to be *at most critical.* To this purpose, recalling (1.15) and (1.19), we have the equivalence for any $\vartheta \in \mathbb{R}$, as $L \to \infty$,

$$\sigma_\beta^2 = \frac{1}{R_L - \vartheta + o(1)} \qquad \Longleftrightarrow \qquad \sigma_\beta^2 = \frac{1}{R_L}\left(1 + \frac{\pi\,\vartheta + o(1)}{\log L}\right). \tag{5.8}$$

The next result is proved in Appendix B.3.

Lemma 5.4. *Given $\hat{t}, \mathsf{c}_1, \mathsf{c}_3 \in (0,\infty)$, there exists $\bar{\vartheta} \in [0,\infty)$ (depending on $\hat{t}, \mathsf{c}_1, \mathsf{c}_3$) such that, for any $\beta \geqslant 0$, $L \in \mathbb{N}$ and any probability mass function φ satisfying* (5.2) *and* (5.5) *with constants $\hat{t}, \mathsf{c}_1$ and c_3, we have (recall σ_β^2 from* (1.16)*)*

$$\sigma_\beta^2 \leqslant \frac{1}{R_L - \bar{\vartheta}} \,. \tag{5.9}$$

We finally define an exponentially dampened version of R_L from (2.5):

$$R_L^{(\hat{\lambda})} := \sum_{n=1}^{L} \mathrm{e}^{-\hat{\lambda}\frac{n}{L}}\, q_{2n}(0) \qquad \text{for } \hat{\lambda} \geqslant 0\,, \tag{5.10}$$

We will use this quantity to give a proxy for the second moment $\mathbb{E}\big[Z_{L/2}^{\beta}(0)^2\big]$, as shown in the next result, proved in Appendix B.4.

Lemma 5.5. *Recall the constant $\mathfrak{a}_+$ from* (2.4)*. If $\beta \geqslant 0$, $L \in \mathbb{N}$ and $\hat{\lambda} \geqslant 0$ satisfy*

$$\sigma_\beta^2 \leqslant \frac{1}{R_L^{(\hat{\lambda})} + 4\,\mathfrak{a}_+} \,, \tag{5.11}$$

then

$$\mathbb{E}\big[Z_{L/2}^{\beta}(0)^2\big] \geqslant \frac{1}{2}\,\frac{1}{1 - \sigma_\beta^2\, R_L^{(\hat{\lambda})}} \,. \tag{5.12}$$

In order to achieve condition (5.11), starting from (5.9), it is enough to take $\hat{\lambda} \geqslant 0$ large enough, as we show in the next elementary result, proved in Appendix B.5.

Lemma 5.6. *Recall the constant $\mathfrak{a}_-$ from* (2.4)*. For any $0 \leqslant \hat{\lambda} \leqslant L$, we have*

$$R_L^{(\hat{\lambda})} \leqslant R_L - \mathfrak{a}_-\, \log \tfrac{\hat{\lambda}}{2} \,. \tag{5.13}$$

We are now ready to describe the strategy of the proof of Theorem 5.2.

5.2. A general estimate. We bound the moments of the partition function exploiting the *functional operator approach* developed in [CSZ23a, LZ23, CCR23]. The following general estimate is extracted from [CCR23, Section 4] (see Appendix B for the details). A comparison with the original bound from [CSZ23a] is discussed in Remark 5.10.

Theorem 5.7 ([CCR23])**.** *Fix any exponent $h \in \mathbb{N}$, $h \geqslant 3$, system size $L \in \mathbb{N}$ and coupling constant $\beta > 0$ small enough, say $\beta \leqslant \beta_0$ for a suitable $\beta_0 = \beta_0(h) > 0$. Given $\hat{t} > 0$ and*

$\hat\lambda \geqslant 0$, *there are constants* $\mathtt{K}_h^{(\hat t)}$, $\mathcal{C}_h^{(\hat t,\hat\lambda)} < \infty$ *such that, assuming* $\sigma_\beta^2\, R_L^{(\hat\lambda)} < 1$ *and defining*

$$\Gamma = \Gamma_{h,\beta,L}^{(\hat t,\hat\lambda)} := \mathtt{K}_h^{(\hat t)}\, \frac{\sigma_\beta^2}{1-\sigma_\beta^2\, R_L^{(\hat\lambda)}}\,, \tag{5.14}$$

the following bound holds for any $1 < p, q < \infty$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and any functions* φ, ψ *on* $\mathbb{Z}^2$*:*

$$\begin{aligned} \big|\mathbb{E}\big[\big(Z_L^\beta(\varphi,\psi) - \mathbb{E}[Z_L^\beta(\varphi,\psi)]\big)^h\big]\big| \leqslant \mathcal{C}_h^{(\hat t,\hat\lambda)} & \sum_{r=1}^\infty \big(p\,q\,\Gamma\big)^{\max\{r,\frac{h}{2}\}} \\ & \times \Big\|\varphi(\cdot)\,\mathrm{e}^{\frac{\hat t}{\sqrt{L}}|\cdot|}\Big\|_{\ell^1} \Big\|\varphi(\cdot)\,\mathrm{e}^{\frac{\hat t}{\sqrt{L}}|\cdot|}\Big\|_{\ell^p}^{h-1} \\ & \times \Big\|\psi(\cdot)\,\mathrm{e}^{-\frac{\hat t}{\sqrt{L}}|\cdot|}\Big\|_{\ell^\infty} \Big\|\psi(\cdot)\,\mathrm{e}^{-\frac{\hat t}{\sqrt{L}}|\cdot|}\Big\|_{\ell^q}^{h-1}. \end{aligned} \tag{5.15}$$

The series converges iff $p\,q\,\Gamma < 1$*, in which case we can bound it by*

$$\sum_{r=1}^\infty \big(p\,q\,\Gamma\big)^{\max\{r,\frac{h}{2}\}} \leqslant \mathrm{e}^h\, \frac{\big(p\,q\,\Gamma\big)^{\frac{h}{2}}}{1-\,p\,q\,\Gamma} \qquad \text{if} \quad 0 < p\,q\,\Gamma < 1\,. \tag{5.16}$$

We will deduce our goal (5.6) from (5.15). We first need to have a suitable control on the second and third lines of (5.15) in order to fit our assumption on φ, see in particular (5.2) and (5.3). Since we will be interested in taking q large, we may assume

$$q \in [4,\infty)\,. \tag{5.17}$$

We will need the following basic interpolation result.

Lemma 5.8. *Fix* $p \in (1,2)$ *and* $q \in (2,\infty)$ *with* $\frac{1}{p}+\frac{1}{q}=1$*. For any* f, g *on* $\mathbb{Z}^2$*,*

$$\|f\,g\|_{\ell^p} \leqslant \Big\|f\,g^{\frac{q}{q-2}}\Big\|_{\ell^1}^{1-\frac{2}{q}}\, \|f\|_{\ell^2}^{\frac{2}{q}}\,. \tag{5.18}$$

Proof. We write $f^p = f^{1-\alpha} f^{2\alpha}$, with $\alpha = p-1 = \frac{p}{q} \in (0,\frac{1}{2})$, and apply Hölder to get

$$\|f\,g\|_{\ell^p}^p = \sum_{z\in\mathbb{Z}^2} |f(z)|^p\,|g(z)|^p \leqslant \bigg(\sum_{z\in\mathbb{Z}^2} |f(z)|\,|g(z)|^{\frac{p}{1-\alpha}}\bigg)^{1-\alpha} \bigg(\sum_{z\in\mathbb{Z}^2} f(z)^2\bigg)^{\alpha}$$

which coincides with (5.19) since $\frac{1-\alpha}{p} = 1-\frac{2}{q}$ and $\frac{\alpha}{p} = \frac{1}{q}$. ☐

Let us look back at the second and third lines of (5.15). We apply (5.18) with $f = \varphi$ and $g(\cdot) = \mathrm{e}^{\frac{\hat t}{\sqrt{L}}|\cdot|}$: for $q \geqslant 4$ we have $\frac{q}{q-2} \leqslant 2$, hence

$$\Big\|\varphi(\cdot)\,\mathrm{e}^{\frac{\hat t}{\sqrt{L}}|\cdot|}\Big\|_{\ell^p} \leqslant \Big\|\varphi(\cdot)\,\mathrm{e}^{\frac{2\hat t}{\sqrt{L}}|\cdot|}\Big\|_{\ell^1}^{1-\frac{2}{q}} \|\varphi\|_{\ell^2}^{\frac{2}{q}} \leqslant \Big\|\varphi(\cdot)\,\mathrm{e}^{\frac{2\hat t}{\sqrt{L}}|\cdot|}\Big\|_{\ell^1} \|\varphi\|_{\ell^2}^{\frac{2}{q}}\,. \tag{5.19}$$

We next make the simple estimate that, for some $c < \infty$,

$$\Big\|\psi(\cdot)\,\mathrm{e}^{-\frac{\hat t}{\sqrt{L}}|\cdot|}\Big\|_{\ell^q} \leqslant \Big\|\psi(\cdot)\,\mathrm{e}^{-\frac{\hat t}{2\sqrt{L}}|\cdot|}\Big\|_{\ell^\infty} \bigg(\sum_{z\in\mathbb{Z}^2} \mathrm{e}^{-\frac{\hat t}{2\sqrt{L}}q|z|}\bigg)^{\frac{1}{q}} \leqslant \Big\|\psi(\cdot)\,\mathrm{e}^{-\frac{\hat t}{2\sqrt{L}}|\cdot|}\Big\|_{\ell^\infty} \bigg(\frac{cL}{\hat t^2}\bigg)^{\frac{1}{q}} \tag{5.20}$$

because $\sum_{z\in\mathbb{Z}^2} \mathrm{e}^{-\frac{s}{2}q|z|} \leqslant \sum_{z\in\mathbb{Z}^2} \mathrm{e}^{-\frac{s}{2}|z|} \leqslant \frac{c}{s^2}$. Plugging these estimates into (5.15), as well as the bound (5.16) with $p \leqslant 2$, we obtain the following corollary of Theorem 5.7.

Proposition 5.9. *For any $h \in \mathbb{N}$, $h \geqslant 3$, there is $\beta_0 = \beta_0(h) > 0$ such that for any $\beta \in [0, \beta_0]$ and $L \in \mathbb{N}$ the following holds. Given $\hat{t} > 0$ and $\hat{\lambda} \geqslant 0$, there are constants $\mathtt{K}_h^{(\hat{t})}$, $\mathcal{C}_h^{(\hat{t},\hat{\lambda})} < \infty$ such that, recalling Γ from (5.14), for any*

$$q \geqslant 4 \quad \text{such that} \quad 0 < 2q\,\Gamma < 1\,, \tag{5.21}$$

we can bound, for any functions φ, ψ on $\mathbb{Z}^2$,

$$\begin{aligned} \left|\mathbb{E}\big[\big(Z_L^\beta(\varphi,\psi) - \mathbb{E}[Z_L^\beta(\varphi,\psi)]\big)^h\big]\right| \leqslant \mathcal{C}_h^{(\hat{t},\hat{\lambda})} \, \frac{(2q\,\Gamma)^{\frac{h}{2}}}{1-2q\,\Gamma} \, \big(L\,\|\varphi\|_{\ell^2}^2\big)^{\frac{h-1}{q}} \\ \times \left\|\varphi(\cdot)\,\mathrm{e}^{\frac{2\hat{t}}{\sqrt{L}}|\cdot - z_0|}\right\|_{\ell^1}^h \left\|\psi(\cdot)\,\mathrm{e}^{-\frac{\hat{t}}{2\sqrt{L}}|\cdot - z_0|}\right\|_{\ell^\infty}^h . \end{aligned} \tag{5.22}$$

In the next subsection we will show that our goal (5.6) follows by (5.22).

Remark 5.10 (Comparison with [CSZ23a]**).** *From the proof of* [CSZ23a, Theorem 6.1] *one can extract a bound similar to* (5.15)*, but with a different dependence on the boundary conditions φ, ψ, namely the second and third lines of* (5.15) *are replaced by*[†]

$$L^{\frac{1}{q}} \left\|\varphi(\cdot)\,\mathrm{e}^{\frac{\hat{t}}{\sqrt{L}}|\cdot|}\right\|_{\ell^p}^h \times \frac{L^{\frac{1}{p}}}{L}\,\|\psi\|_\infty^h \left\|\mathrm{e}^{-\frac{\hat{t}}{\sqrt{L}}|\cdot|}\right\|_{\ell^q}^h = \left\|\varphi(\cdot)\,\mathrm{e}^{\frac{\hat{t}}{\sqrt{L}}|\cdot|}\right\|_{\ell^p}^h \|\psi\|_\infty^h \left\|\mathrm{e}^{-\frac{\hat{t}}{\sqrt{L}}|\cdot|}\right\|_{\ell^q}^h . \tag{5.23}$$

The bound (5.15) *is better for two reasons:*

- *the quantity $\left\|\psi(\cdot)\,\mathrm{e}^{-\frac{\hat{t}}{\sqrt{L}}|\cdot|}\right\|_{\ell^q}$ is smaller than $\|\psi\|_\infty \left\|\mathrm{e}^{-\frac{\hat{t}}{\sqrt{L}}|\cdot|}\right\|_{\ell^q}$ and it allows for unbounded functions ψ;*
- *the power $h-1$ in* (5.15) *is better than h in* (5.23) *in case $D = \mathbb{D}[\varphi] \ll L$, see* (1.28)*: for instance, if φ is a probability mass function supported on a ball of radius $\sqrt{D}$ with $\|\varphi\|_\infty = O(\frac{1}{D})$ and $\psi \equiv 1$ is constant (for simplicity), we have for some $c > 0$*

$$\left\|\varphi(\cdot)\,\mathrm{e}^{\frac{\hat{t}}{\sqrt{L}}|\cdot|}\right\|_{\ell^p} \geqslant c\,\frac{D^{\frac{1}{p}}}{D} = c\,D^{-\frac{1}{q}}\,, \qquad \left\|\psi(\cdot)\,\mathrm{e}^{-\frac{\hat{t}}{\sqrt{L}}|\cdot|}\right\|_{\ell^q} \geqslant c\,L^{\frac{1}{q}}\,,$$

hence (5.23) *is larger than* (5.15) *by a factor $\geqslant c^2\,(\frac{L}{D})^{\frac{1}{q}} \gg 1$.*

5.3. Proof of Theorem 5.2. Given $h \in \mathbb{N}$ and constants $\hat{t}$, c_1, c_2', $\mathsf{c}_3 \in (0,\infty)$, we need to prove the bound (5.6) for all $\beta \geqslant 0$ and L large enough, uniformly over probability mass functions φ satisfying (5.2), (5.3), (5.5) and over arbitrary functions ψ on $\mathbb{Z}^2$. For simplicity, we assume $z_0 = 0$ (it suffices to replace φ by $\varphi(\,\cdot + z_0)$ and likewise for ψ).

We will deduce (5.6) from (5.22). To apply Proposition 5.9, we note that $\beta \leqslant \beta_0(h)$ is guaranteed by (5.9) if we take $L \geqslant L_h$ for L_h large enough (depending on h and $\bar{\vartheta}$ in (5.9), hence on $\hat{t}, \mathsf{c}_1, \mathsf{c}_3$). We show below that condition (5.21) can be satisfied as follows:

(a) we first fix $\hat{\lambda} \geqslant 0$ such that $\Gamma \in (0, \frac{1}{16}]$;

(b) then we pick $q \geqslant 4$ with $2q\,\Gamma \leqslant \frac{1}{2}$ (to discard the denominator $1 - 2q\,\Gamma \geqslant \frac{1}{2}$ in (5.22)).

We can now apply the bound (5.15). We show below that

[†]The factors $L^{\frac{1}{q}}$, $L^{\frac{1}{p}}$ in (5.23) arise from operator norms, see [CSZ23a, Proposition 6.6], while $\frac{1}{L}$ is due to an averaging on the system size from L to $2L$ which is performed in the proof.

(c) for suitable constants $C, C' < \infty$ (depending on h, $\hat{t}$, c_1, c_2', c_3),

$$2q\,\Gamma \leqslant C\ \mathbb{V}\mathrm{ar}[Z_L^\beta(\varphi)]\,, \qquad \big(L\,\|\varphi\|_{\ell^2}^2\big)^{\frac{1}{q}} \leqslant C'\,. \tag{5.24}$$

Plugging these bounds into (5.22), in view of (5.2), we see that the RHS of (5.15) gives precisely (5.6) with the constant

$$\mathfrak{C}_h = 2\,\mathcal{C}_h^{(\hat{t},\hat{\lambda})}\,C^{\frac{h}{2}}\,(C')^{h-1}\,\mathsf{c}_1^h \tag{5.25}$$

We complete the proof of Theorem 5.2 proving the steps (a), (b) and (c).

Step (a). To find the desired $\hat{\lambda} \geqslant 0$, we note that by (5.14) we have the equivalence

$$0 < \Gamma \leqslant \tfrac{1}{16} \qquad \Longleftrightarrow \qquad \sigma_\beta^2 \leqslant \frac{1}{R_L^{(\hat{\lambda})} + 16\,\mathtt{K}_h^{(\hat{t})}}\,. \tag{5.26}$$

We will later need the similar condition (5.11). Both conditions hold if we take $\hat{\lambda}$ large enough, thanks to Lemmas 5.4 and 5.6: more explicitly, by (5.9) and (5.13), we can fix

$$\hat{\lambda} := 2\,\mathrm{e}^{2\pi\left(\bar{\vartheta}+\max\left\{16\,\mathtt{K}_h^{(\hat{t})},\,4\,\mathfrak{a}_+\right\}\right)}\,. \tag{5.27}$$

Step (b). We will see below that it would be convenient to take $q \approx R_{L/2}(\varphi,\varphi)$. To ensure that $q \geqslant 4$ and also $2q\,\Gamma \leqslant \frac{1}{2}$ i.e. $q \leqslant \frac{1}{4\Gamma}$, since $R_{L/2}(\varphi,\varphi) \geqslant \eta$ by (5.7), we define

$$q := \min\left\{4\,\frac{R_{L/2}(\varphi,\varphi)}{\eta}\,,\,\frac{1}{4\,\Gamma}\right\}. \tag{5.28}$$

In particular, we have by definition

$$q\,\Gamma \leqslant \frac{4}{\eta}\,R_{L/2}(\varphi,\varphi)\,\Gamma\,. \tag{5.29}$$

Step (c). Let us prove the first relation in (5.24). By definition of Γ, see (5.14), and by Lemma 5.5 (note that condition (5.11) is ensured by our choice of $\hat{\lambda}$), we can bound

$$R_{L/2}(\varphi,\varphi)\,\Gamma \leqslant \mathtt{K}_h^{(\hat{t})}\,R_{L/2}(\varphi,\varphi)\,\frac{\sigma_\beta^2}{1-\sigma_\beta^2\,R_L^{(\hat{\lambda})}} \leqslant 2\,\mathtt{K}_h^{(\hat{t})}\,R_{L/2}(\varphi,\varphi)\,\sigma_\beta^2\,\mathbb{E}\big[Z_{L/2}^\beta(0)^2\big]\,.$$

Applying the first bound in (2.16) we then obtain

$$R_{L/2}(\varphi,\varphi)\,\Gamma \leqslant 2\,\mathtt{K}_h^{(\hat{t})}\ \mathbb{V}\mathrm{ar}[Z_L^\beta(\varphi)]\,, \tag{5.30}$$

hence, recalling (5.29), the first relation in (5.24) holds with $C := \frac{16\,\mathtt{K}_h^{(\hat{t})}}{\eta}$.

Let us finally prove the second relation in (5.24). By assumption (5.3) we can bound

$$\big(L\,\|\varphi\|_{\ell^2}^2\big)^{\frac{1}{q}} = \mathrm{e}^{\frac{1}{q}\log(L\,\|\varphi\|_{\ell^2}^2)} \leqslant \mathrm{e}^{\frac{1}{q}\,\mathsf{c}_2'\,R_L(\varphi,\varphi)}\,.$$

It remains to show that, for some constant $c > 0$ (depending on h, $\hat{t}$, c_1, c_2', c_3),

$$q \geqslant c\,R_L(\varphi,\varphi)\,, \tag{5.31}$$

so that the second relation in (5.24) holds with $C' = \mathrm{e}^{\frac{1}{c}\,\mathsf{c}_2'}$.

Let us finally prove (5.31). It follows by (5.30) and assumption (5.5) that

$$\frac{1}{\Gamma} \geqslant \frac{1}{2\,\mathtt{K}_h^{(\hat{t})}\,\mathsf{c}_3}\,R_{L/2}(\varphi,\varphi)\,,$$

hence, recalling (5.28), we can bound

$$q \geqslant a\, R_{L/2}(\varphi,\varphi) \qquad \text{with} \quad a := \tfrac{4}{\eta} \wedge \frac{1}{8\, \mathtt{K}_h^{(t)}\, \mathtt{c}_3}\,. \tag{5.32}$$

At last, by (2.7) and (5.7), for any probability mass function φ we can write

$$\frac{R_L(\varphi,\varphi)}{R_{L/2}(\varphi,\varphi)} \leqslant 1 + \frac{R_L(\varphi,\varphi) - R_{L/2}(\varphi,\varphi)}{R_{L/2}(\varphi,\varphi)} \leqslant 1 + \frac{\mathfrak{a}_+}{\eta}\,. \tag{5.33}$$

Combining (5.33) with (5.32) then gives (5.31) with $c := a/(1+\mathfrak{a}_+/\eta)$. □

Appendix A. Second moment computations

We first prove Lemma 2.1 about Green's functions. Then we prove Proposition 2.3 on the second moment of the point-to-plane partition function. Finally, we prove Theorem 2.5 on the variance of the averaged partition function, together with Propositions 2.7 and 2.8.

Proof of Lemma 2.1. From (2.8), we see that $\mathcal{G}(a) = \frac{1}{2\pi}\log(1+a^{-2}) + O(1)$ uniformly over $a > 0$. This already proves the second equality in (2.9). To prove the first equality, we plug the first line of (2.3) into (2.5) to get

$$R_L(z) = \left\{ \sum_{n=1}^{L} g_n(z) \right\} 2 \cdot \mathbb{1}_{\mathbb{Z}^2_{\mathrm{even}}}(z) \;+\; O(1)\,. \tag{A.1}$$

We can replace $g_n(z) = g_n(|z|)$ by $g_n(|z|+1)$, because their difference is $O(n^{-3/2})$ and can be absorbed in the $O(1)$ term. We next write

$$\sum_{n=1}^{L} g_n(|z|+1) = \frac{1}{L}\sum_{n=1}^{L} g_{\frac{n}{L}}\Big(\tfrac{|z|+1}{\sqrt{L}}\Big) = \mathcal{G}\Big(\tfrac{|z|+1}{\sqrt{L}}\Big) + \Delta_L\Big(\tfrac{|z|+1}{\sqrt{L}}\Big)$$

where we set

$$\Delta_L(x) := \sum_{n=1}^{L} \int_{\frac{n-1}{L}}^{\frac{n}{L}} \big\{ g_{\frac{n}{L}}(x) - g_t(x) \big\}\, \mathrm{d}t\,.$$

It remains to show that $\Delta_L(x) = O(1)$ is uniformly bounded for $|x| = \frac{|z|+1}{\sqrt{L}} \geqslant \frac{1}{\sqrt{L}}$.

By direct computation we see that $|\partial_u g_u(x)| \leqslant \frac{c}{u^2}\, \mathrm{e}^{-\frac{|x|^2}{cu}}$ for some $c < \infty$. Then, uniformly for $|x| \geqslant \frac{1}{\sqrt{L}}$, we can bound the term $n=1$ in $\Delta_L(x)$ by an absolute constant $C < \infty$:

$$\int_0^{\frac{1}{L}} \bigg(\int_t^{\frac{1}{L}} \frac{c}{u^2}\, \mathrm{e}^{-\frac{1}{cLu}}\, \mathrm{d}u \bigg) \mathrm{d}t = \int_0^{\frac{1}{L}} \frac{c}{u}\, \mathrm{e}^{-\frac{1}{cLu}}\, \mathrm{d}u = \int_0^{1} \frac{c}{v}\, \mathrm{e}^{-\frac{1}{cv}}\, \mathrm{d}v =: C < \infty\,.$$

For the terms $n \geqslant 2$ in $\Delta_L(x)$, we simply use $|\partial_u g_u(x)| \leqslant \frac{c}{u^2}$ to bound $|g_{\frac{n}{L}}(x) - g_t(x)| \leqslant \frac{c}{t^2}\frac{1}{L}$ (because $\frac{n}{L} - t \leqslant \frac{1}{L}$) and we obtain, uniformly over $x \in \mathbb{R}^2$,

$$\sum_{n=2}^{L} \int_{\frac{n-1}{L}}^{\frac{n}{L}} \big| g_{\frac{n}{L}}(x) - g_t(x) \big|\, \mathrm{d}t \leqslant \frac{c}{L} \sum_{n=2}^{L} \int_{\frac{n-1}{L}}^{\frac{n}{L}} \frac{1}{t^2}\, \mathrm{d}t = \frac{c}{L} \int_{\frac{1}{L}}^{1} \frac{1}{t^2}\, \mathrm{d}t \leqslant c < \infty\,.$$

This completes the proof that $|\Delta_L(x)| \leqslant C + c$ uniformly for $L \in \mathbb{N}$ and $|x| \geqslant \frac{1}{\sqrt{L}}$.

We finally prove (2.10), for which we may assume that $z \neq 0$ and L is large enough. By the second line of (2.3), uniformly over $|z|^2 \leqslant 2(1+t^2)n$ we can write, for a suitable $c > 0$,

$$q_{2n}(z) = g_n(z)\,\mathrm{e}^{O(\frac{(1+t^2)^2}{n})}\, 2\,\mathbb{1}_{\mathbb{Z}^2_{\mathrm{even}}}(z) \geqslant \frac{c\,\mathrm{e}^{-\frac{t^4}{c}}}{n}\,\mathbb{1}_{\mathbb{Z}^2_{\mathrm{even}}}(z)\,,$$

hence, restricting the sum in (2.5) to $n \geqslant \bar{n}(z) := \lceil \frac{1}{2}(\frac{|z|^2}{1+t^2}+1)\rceil$, we get

$$R_L(z) \geqslant \sum_{n=\bar{n}(z)}^{L} \frac{c\,\mathrm{e}^{-\frac{t^4}{c}}}{n} \geqslant \int_{\bar{n}(z)}^{L+1} \frac{c\,\mathrm{e}^{-\frac{t^4}{c}}}{u}\,\mathrm{d}u = c\,\mathrm{e}^{-\frac{t^4}{c}}\log\frac{L+1}{\bar{n}(z)}\,.$$

For $|z| \leqslant t\sqrt{L}$ we have $\bar{n}(z) \leqslant \frac{1}{2}(L+1)$, equivalently $\frac{L+1}{\bar{n}(z)} \geqslant 1+\frac{L+1}{2\bar{n}(z)}$, and since $\bar{n}(z) \leqslant \frac{|z|^2+1}{2}$ we finally obtain $\log\frac{L+1}{\bar{n}(z)} \geqslant \log(1+\frac{L+1}{1+|z|^2})$, hence (2.10) holds with $c_t = c\,\mathrm{e}^{-\frac{t^4}{c}}$. □

The key tool for the proof of Proposition 2.3 is the following renewal representation of the second moment of the partition function developed in [CSZ19a].

Remark A.1 (Renewal interpretation). *Given $N \in \mathbb{N}$, we define the integer valued renewal process*

$$\tau_k^{(N)} = T_1^{(N)} + \ldots + T_k^{(N)}$$

with $\tau_0^{(N)} := 0$ and i.i.d. increments $(T_i^{(N)})_{i\in\mathbb{N}}$ with distribution (recall (1.14) and (1.15))

$$\mathrm{P}\big(T_i^{(N)} = n\big) = \frac{1}{R_N}\,q_{2n}(0), \qquad 1 \leqslant i \leqslant N. \tag{A.2}$$

For $\beta = \beta_N$ in the quasi-critical regime (1.22), we can then write (2.12) as follows: for every $n \in \mathbb{N}_0$,

$$U_{\beta_N}(n) = \sum_{k\geqslant 0}\Big(1-\tfrac{|\vartheta_N|}{\log N}\Big)^k\,\mathrm{P}\big(\tau_k^{(N)} = n\big) = \frac{\log N}{|\vartheta_N|}\,\mathrm{P}\big(\tau_{\mathcal{K}_N}^{(N)} = n\big)\,, \tag{A.3}$$

where $\mathcal{K}_N$ is an independent Geometric random variable with mean $\frac{\log N}{|\vartheta_N|}$.

Proof of Proposition 2.3. Note that (recall (A.2) and (1.15))

$$\mathrm{P}\big(\tau_k^{(N)} \leqslant L\big) \leqslant \mathrm{P}\big(T_i^{(N)} \leqslant L\ \forall i=1,\ldots,L\big) = \mathrm{P}\big(T_1^{(N)} \leqslant L\big)^k = \Big(\frac{R_L}{R_N}\Big)^k.$$

Then it follows by (2.12) and (2.13) that

$$\mathbb{E}\big[Z_L^{\beta_N}(0)^2\big] = \overline{U}_{\beta_N}(L) = \sum_{k\geqslant 0}\Big(1-\tfrac{|\vartheta_N|}{\log N}\Big)^k\,\mathrm{P}\big(\tau_k^{(N)} \leqslant L\big) \leqslant \sum_{k\geqslant 0}\Big(1-\tfrac{|\vartheta_N|}{\log N}\Big)^k\Big(\tfrac{R_L}{R_N}\Big)^k\,,$$

which yields the RHS of (2.14) as an upper bound.

To get a lower bound, note that we can write $\mathrm{P}\big(\tau_k^{(N)} \leqslant L\big) = (\frac{R_L}{R_N})^k\,\mathrm{P}\big(\tau_k^{(L)} \leqslant L\big)$ because the law of $T_i^{(N)}$ conditionally on $T_i^{(N)} \leqslant L$ is just the law of $T_i^{(L)}$, see (A.2). By Markov's inequality, we have $\mathrm{P}\big(\tau_k^{(L)} \leqslant L\big) \geqslant 1-\frac{1}{L}\mathrm{E}[\tau_k^{(L)}]$, where

$$\mathrm{E}[\tau_k^{(L)}] = k\,\mathrm{E}[\tau_1^{(L)}] = \frac{k}{R_L}\sum_{n=1}^{L} n\,q_{2n}(0) \leqslant c\,\frac{k}{R_L}\,L \quad \text{for some} \quad c < \infty.$$

Using $\sum_{k=0}^{\infty} k\, x^k = \frac{x}{(1-x)^2}$, we obtain

$$\begin{aligned}\mathbb{E}\big[Z_L^{\beta_N}(0)^2\big] &\geqslant \sum_{k\geqslant 0} \Big(1-\tfrac{|\vartheta_N|}{\log N}\Big)^k \Big(\tfrac{R_L}{R_N}\Big)^k \Big(1-c\,\tfrac{k}{R_L}\Big) \\ &= \frac{1}{1-\frac{R_L}{R_N}(1-\frac{|\vartheta_N|}{\log N})}\left(1-\frac{c}{R_L}\frac{\frac{R_L}{R_N}(1-\frac{|\vartheta_N|}{\log N})}{1-\frac{R_L}{R_N}(1-\frac{|\vartheta_N|}{\log N})}\right) \\ &\geqslant \frac{1}{1-\frac{R_L}{R_N}(1-\frac{|\vartheta_N|}{\log N})}\left(1-\frac{c'}{|\vartheta_N|}\right),\end{aligned}$$

where the last inequality is obtained using $(1-\frac{|\vartheta_N|}{\log N}) \leqslant 1$ in the numerator, $\frac{R_L}{R_N} \leqslant 1$ in the denominator, and $R_N \sim \frac{\log N}{\pi}$ by (1.15). This completes the proof of (2.14). □

Proof of Theorem 2.5. We are going to exploit the upper and lower bounds in (2.16).

We recall from (1.24) that $\log \frac{1}{\delta_N^2} = |\vartheta_N|$, hence by (1.15),

$$\frac{R_{L_N}}{R_N} = \frac{\log N - \ell \log \frac{1}{\delta_N^2} + O(1)}{\log N + O(1)} = 1 - \ell\,\frac{|\vartheta_N| + O(1)}{\log N}\,.$$

Applying (2.14) then gives

$$\mathbb{E}\big[Z_{L_N}^{\beta_N}(0)^2\big] \sim \frac{1}{1-\frac{R_{L_N}}{R_N}(1-\frac{|\vartheta_N|}{\log N})} \sim \frac{1}{1+\ell}\,\frac{\log N}{|\vartheta_N|}\,, \tag{A.4}$$

and the same holds for $\mathbb{E}\big[Z_{\frac12 L_N}^{\beta_N}(0)^2\big]$ since $\frac12 L_N = N\,(\delta_N^2)^{\ell+o(1)}$, just as L_N.

The difference $R_L(\varphi,\varphi) - R_{\frac12 L}(\varphi,\varphi)$ is uniformly bounded by a constant, see (2.7). Then assumption (2.20) on φ_N implies, since $|\vartheta_N| \to \infty$,

$$R_{\frac12 L_N}(\varphi_N,\varphi_N) = R_{L_N}(\varphi_N,\varphi_N) + O(1) = \frac{1}{\pi}\,(w-\ell)\,|\vartheta_N| + o(|\vartheta_N|)\,. \tag{A.5}$$

Since $\sigma_{\beta_N}^2 \sim \frac{1}{R_N} \sim \frac{\pi}{\log N}$, see (1.22) and (1.15), the bounds in (2.16) yield (2.21). □

Remark A.2 (Sub-critical regime). *The proof of Theorem 2.5 also applies to the sub-critical regime* (1.26) *if we take* $\vartheta_N \sim -(1-\hat\beta^2)\log N$ *with* $\hat\beta^2 \in (0,1)$. *In this case* $\frac{|\vartheta_N|}{\log N} \to (1-\hat\beta^2) > 0$ *and we must take into account second order terms in* (A.4), *namely*

$$\mathbb{E}\big[Z_{L_N}^{\beta_N}(0)^2\big] \sim \frac{1}{1-(1-\ell\,(1-\hat\beta^2))\,(1-(1-\hat\beta^2))} \sim \frac{1}{(1+\ell\,\hat\beta^2)\,(1-\hat\beta^2)}\,. \tag{A.6}$$

Since $\sigma_{\beta_N}^2 \sim \frac{\hat\beta^2}{R_N} \sim \frac{\pi\,\hat\beta^2}{\log N}$, *the bounds in* (2.16) *yield* (2.22).

Proof of Proposition 2.7. Recalling (2.6), we rewrite condition (2.20) as

$$\sum_{x,y\in\mathbb{Z}^2_{\mathrm{even}}} \varphi_N(x)\,\varphi_N(y)\,R_{L_N}(x-y) = \frac{1}{\pi}\,\log\frac{L_N}{W_N} + o(|\vartheta_N|)\,. \tag{A.7}$$

To obtain our goal (2.23), we simply replace $R_{L_N}(x-y)$ in this sum by $\frac{1}{\pi}\log(1+\frac{L_N}{1+|x-y|^2})$ because their difference is uniformly bounded, see (2.9), and hence their contributions to the sum differ by $O(1) = o(|\vartheta_N|)$, which is negligible for the RHS of (A.7). □

Proof of Proposition 2.8. We need to show that condition (2.24) implies (2.23).

We assume for simplicity of notation that $z_N = 0$. We then rewrite (2.24) as follows:

$$\sum_{|x| \leqslant \sqrt{W_N^+}} \varphi_N(x) = 1 - o(1)\,, \qquad \sup_{x \in \mathbb{Z}^2} \varphi_N(x) \leqslant \frac{1}{W_N^-}\,, \qquad \text{with } \; W_N^{\pm} := W_N \, \mathrm{e}^{\pm t_N}\,, \tag{A.8}$$

where we recall that $0 \leqslant t_N = o(|\vartheta_N|)$. Henceforth

If φ_N satisfies (A.8), then we can get a lower bound on the sum in (2.23) by restricting to the ranges $|x|, |y| \leqslant \sqrt{W_N^+}$, which have probability $1 - o(1)$ by the first relation in (A.8). For such values of x, y we have $|x - y| \leqslant 2\sqrt{W_N^+}$, so the logarithm in the LHS of (2.23) is bounded from below by

$$\log\Big(1 + \tfrac{L_N}{1 + 4\,W_N^+}\Big) \geqslant \log \tfrac{L_N}{5\,W_N^+} = \log \tfrac{L_N}{W_N} - t_N - \log 5 = \log \tfrac{L_N}{W_N} - o(|\vartheta_N|)\,, \tag{A.9}$$

where the first inequality holds for large N because $W_N^+ \geqslant W_N \to \infty$, see (2.19). This yields the RHS of (2.23).

To get an upper bound, we fix $\xi_N \to \infty$ with $\xi_N = o(|\vartheta_N|)$ and we define the scale $V_N := W_N^- \, \mathrm{e}^{-\xi_N} = W_N \, \mathrm{e}^{-t_N - \xi_N}$ which is smaller than W_N^-. For $|x - y| \geqslant \sqrt{V_N}$, we can bound from above the logarithm in (2.23) by $\log(1 + \frac{L_N}{1 + V_N}) \leqslant \log(1 + \frac{L_N}{V_N})$. In case $\frac{L_N}{V_N} \leqslant 1$ this is at most $\log 2 = o(|\vartheta_N|)$, while in case $\frac{L_N}{V_N} > 1$ we obtain the upper bound

$$\log\Big(2\,\tfrac{L_N}{V_N}\Big) = \log \tfrac{L_N}{W_N} + t_N + \xi_N + \log 2 = \log \tfrac{L_N}{W_N} + o(|\vartheta_N|)\,,$$

which agrees with the RHS of (2.23). We are left with showing that the range $|x - y| < \sqrt{V_N}$ gives a negligible contribution of order $o(|\vartheta_N|)$ to the sum in (2.23).

For fixed y, we apply the second relation in (A.8) to estimate the sum over x in (2.23):

$$\begin{aligned} \sum_{x \in B(y, \sqrt{V_N})} \varphi_N(x) \, \log\Big(1 + \tfrac{L_N}{1 + |x-y|^2}\Big) &\leqslant \frac{1}{W_N^-} \sum_{x \in B(y, \sqrt{V_N})} \log\Big(1 + \tfrac{L_N}{1 + |x-y|^2}\Big) \\ &\leqslant C\, \frac{V_N}{W_N^-} \int_{|z| \leqslant 1} \log\Big(1 + \tfrac{L_N}{V_N}\, \tfrac{1}{|z|^2}\Big)\, \mathrm{d}z \end{aligned}$$

for some $C < \infty$, by Riemann sum approximation. Since $\frac{V_N}{W_N^-} = \mathrm{e}^{-\xi_N} = o(1)$ by definition of V_N, it remains to show that the integral is $O(|\vartheta_N|)$. If $\frac{L_N}{V_N} \leqslant 1$ then the integral is at most

$$\int_{|z| \leqslant 1} \log\Big(1 + \tfrac{1}{|z|^2}\Big)\, \mathrm{d}z = O(1) = o(|\vartheta_N|)\,,$$

while if $\frac{L_N}{V_N} > 1$ then, recalling that $V_N = W_N \, \mathrm{e}^{-o(|\vartheta_N|)}$, we can bound the integral by

$$\int_{|z| \leqslant 1} \log\Big(\tfrac{L_N}{V_N}\, \tfrac{2}{|z|^2}\Big)\, \mathrm{d}z = \log \tfrac{L_N}{V_N} + O(1) = \log \tfrac{L_N}{W_N} + o(|\vartheta_N|) = O(|\vartheta_N|)\,,$$

where the last equality holds by (2.18) and (2.19). The proof is completed. □

Appendix B. Auxiliary proofs for high moments bounds

In this appendix, we collect the proofs of some results from Section 5.

B.1. Proof of Lemma 5.1. We can rewrite $\mathbb{D}[\varphi]$, see (1.28), as follows:

$$\mathbb{D}[f] = \frac{1}{2} \sum_{z,w\in\mathbb{Z}^2} |z-w|^2 \, f(z)\, f(w)\,. \tag{B.1}$$

To prove (5.4), we first estimate

$$p_> := \sum_{|z-z_0|>s\sqrt{L/2}} \varphi(z) \leqslant \mathrm{e}^{-\sqrt{2}\,\hat{t}\,s} \sum_{z\in\mathbb{Z}^2} \varphi(z)\,\mathrm{e}^{2\,\hat{t}\,\frac{|z-z_0|}{\sqrt{L}}} \leqslant \mathsf{c}_1\,\mathrm{e}^{-\sqrt{2}\,\hat{t}\,s} \leqslant \frac{1}{4} \qquad \text{for } s := \tfrac{\log(4\mathsf{c}_1)}{\sqrt{2}\,\hat{t}}\,,$$

therefore $p_\leqslant := 1 - p_> = \sum_{|z-z_0|\leqslant s\sqrt{L/2}} \varphi(z) \geqslant \frac{3}{4}$. We now restrict the sums in the definition (2.6) of $R_L(\varphi,\varphi)$ to $|z-z_0| \leqslant s\sqrt{L/2}$, $|w-z_0| \leqslant s\sqrt{L/2}$: defining the probability mass function $\tilde{\varphi}(z) := p_\leqslant^{-1}\,\varphi(z)\,\mathbb{1}_{|z-z_0|\leqslant s\sqrt{L/2}}$ and recalling (2.10), we can write

$$R_L(\varphi,\varphi) \geqslant R_{L/2}(\varphi,\varphi) \geqslant p_\leqslant^2\, R_{L/2}(\tilde{\varphi},\tilde{\varphi}) \geqslant p_\leqslant^2\, c_s \log\Big(1 + \tfrac{L/2}{1+2\mathbb{D}[\tilde{\varphi}]}\Big)\,,$$

where we applied (B.1) and Jensen's inequality, since $x \mapsto \log(1+\frac{L}{x})$ is convex for $x > 0$. Since $\mathbb{D}[\tilde{\varphi}] \leqslant p_\leqslant^{-2}\,\mathbb{D}[\varphi] \leqslant 2\,\mathbb{D}[\varphi]$, the proof of (5.4) is complete.

In order to prove that (1.30) implies (5.3), we first observe that

$$\|\varphi\|_{\ell^2}^2 = \sum_{z\in\mathbb{Z}^2} \varphi(z)^2 \leqslant \|\varphi\|_{\ell^\infty} \sum_{z\in\mathbb{Z}^2} \varphi(z) = \|\varphi\|_{\ell^\infty}\,, \tag{B.2}$$

since φ is a probability mass function. We next apply (5.4), that we rewrite for convenience:

$$R_L(\varphi,\varphi) \geqslant \mathfrak{c}\,\log\Big(1 + \tfrac{L/2}{1+4\,\mathbb{D}[\varphi]}\Big)\,. \tag{B.3}$$

We now assume that (1.30) holds and we distinguish two cases:

- if $\mathbb{D}[\varphi] \leqslant 1$ then $R_L(\varphi,\varphi) \geqslant \mathfrak{c}\log\frac{L}{10}$ by (B.3), while $\log(L\,\|\varphi\|_{\ell^2}^2) \leqslant \log(L)$ by (B.2) and $\|\varphi\|_{\ell^\infty} \leqslant 1$, hence (5.3) is satisfied;
- if $\mathbb{D}[\varphi] > 1$, then $R_L(\varphi,\varphi) \geqslant \mathfrak{c}\log(\frac{1}{10}\,\frac{L}{\mathbb{D}[\varphi]})$ by (B.3) and $\log(L\,\|\varphi\|_{\ell^2}^2) \leqslant \log(\mathsf{c}_1\mathsf{c}_2\,\frac{L}{\mathbb{D}[\varphi]})$ by (B.2) and (1.30), so (5.3) holds again.

The proof is completed. ☐

B.2. Proof of Lemma 5.3. Recalling (B.1), by the (squared) triangle inequality $|z-w|^2 \leqslant 2(|z-z_0|^2 + |w-z_0|^2)$ and $x^2 \leqslant \mathrm{e}^x$ for $x \geqslant 0$, we can bound

$$\mathbb{D}[\varphi] \leqslant \sum_{z\in\mathbb{Z}^2} |z-z_0|^2\,\varphi(z) \leqslant \frac{L}{2\,\hat{t}} \sum_{z\in\mathbb{Z}^2} \mathrm{e}^{\frac{2\,\hat{t}}{L}|z-z_0|^2}\,\varphi(z) \leqslant \frac{\mathsf{c}_1}{2\,\hat{t}}\,L\,.$$

It then suffices to apply (5.4) to prove (5.7) with $\eta = \mathfrak{c}\,\log\Big(1 + \frac{\hat{t}}{2(\hat{t}+2\mathsf{c}_1)}\Big)$. ☐

B.3. Proof of Lemma 5.4. We fix $\beta \geqslant 0$, $L \in \mathbb{N}$ and a probability mass function φ satisfying (5.2) and (5.5). By the first inequality in (2.16), in view of (5.7) and (1.32), we obtain

$$\sigma_\beta^2\,\mathbb{E}\big[Z_{L/2}^\beta(0)^2\big] \leqslant \frac{\mathsf{c}_3}{\eta}\,. \tag{B.4}$$

We are going to obtain a lower bound on $\mathbb{E}\big[Z_{L/2}^\beta(0)^2\big]$ which will yield our goal (5.9).

If $\sigma_\beta^2 R_L < 1$ then (5.9) holds with $\bar\vartheta = 0$. We then assume $\sigma_\beta^2 R_L \geqslant 1$, which lets us write

$$\sigma_\beta^2 \, R_L = \frac{1}{1-\frac{\vartheta}{R_L}} \qquad \text{for a suitable} \quad 0 \leqslant \vartheta = \vartheta(\beta, L) < R_L \,. \tag{B.5}$$

Our goal (5.9) is to show that $\vartheta \leqslant \bar\vartheta$ for some $\bar\vartheta = \bar\vartheta(\hat{t}\,, \mathsf{c}_1, \mathsf{c}_3) \in [0,\infty)$.

Let $\tau_k^{(L)}$ be the random walk with $\mathrm{P}(\tau_1^{(L)} = n) = \frac{1}{R_L}\, q_{2n}(0)\, \mathbb{1}_{\{1\leqslant n\leqslant L\}}$. We can write

$$\begin{aligned}\mathbb{E}\big[Z_{L/2}^\beta(0)^2\big] &= 1 + \sum_{k=1}^\infty (\sigma_\beta^2)^k \sum_{0<n_1<\ldots<n_k\leqslant L/2} \prod_{i=1}^k q_{2(n_i-n_{i-1})}(0) \\ &= 1 + \sum_{k=1}^\infty (\sigma_\beta^2 R_L)^k \, \mathrm{P}(\tau_k^{(L)} \leqslant L/2) \geqslant 1 + \sum_{k=1}^\infty (\sigma_\beta^2 R_L)^k \left(1 - \frac{\mathrm{E}[\tau_k^{(L)}]}{L/2}\right).\end{aligned}$$

Note that $\mathrm{E}[\tau_k^{(L)}] = k\,\mathrm{E}[\tau_1^{(L)}] = \frac{k}{R_L}\sum_{n=1}^L n\, q_{2n}(0) \leqslant \frac{k}{R_L}\,\mathfrak{a}_+\, L$ by (2.4). Restricting to $k \leqslant \lceil \frac{1}{4\mathfrak{a}_+} R_L\rceil$ we then have $\mathrm{E}[\tau_k^{(L)}] \leqslant L/4$. Since $\sigma_\beta^2 R_L \geqslant \mathrm{e}^{\frac{\vartheta}{R_L}}$ by (B.5) and $\frac{1}{1-x} \geqslant \mathrm{e}^x$ for $0 \leqslant x \leqslant 1$, we get

$$\mathbb{E}\big[Z_{L/2}^\beta(0)^2\big] \geqslant \frac{1}{2}\sum_{k=1}^K (\sigma_\beta^2 R_L)^k \geqslant \frac{1}{2}\sum_{k=1}^K \mathrm{e}^{\frac{\vartheta}{R_M}k} \qquad \text{with} \qquad K := \lceil \tfrac{1}{4\mathfrak{a}_+} R_L \rceil \,.$$

Applying Jensen we then obtain

$$\mathbb{E}\big[Z_{L/2}^\beta(0)^2\big] \geqslant \frac{K}{2}\,\mathrm{e}^{\frac{\vartheta}{R_L}\frac{1}{K}\sum_{k=1}^K k} \geqslant \frac{K}{2}\,\mathrm{e}^{\frac{\vartheta}{R_L}\frac{K}{2}} \geqslant \frac{R_L}{8\mathfrak{a}_+}\,\mathrm{e}^{\frac{\vartheta}{8\mathfrak{a}_+}} \geqslant \frac{1}{\sigma_\beta^2}\,\frac{1}{8\mathfrak{a}_+}\,\mathrm{e}^{\frac{\vartheta}{8\mathfrak{a}_+}} \,,$$

where in the last inequality we used $\sigma_\beta^2\, R_L \geqslant 1$. Recalling (B.4), we finally obtain

$$\mathrm{e}^{\frac{\vartheta}{8\mathfrak{a}_+}} \leqslant 8\mathfrak{a}_+\,\frac{\mathsf{c}_3}{\eta}\,, \qquad \text{hence} \qquad \vartheta \leqslant \bar\vartheta := 8\mathfrak{a}_+ \, \log^+ \frac{8\mathfrak{a}_+\,\mathsf{c}_3}{\eta}\,,$$

which completes the proof of (5.9). □

B.4. Proof of Lemma 5.5. Recalling (5.10), we denote by $\tau_k^{(L,\hat\lambda)}$ the random walk with step distribution $\mathrm{P}(\tau_1^{(L,\hat\lambda)} = n) = \frac{1}{R_L^{(\hat\lambda)}}\mathrm{e}^{-\frac{\hat\lambda}{L}n}\, q_{2n}(0)\,\mathbb{1}_{\{1,\ldots,L\}}(n)$. Then we can write

$$\begin{aligned}\mathbb{E}\big[Z_{L/2}^\beta(0)^2\big] &= 1 + \sum_{k=1}^\infty (\sigma_\beta^2)^k \sum_{0<n_1<\ldots<n_k\leqslant L/2} \prod_{i=1}^k q_{2(n_i-n_{i-1})}(0) \\ &\geqslant 1 + \sum_{k=1}^\infty (\sigma_\beta^2)^k \sum_{0<n_1<\ldots<n_k\leqslant L/2} \prod_{i=1}^k \mathrm{e}^{-\frac{\hat\lambda}{L}(n_i-n_{i-1})}\, q_{2(n_i-n_{i-1})}(0) \\ &= 1 + \sum_{k=1}^\infty (\sigma_\beta^2 R_L^{(\hat\lambda)})^k \, \mathrm{P}(\tau_k^{(L,\hat\lambda)} \leqslant L/2) \geqslant 1 + \sum_{k=1}^\infty (\sigma_\beta^2 R_L^{(\hat\lambda)})^k \left(1 - \frac{\mathrm{E}[\tau_k^{(L,\hat\lambda)}]}{L/2}\right)\end{aligned}$$

and note that $\mathrm{E}[\tau_k^{(L,\hat\lambda)}] = k\,\mathrm{E}[\tau_1^{(L,\hat\lambda)}] = \frac{k}{R_L^{(\hat\lambda)}} \sum_{n=1}^L n\,\mathrm{e}^{-\frac{\hat\lambda}{L}n}\, q_{2n}(0) \leqslant \frac{k}{R_L^{(\hat\lambda)}}\, \mathfrak{a}_+ L$ by (2.4). Note that assumption (5.11) ensures that $\sigma_\beta^2 R_L^{(\hat\lambda)} < 1$. By $\sum_{k=1}^\infty k\,x^k = \frac{x}{(1-x)^2}$ we then obtain

$$\mathbb{E}\big[Z_{L/2}^\beta(0)^2\big] \geqslant \frac{1}{1-\sigma_\beta^2 R_L^{(\hat\lambda)}} - \frac{2\mathfrak{a}_+}{R_L^{(\hat\lambda)}}\, \frac{\sigma_\beta^2 R_L^{(\hat\lambda)}}{(1-\sigma_\beta^2 R_L^{(\hat\lambda)})^2} = \frac{1}{1-\sigma_\beta^2 R_L^{(\hat\lambda)}} \bigg(1 - \frac{2\mathfrak{a}_+\,\sigma_\beta^2}{1-\sigma_\beta^2 R_L^{(\hat\lambda)}}\bigg).$$

Note that assumption (5.11) is equivalent to $\frac{2\mathfrak{a}_+\,\sigma_\beta^2}{1-\sigma_\beta^2 R_L^{(\hat\lambda)}} \leqslant \frac{1}{2}$, which proves (5.12). □

B.5. Proof of Lemma 5.6. Since $1-\mathrm{e}^{-x} \geqslant \frac{1}{2}$ for $x \geqslant 1$, recalling (2.4) and bounding $\sum_{n=a}^b \frac{1}{n} \geqslant \log\frac{b}{a}$, for $\hat\lambda \geqslant 1$ we can write

$$R_L^{(\hat\lambda)} := R_L - \sum_{n=1}^L (1-\mathrm{e}^{-\hat\lambda\frac{n}{L}})\, q_{2n}(0) \leqslant R_L - \frac{1}{2}\sum_{n=\lceil\hat\lambda^{-1}L\rceil}^L q_{2n}(0) \leqslant R_L - \frac{\mathfrak{a}_-}{2}\log\frac{L}{\lceil\hat\lambda^{-1}L\rceil}.$$

We finally note that

$$\frac{L}{\lceil\hat\lambda^{-1}L\rceil} \geqslant \frac{L}{\hat\lambda^{-1}L+1} = \hat\lambda\,\frac{L}{L+\hat\lambda} \geqslant \frac{\hat\lambda}{2} \qquad \text{for } L \geqslant \hat\lambda\,,$$

which completes the proof of (5.13). □

B.6. Proof of Theorem 5.7. For the final relation (5.16), we simply note that

$$\forall 0 \leqslant x < 1\,,\ h \geqslant 2: \qquad \sum_{r=1}^\infty x^{\max\{r,\frac{h}{2}\}} \leqslant \Big(\frac{h}{2} + \frac{1}{1-x}\Big)\, x^{\frac{h}{2}} \leqslant \mathrm{e}^h\, \frac{x^{\frac{h}{2}}}{1-x}.$$

It remains to show that (5.15) holds. This bound is already proved in [CCR23], though it is not stated in this form. For this reason, in the next lines we state the needed results from [CCR23, Section 4] and we put them together to deduce (5.15). *The purpose is to provide a roadmap for an interested reader to check that* (5.15) *is a direct consequence of the results in* [CCR23]. We refrain from introducing the notation involved, see [CCR23] for details.

We start from [CCR23, Theorems 4.8 and 4.11]: by equations (4.18)-(4.19) and (4.24)-(4.25) with $\lambda = \hat\lambda/L$, the following inequality holds:

$$\big|\mathbb{E}\big[\big(Z_L^\beta(\varphi,\psi) - \mathbb{E}[Z_L^\beta(\varphi,\psi)]\big)^h\big]\big| \leqslant \Big(\max_{I\neq *} \big\|\hat{\mathsf{q}}_L^{|\varphi|,I}\, \tfrac{1}{\mathcal{W}}\big\|_{\ell^p}\Big)\Big(\max_{J\neq *}\big\|\mathcal{W}\,\bar{\mathsf{q}}_L^{|\psi|,J}\big\|_{\ell^q}\Big)\, \mathrm{e}^{\hat\lambda} \sum_{r=1}^\infty \Xi^{\mathrm{bulk}}(r) \tag{B.6}$$

where

$$\Xi^{\mathrm{bulk}}(r) := \sum_{\substack{I_1,\ldots,I_r \vdash \{1,\ldots,h\} \\ \text{with full support} \\ \text{and } I_i \neq I_{i-1},\ I_i \neq *\ \forall i}} \bigg\{\prod_{i=1}^r \big|\mathbb{E}[\xi_\beta^{I_i}]\big|\bigg\}\, \big(\|\widehat{\mathsf{Q}}_L\|_{\ell^q\to\ell^q}^{\mathcal{W}}\big)^{r-1}\, \big(\| |\widehat{\mathsf{U}}|_{L,\hat\lambda,\beta}\|_{\ell^q\to\ell^q}^{\mathcal{W}}\big)^r. \tag{B.7}$$

The terms in these expressions are defined in [CCR23, Section 4]. In a nutshell, $\mathcal{W}$ is a weight function† and $\hat{\mathsf{q}}_L^{|\varphi|,I}$, $\bar{\mathsf{q}}_L^{|\varphi|,I}$ denote suitable averages of the boundary conditions φ, ψ with respect to the random walk kernel, while $\widehat{\mathsf{Q}}$ and $\widehat{\mathsf{U}}$ denote linear operators acting on

†We choose $\mathcal{W} = \mathcal{W}_t$ with $t = \hat t/\sqrt{L}$ as in [CCR23, Remark 4.12].

functions on $(\mathbb{Z}^2)^h$. *We will next state estimates on each term in* (B.6)-(B.7)*, taken verbatim from* [CCR23]*, which combined together will lead to* (5.15)*.*

We denote by $\mathscr{C}_{\hat t}$, $\overline{\mathscr{C}_{\hat t}}$, $\widehat{\mathscr{C}_{\hat t}}$, $\check{\mathscr{C}_{\hat t}}$ suitable constants. Let us first bound the terms in (B.6):

- by [CCR23, Proposition 4.19], equation (4.44) with $r = 1$ gives

$$\Big(\max_{I\neq *} \big\|\widehat{\mathsf{q}}_L^{|\varphi|,I}\tfrac{1}{\mathcal{W}}\big\|_{\ell^p}\Big) \leqslant 4\,\mathscr{C}_{\hat t}^h\, q\, \Big\|\varphi(\cdot)\,\mathrm{e}^{\frac{\hat t}{\sqrt{L}}|\cdot|}\Big\|_{\ell^1} \Big\|\varphi(\cdot)\,\mathrm{e}^{\frac{\hat t}{\sqrt{L}}|\cdot|}\Big\|_{\ell^p}^{h-1}\,; \tag{B.8}$$

- by [CCR23, Proposition 4.21], equation (4.49) with $w_t(\cdot) = \mathrm{e}^{-\frac{\hat t}{\sqrt{L}}|\cdot|}$ gives

$$\Big(\max_{J\neq *} \big\|\mathcal{W}\,\overline{\mathsf{q}}_L^{|\psi|,J}\big\|_{\ell^q}\Big) \leqslant \overline{\mathscr{C}_{\hat t}}^h\, p\, \Big\|\psi(\cdot)\,\mathrm{e}^{-\frac{\hat t}{\sqrt{L}}|\cdot|}\Big\|_{\ell^\infty} \Big\|\psi(\cdot)\,\mathrm{e}^{-\frac{\hat t}{\sqrt{L}}|\cdot|}\Big\|_{\ell^q}^{h-1}\,. \tag{B.9}$$

Plugging these bounds into (B.6), we obtain the second and third line in (5.15).

We next attach the factors p and q from (B.8) and (B.9) to $\Xi^{\mathrm{bulk}}(r)$. To complete the proof of (5.15), recalling (5.14), it suffices to show that for a suitable constant $\mathtt{K}_h^{(\hat t)} < \infty$

$$(p\,q)\,\Xi^{\mathrm{bulk}}(r) \leqslant \big(p\,q\,\Gamma\big)^{\max\{r,\frac{h}{2}\}} \qquad \text{with} \quad \Gamma = \mathtt{K}_h^{(\hat t)}\,\frac{\sigma_\beta^2}{1-\sigma_\beta^2\,R_L^{(\hat\lambda)}}\,. \tag{B.10}$$

We bound the terms in (B.7) as follows:[†]

- by [CCR23, Proposition 4.23], equation (4.45) gives

$$\|\widehat{\mathsf{Q}}_L\|_{\ell^q\to\ell^q}^{\mathcal{W}} \leqslant h!\,\widehat{\mathscr{C}_{\hat t}}^h\, p\,q\,;$$

- by [CCR23, Proposition 4.24], equation (4.58) gives

$$\||\widehat{\mathsf{U}}|_{L,\hat\lambda,\beta}\|_{\ell^q\to\ell^q}^{\mathcal{W}} \leqslant 1 + \check{\mathscr{C}_{\hat t}}^h\,\frac{\sigma_\beta^2 R_L^{(\hat\lambda)}}{1-\sigma_\beta^2 R_L^{(\hat\lambda)}} \leqslant \frac{\check{\mathscr{C}_{\hat t}}^h}{1-\sigma_\beta^2 R_L^{(\hat\lambda)}}\,;$$

- by [CCR23, Proposition 4.13], there is $C(h) < \infty$ such that

$$\forall I_1,\ldots,I_r \vdash \{1,\ldots,h\} \textit{ with full support}: \qquad \prod_{i=1}^r \big|\mathbb{E}[\xi_\beta^{I_i}]\big| \leqslant C(h)^r\,\big(\sigma_\beta^2\big)^{\max\{r,\frac{h}{2}\}}\,.$$

Overall, applying (B.7) we can bound

$$(p\,q)\,\Xi^{\mathrm{bulk}}(r) \leqslant (p\,q)^r\,\big(h!\,\widehat{\mathscr{C}_{\hat t}}^h\big)^{r-1}\left(\frac{\check{\mathscr{C}_{\hat t}}^h}{1-\sigma_\beta^2 R_L^{(\hat\lambda)}}\right)^r C(h)^r\,\big(\sigma_\beta^2\big)^{\max\{r,\frac{h}{2}\}}\,.$$

We increase the RHS replacing r by $\max\{r,\frac{h}{2}\}$, which shows that (B.10) holds with

$$\mathtt{K}_h^{(\hat t)} := h!\,\widehat{\mathscr{C}_{\hat t}}^h\,\check{\mathscr{C}_{\hat t}}^h\,C(h)\,. \tag{B.11}$$

This completes the proof. □

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[†] Propositions 4.13 and 4.24 in [CCR23] require that $\beta \leqslant \beta_0(h)$ for a suitable $\beta_0(h) > 0$ depending on the disorder distribution (to ensure that $\mathbb{E}[(\mathrm{e}^{\beta\omega-\lambda(\beta)})^p] \leqslant \mathbb{E}[(\mathrm{e}^{\beta\omega-\lambda(\beta)})^2] = \sigma_\beta^2$ for all $3 \leqslant p \leqslant h$). This is why the same requirement appears in Theorem 5.7.

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