

The critical 2d Stochastic Heat Flow

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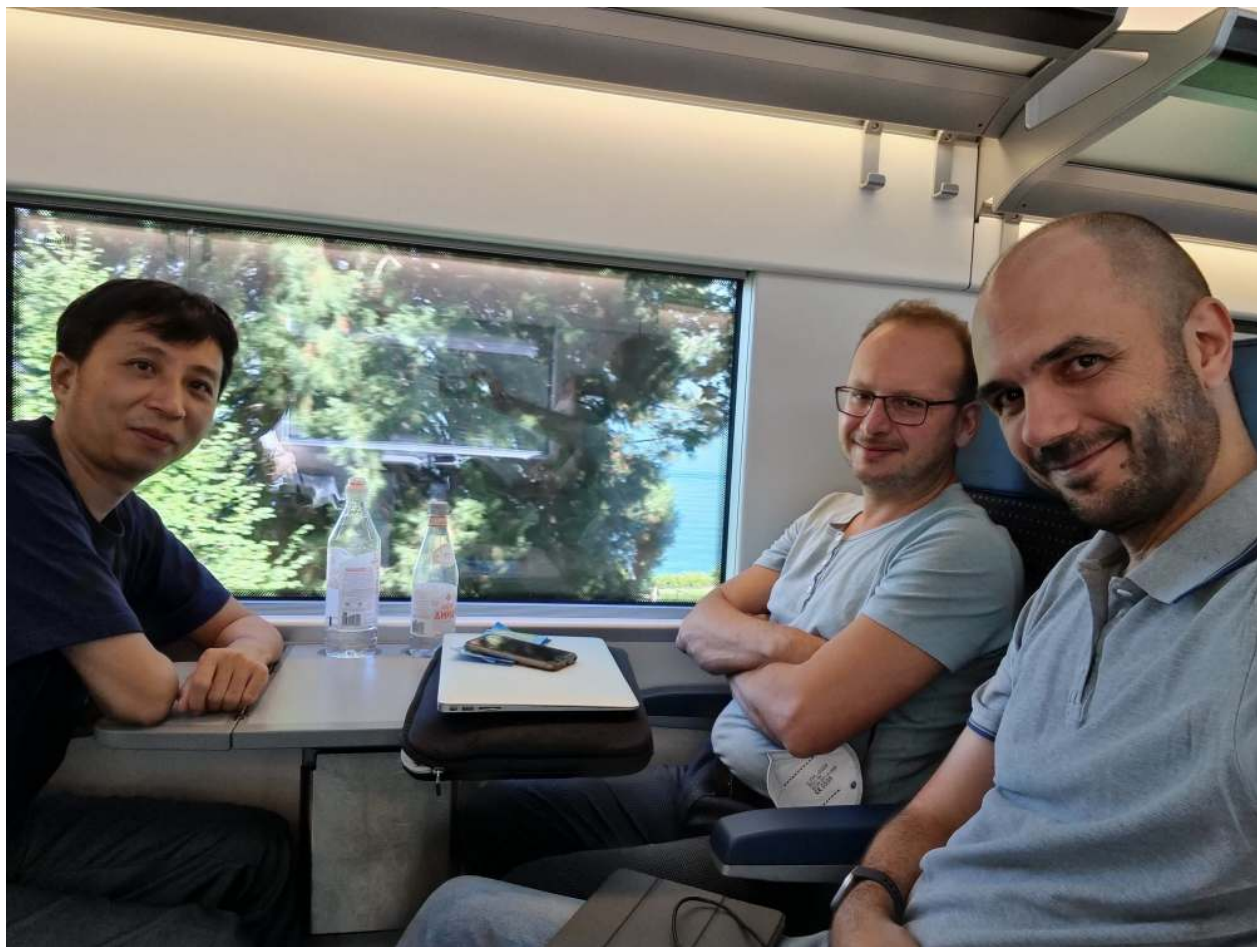
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Spring School on Critical Singular SPDEs

BEIJING INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH

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PLAN OF THE COURSE

LECTURE 1. OVERLOOK AND SUB-CRITICAL REGIME [Caravenna]

Basic tools • Gaussian behavior

LECTURE 2. CRITICAL REGIME I [Sun]

Coarse-graining • Universality

LECTURE 3. CRITICAL REGIME II [Zygouras]

Moment estimates • Key features •
Questions and outlook

REFERENCES

- [CSZ 23a] F. Caravenna, R. Sun, N. Zygouras
THE CRITICAL 2D STOCHASTIC HEAT FLOW
Invent. Math. 233 (2023), 325-460
- [CSZ 23b] Ann. Probab. 51 (2023)
- [CSZ 20] Ann. Probab. 48 (2020)
- [CSZ 19b] Commun. Math. Phys. 372 (2019)
- [CSZ 19a] Electron. J. Probab. 24 (2019)
- [CSZ 17b] Ann. Appl. Probab. 27 (2017)
- [CSZ 17a] J. Eur. Math. Soc. 19 (2017)

INTRODUCTION AND MOTIVATION

THE STOCHASTIC HEAT EQUATION

(MULTIPLICATIVE!)

$$(SHE) \quad \begin{cases} \partial_t U(t, x) = \Delta U(t, x) + \beta \xi(t, x) U(t, x) \\ U(0, x) \equiv 1 \end{cases} \quad t > 0, x \in \mathbb{R}^d$$

- $\beta > 0$ coupling constant
- $\xi(t, x)$ "space-time white noise" (δ -correlated Gaussian)

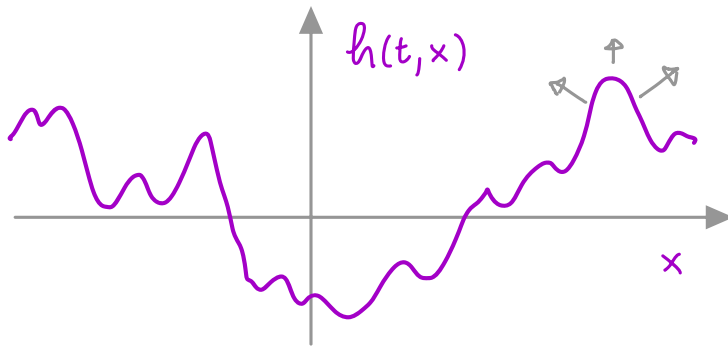
GOAL: Construct the solution $U(t, x)$ for $d=2$

THE KARDAR - PARISI - ZHANG EQUATION

[PRL 1986]

COLE-HOPF TRANSFORMATION: $h(t,x) := \log U(t,x)$ solves (formally)

$$(KPZ) \quad \partial_t h(t,x) = \underbrace{\Delta h(t,x)}_{\text{SMOOTHING}} + \underbrace{|\nabla h(t,x)|^2}_{\perp \text{ GROWTH}} + \underbrace{\beta \xi(t,x)}_{\text{NOISE}}$$



SHE can help us
make sense of KPZ

WHITE NOISE

Formally: $(\xi(t, x))_{t \geq 0, x \in \mathbb{R}^d}$ centered Gaussian

$$\mathbb{E}[\xi(t, x) \xi(s, y)] = \delta(t-s) \delta(x-y) \quad \text{"i.i.d. in space-time"}$$

Rigorously: $(\langle \xi, \varphi \rangle = \int \varphi(t, x) \xi(t, x) dt dx)_{\varphi \in C_c^\infty}$ centred Gaussian

$$\mathbb{E}[\langle \xi, \varphi \rangle \langle \xi, \psi \rangle] = \langle \varphi, \psi \rangle = \int \varphi(t, x) \psi(t, x) dt dx$$

$\varphi \mapsto \langle \xi, \varphi \rangle$ random distribution (not a function, nor a measure)

($d=0$)

$$\xi(t) = \frac{d}{dt} B(t)$$

"derivative of BM"

SINGULARITY

(SHE) and (KPZ) are ill-defined due to singular products

$$\xi(t, x) \quad U(t, x)$$

$$|\nabla h(t, x)|^2$$

$\xi(t, x)$ is a distribution $\rightsquigarrow$ $U(t, x)$ and $h(t, x)$ expected to be:

- non-smooth functions ($d=1$)
- genuine distributions ($d \geq 2$)

We mainly focus on (SHE)

THE ROLE OF DIMENSION

Space-time blow up: $\tilde{U}(t,x) := U(\varepsilon^2 t, \varepsilon x)$ solves

$$\partial_t \tilde{U}(t,x) = \Delta \tilde{U}(t,x) + \beta \varepsilon^{\frac{2-d}{2}} \tilde{\xi}(t,x) \tilde{U}(t,x)$$

As $\varepsilon \downarrow 0$ the noise formally $\left\{ \begin{array}{ll} \text{vanishes} & (d < 2) \\ \text{stays constant} & (d = 2) \\ \text{diverges} & (d > 2) \end{array} \right.$

$d=2$ is CRITICAL DIMENSION for SHE

DIMENSION $d=1$

(1980s) $u(t,x)$ well-posed by stochastic integration (Ito-Walsh)

(2010s) Robust solution theories for "sub-critical" singular PDEs

- REGULARITY STRUCTURES [Hairer]
- PARACONTROLLED CALCULUS [Gubinelli, Imkeller, Perkowski]
- ENERGY SOLUTIONS [Goncalves, Jara] • RENORMALIZATION [Kupiainen]

DIMENSIONS $d \geq 3$

(2015- ...) Results by several authors

Magnen, Unterberger, Chatterjee, Dunlap, Gu, Ryzhik, Zeitouni, Comets, Cosco, Mukherjee, Lygkonis, Zygouras, Nakajima, Nakashima, Junk, ...

DISCRETIZED WHITE NOISE

alternative to mollification or
Fourier cutoff (Fabio & Giuseppe)

Discrete approximation via i.i.d. random variables

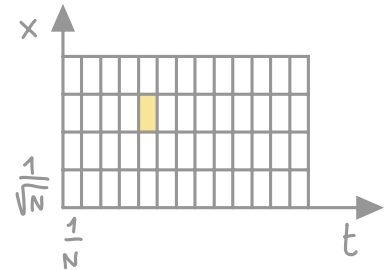
$$(X(k, z))_{k \in \mathbb{N}, z \in \mathbb{Z}^d}$$

$$\mathbb{E}[X] = 0 \quad \mathbb{E}[X^2] = 1$$

$$\text{For } (t, x) \sim \left(\frac{k}{N}, \frac{z}{\sqrt{N}}\right) \in \frac{\mathbb{N}}{N} \times \frac{\mathbb{Z}^d}{\sqrt{N}}$$

$$\left(\frac{k}{N}, \frac{z}{\sqrt{N}}\right) \leq (t, x) \leq \left(\frac{k+1}{N}, \frac{z+1}{\sqrt{N}}\right)$$

$$\xi_N(t, x) := N^{\frac{1}{2} + \frac{d}{4}} \frac{X(k, z)}{(\text{vol})^{-\frac{1}{2}}} \xrightarrow{N \rightarrow \infty} \xi(t, x)$$



$$\underline{E_x.} \quad \int \xi_N(t, x) \varphi(t, x) dt dx \xrightarrow{d} N(0, \|\varphi\|_{L^2}^2)$$

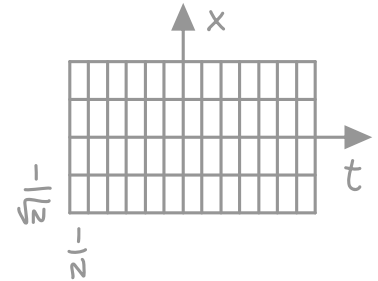
$\varphi = \mathbb{1}_A$ A rectangle

REGULARIZING SHE VIA DISCRETIZATION

We fix $d=2$ and restrict to the lattice

$$(t, x) = \left(\frac{k}{N}, \frac{z}{\sqrt{N}} \right) \in \frac{\mathbb{N}}{N} \times \frac{\mathbb{Z}^2}{\sqrt{N}}$$

Discretized SHE



$$\partial_t^N U_N(t, x) = \Delta^N U_N(t, x) + \beta \cdot N \cdot X(k+1, z) \langle U_N(t, x) \rangle$$

DISCR. DERIVATIVE

DISCR. LAPLACIAN

I.I.D. RVs

$$N \left\{ U(t + \frac{1}{N}, x) - U(t, x) \right\}$$

$$\frac{N}{4} \sum_{x' \sim x} \{ U(t, x') - U(t, x) \}$$

$$\mathbb{E}[X] = 0 \quad \mathbb{E}[X^2] = 1$$

$$\frac{1}{4} \sum_{x' \sim x} U(t, x')$$

Well-defined solution $U_N(t, x) \geq 0$

(if $\beta X \geq -1$)

$$[U_N(0, \cdot) \equiv 1]$$

CONVERGENCE ?

Can we hope that $U_N(t, x) \xrightarrow[N \rightarrow \infty]{} \mathcal{U}(t, x)$? (non-trivial limit)

YES ! But...

① Convergence as (random) distributions :

$$\int_{\mathbb{R}^2} \varphi(x) U_N(t, x) dx \xrightarrow[N \rightarrow \infty]{d} \int_{\mathbb{R}^2} \varphi(x) \underbrace{\mathcal{U}(t, dx)}_{\text{random measure on } \mathbb{R}^2}$$

② Rescale coupling

$$\beta = \beta_N = O\left(\frac{1}{\sqrt{\log N}}\right) \xrightarrow[N \rightarrow \infty]{} 0$$

WHY ?!

Why do we rescale $\beta = \beta_N \sim \frac{\hat{\beta}}{\sqrt{\log N}}$?

$$\mathbb{E} \left[\int \varphi(x) U_N(t, x) dx \right] \equiv \int \varphi(x) dx$$

$$\text{VAR} \left[\int \varphi(x) U_N(t, x) dx \right] \longrightarrow \begin{cases} 0 & \text{if } \hat{\beta} < \sqrt{\pi} \\ \sigma^2(\varphi) & \text{if } \hat{\beta} = \sqrt{\pi} \\ \infty & \text{if } \hat{\beta} > \sqrt{\pi} \end{cases} \quad \text{PHASE TRANSITION}$$

For $\hat{\beta} < \sqrt{\pi}$: $U_N(t, x) dx \xrightarrow{d} dx = \text{Lebesgue measure}$ (" $U_N \rightarrow 1$ ")

For $\hat{\beta} > \sqrt{\pi}$ we conjecture that $U_N(t, x) dx \xrightarrow{d} 0$

For $\hat{\beta} = \sqrt{\pi}$ do we have $U_N(t, x) dx \xrightarrow{d} \mathcal{U}(t, dx)$?

MAIN THEOREM (Lectures 2 + 3)

[CSE 23]

Rescale $\beta \sim \frac{\sqrt{\pi}}{\sqrt{\log N}}$ (critical!)

$$\textcircled{\star} \quad \beta = \frac{\sqrt{\pi}}{\sqrt{\log N}} \left(1 + \frac{g}{\log N} \right) \quad g \in \mathbb{R}$$

Then $U_N(t, x)$ converges to a unique & non-trivial limit:

$$(U_N(t, x) dx)_{t \geq 0} \xrightarrow[N \rightarrow \infty]{\text{f.d.d.}} \mathcal{U}^g = (\mathcal{U}_t^g(dx))_{t \geq 0}$$

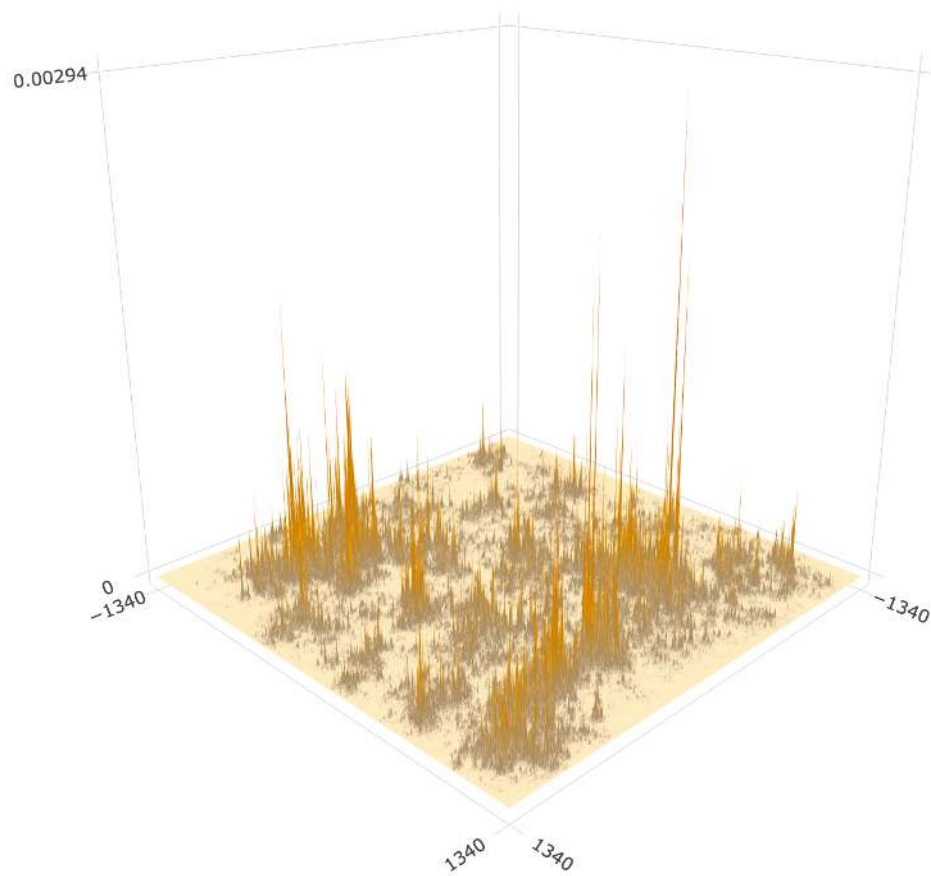
$\mathcal{U}^g$ is a stochastic process of random measures on $\mathbb{R}^2$:

critical 2d STOCHASTIC HEAT FLOW (SHF)

SHF for $N = 5e+04$, $hb2 = 1$ i.e. $\theta = 0$ ($r_0 = 3$, $\text{range} = 0$)

zoom=1.00

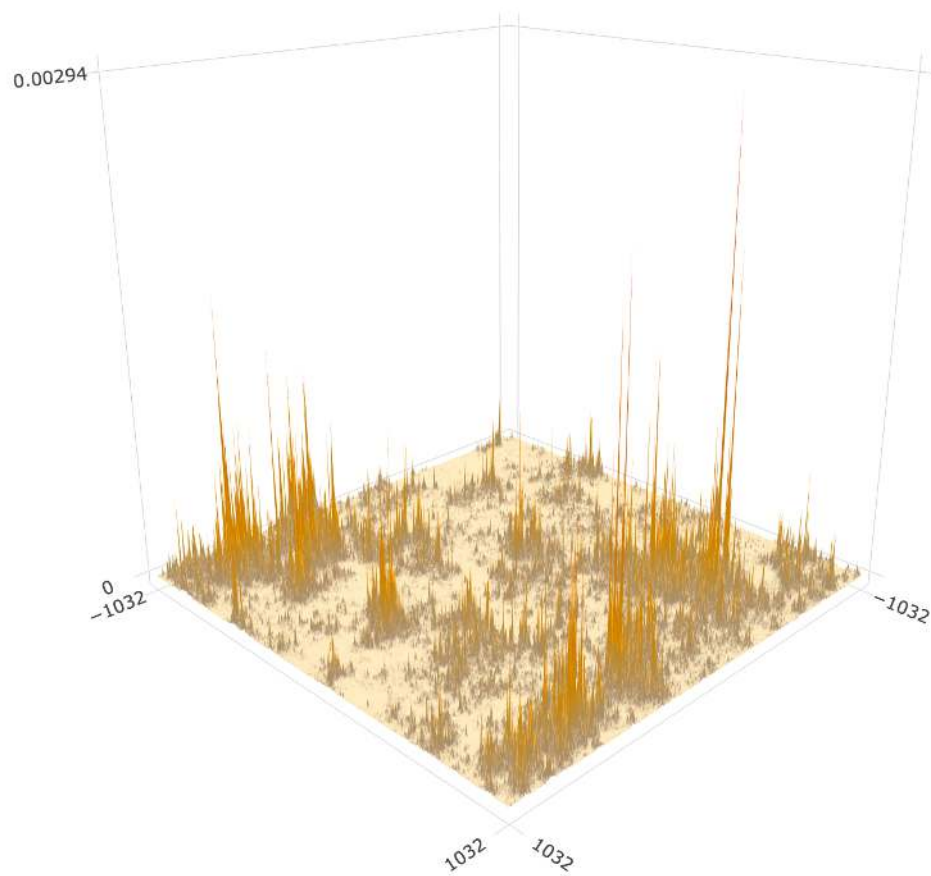
Sum= $1.79e+01$ Max= $2.94e-03$



SHF for $N = 5e+04$, $hb2 = 1$ i.e. $\theta = 0$ ($r0 = 3$, $range = 0$)

zoom=1.30

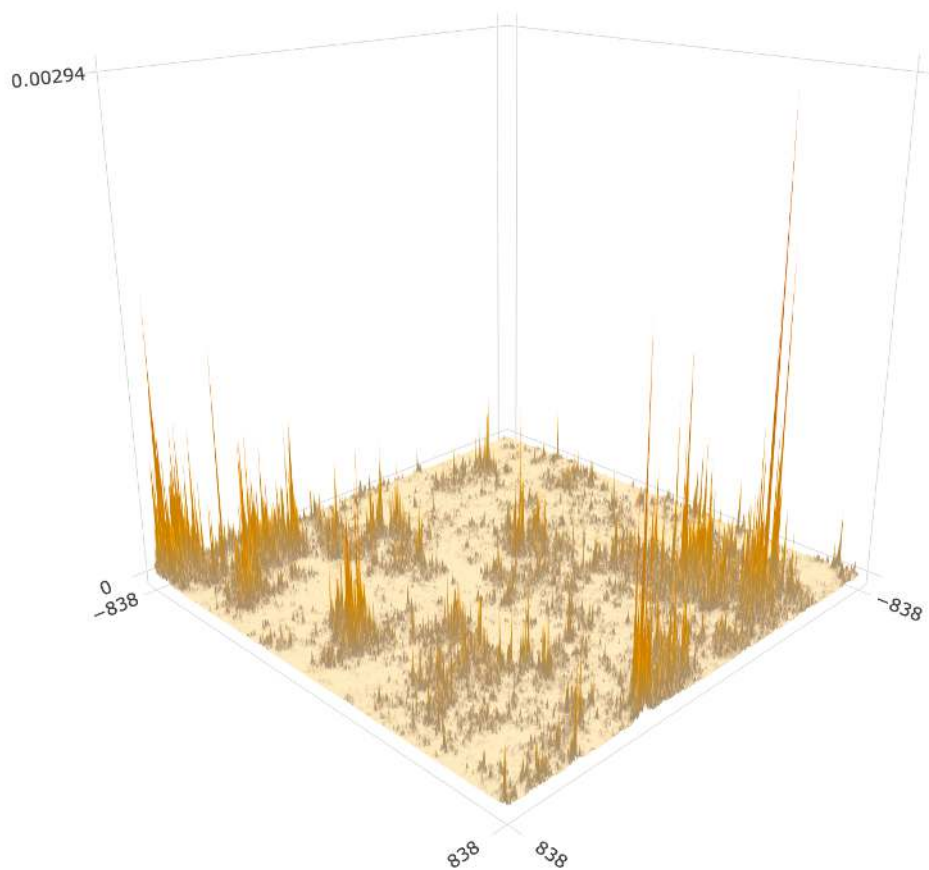
Sum= $1.40e+01$ Max= $2.94e-03$



SHF for $N = 5e+04$, $hb2 = 1$ i.e. $\theta = 0$ ($r0 = 3$, $range = 0$)

zoom=1.60

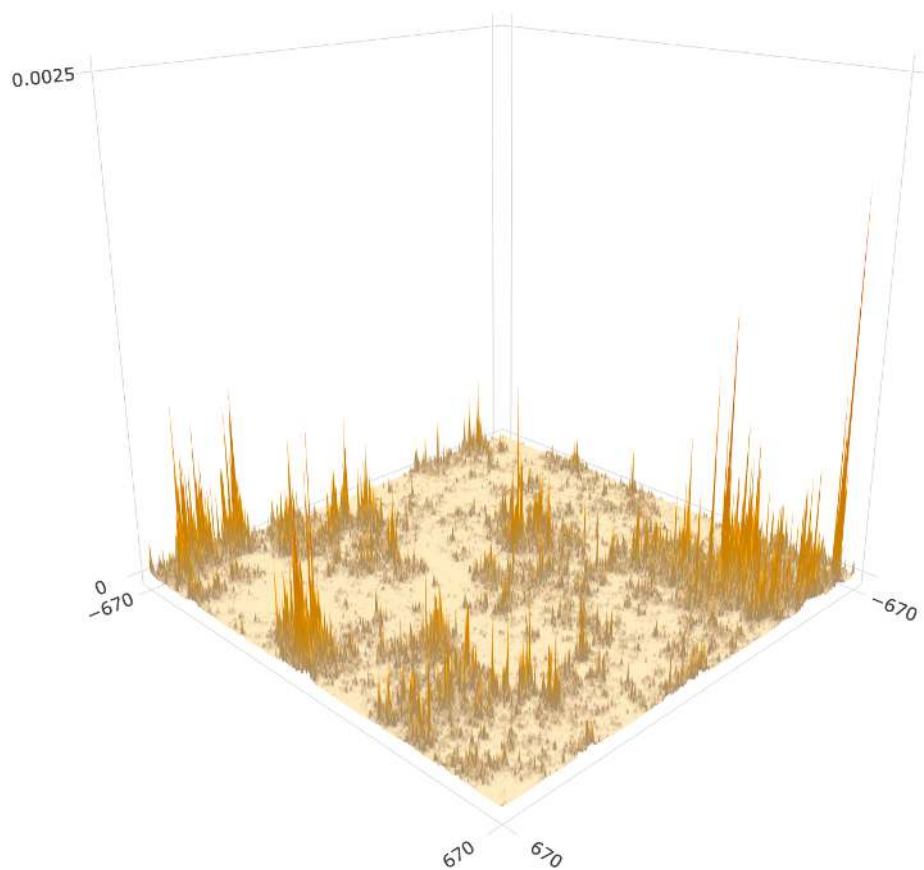
Sum= $9.47e+00$ Max= $2.94e-03$



SHF for $N = 5e+04$, $hb2 = 1$ i.e. $\theta = 0$ ($r0 = 3$, $range = 0$)

zoom=2.00

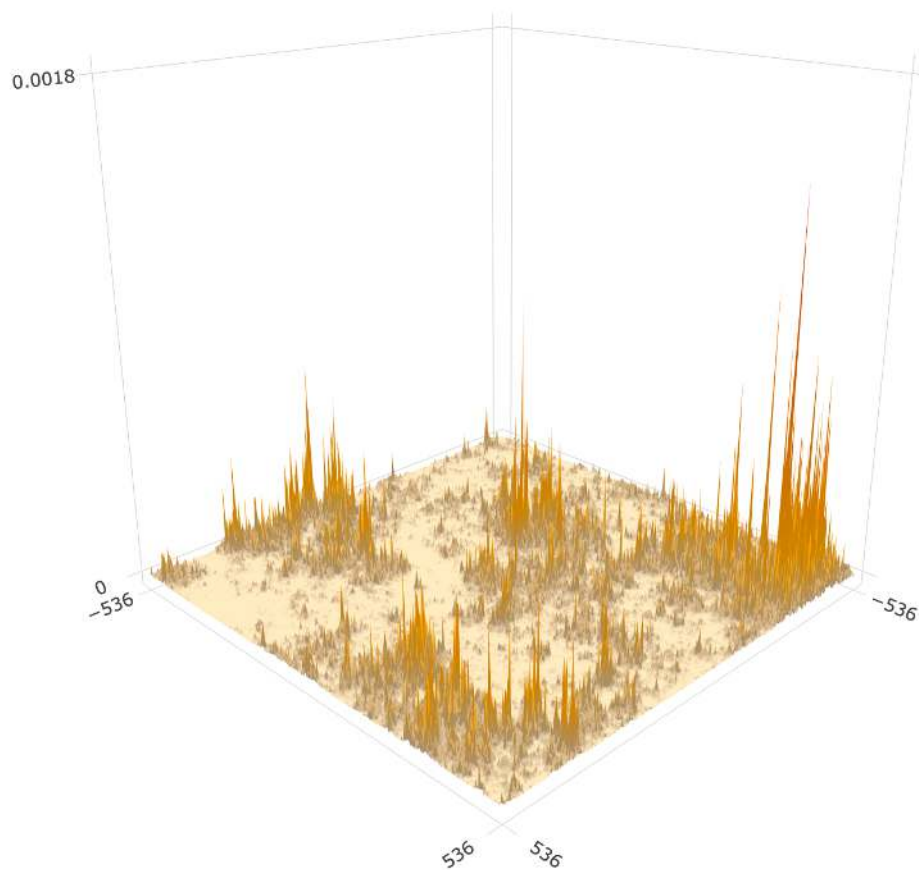
Sum=5.90e+00 Max=2.06e-03



SHF for $N = 5e+04$, $hb2 = 1$ i.e. $\theta = 0$ ($r0 = 3$, $range = 0$)

zoom= 2.50

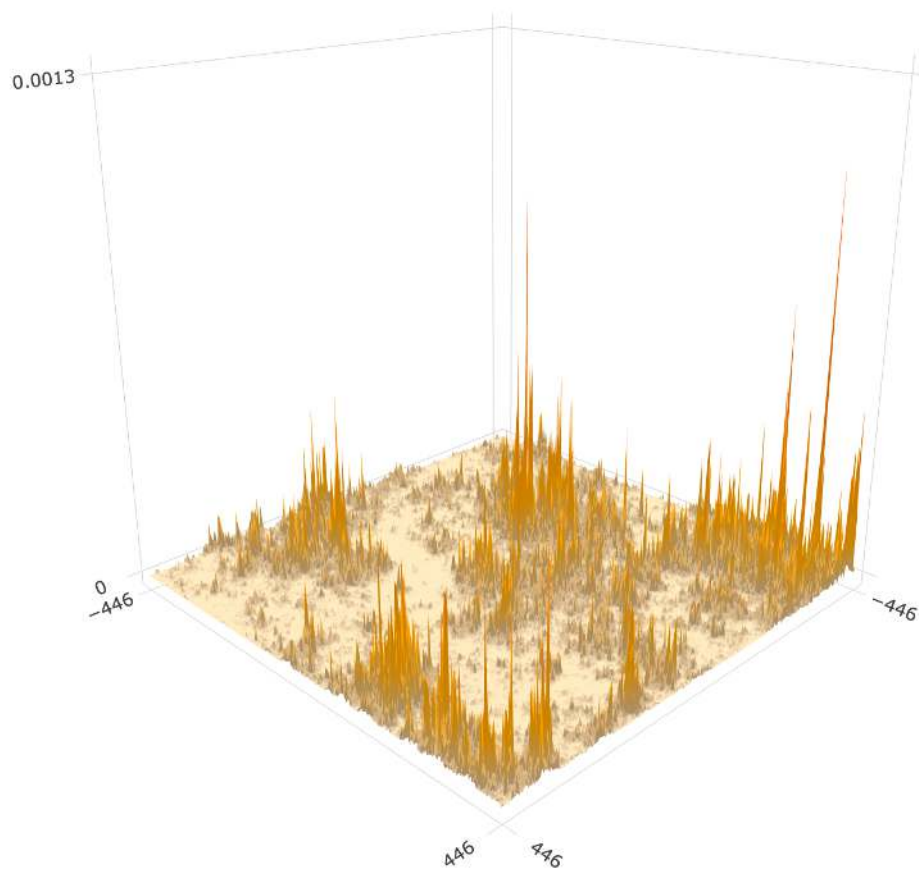
Sum=3.46e+00 Max=1.49e-03



SHF for $N = 5e+04$, $hb2 = 1$ i.e. $\theta = 0$ ($r0 = 3$, $range = 0$)

zoom= 3.00

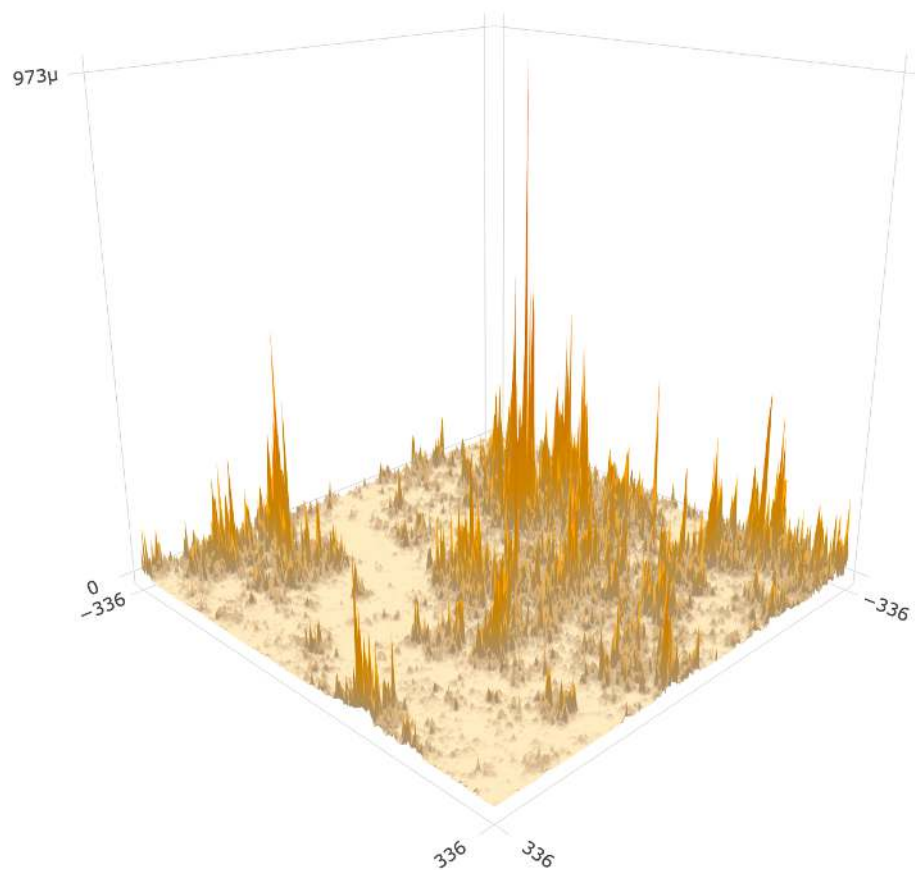
Sum= $2.18e+00$ Max= $1.10e-03$



SHF for N = 5e+04, hb2 = 1 i.e. theta = 0 (r0 = 3, range = 0)

zoom= 4.00

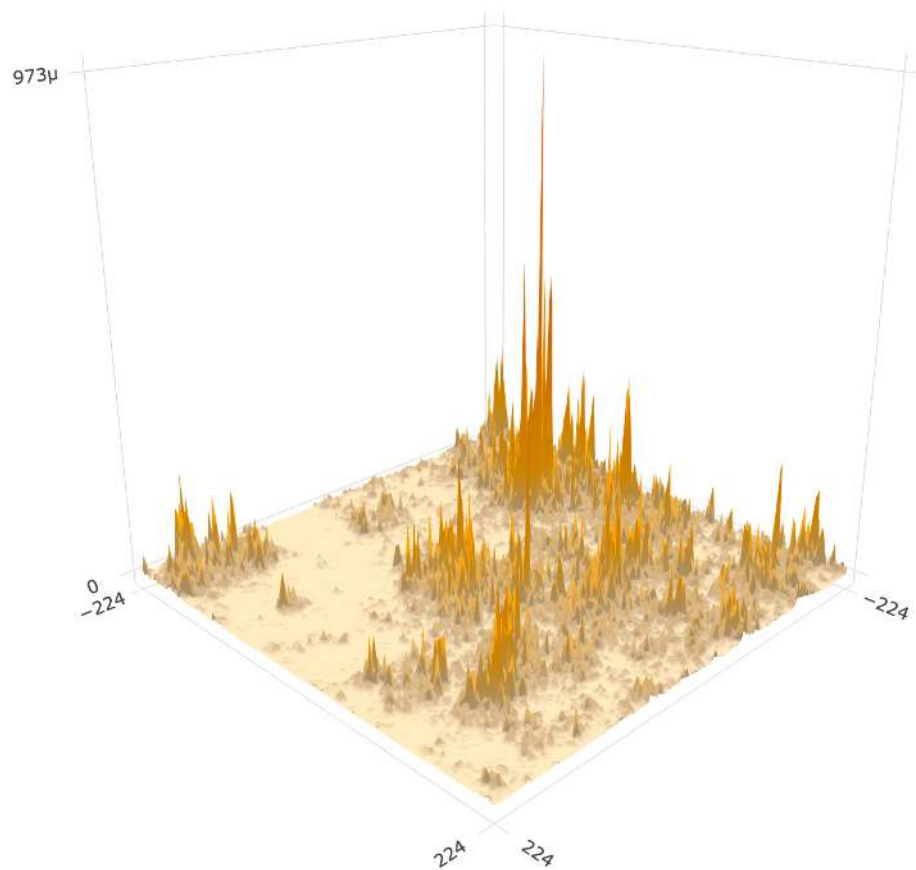
Sum=1.17e+00 Max=9.73e-04



SHF for $N = 5e+04$, $hb2 = 1$ i.e. $\theta = 0$ ($r0 = 3$, range = 0)

zoom= 6.00

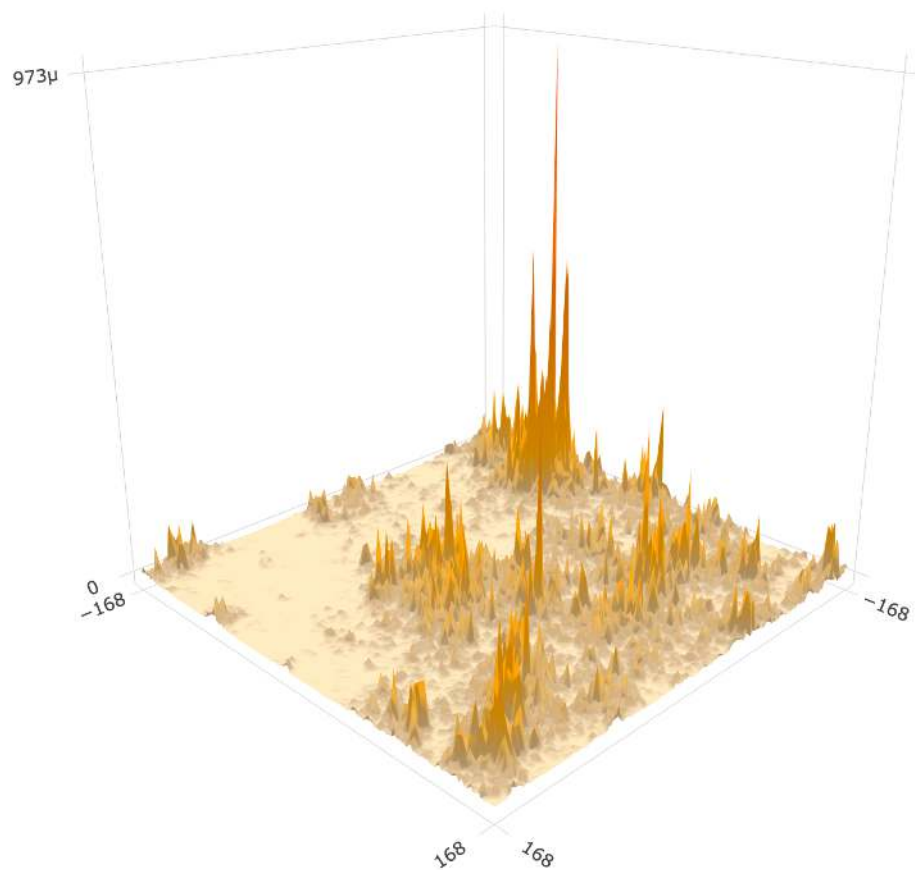
Sum= $5.78e-01$ Max= $9.73e-04$



SHF for N = 5e+04, hb2 = 1 i.e. theta = 0 (r0 = 3, range = 0)

zoom= 8.00

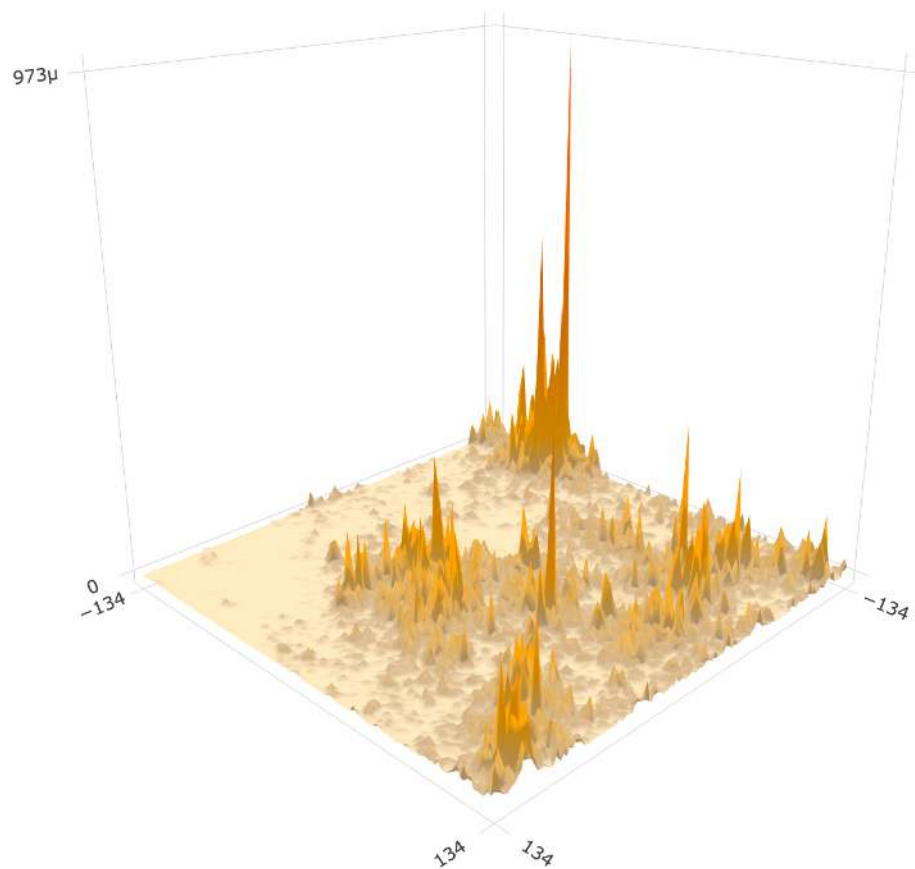
Sum=3.45e-01 Max=9.73e-04



SHF for $N = 5e+04$, $hb2 = 1$ i.e. $\theta = 0$ ($r_0 = 3$, range = 0)

zoom=10.00

Sum= $2.31e-01$ Max= $9.73e-04$



PLAN OF THIS LECTURE

I. DIRECTED POLYMERS & PHASE TRANSITION

Statistical mechanics • Disorder strength

II. TOOLS & PROOFS

Polynomial chaos • Variance computations

III. SUB-CRITICAL RESULTS

One-point distribution • Gaussian fluctuations

I. DIRECTED POLYMERS & PHASE TRANSITION

DIRECTED POLYMER IN RANDOM ENVIRONMENT

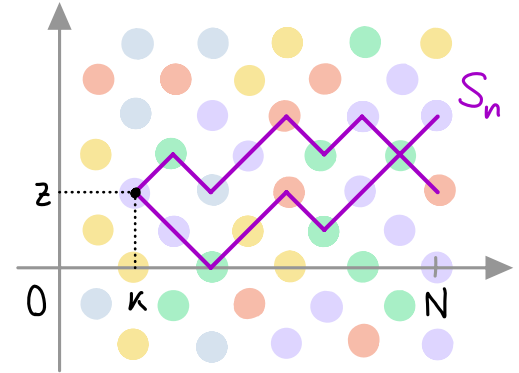
- $(S_n)_{n \geq 0}$ simple random walk on $\mathbb{Z}^2$

$$P\left(S_n - S_{n-1} = \begin{pmatrix} \pm 1, 0 \\ 0, \pm 1 \end{pmatrix}\right) = \frac{1}{4}$$

- $(\omega(n, z))_{n \in \mathbb{N}, z \in \mathbb{Z}^2}$ i.i.d. environment

$$\mathbb{E}[\omega] = 0 \quad \mathbb{E}[\omega^2] = 1 \quad \lambda(\beta) = \log \mathbb{E}[e^{\beta \omega}] < \infty$$

$$\longrightarrow \mathbb{E}[Z_N] = 1$$



PARTITION
FUNCTIONS

$$Z_N^\omega(k, z) := \mathbb{E} \left[e^{\sum_{n=k}^N \beta \omega(n, S_n) - \lambda(\beta)} \mid S_k = z \right]$$

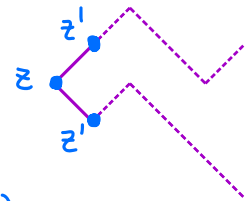
Discrete Feynman-Kac

$$H_N(S, \omega)$$

FEYNMAN-KAC FORMULA

Discretized SHE solution $U_N^X(t, x) = Z_N^\omega(N(1-t), \sqrt{N}x)$

Proof. Markov property of SRW:



$$Z_N^\omega(k, z) = e^{\beta \omega(k, z) - \lambda(\beta)} \frac{1}{4} \sum_{z' \sim z} Z_N^\omega(k+1, z')$$

$$-Z_N(k+1, z) \quad 1 + \{e^{\dots} - 1\}$$

$$-\partial_{k, k+1} Z_N^\omega(k, z) = \Delta_z Z_N^\omega(k+1, z) + \left\{ e^{\beta \omega(k, z) - \lambda(\beta)} - 1 \right\} \bar{Z}_N^\omega(k+1, z)$$

(time-reversed) discretized SHE

i.i.d. $\beta X(k, z) \geq -1$

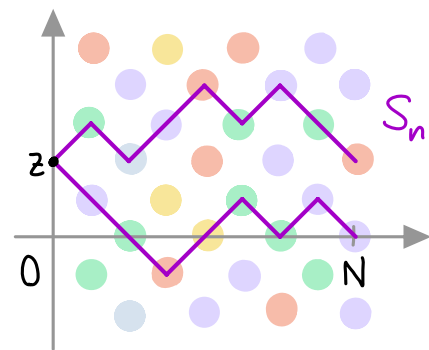
SCALING LIMITS OF PARTITION FUNCTIONS

Henceforth we fix $t=1$ and we focus on partition functions:

$$Z_N^\omega(z) := E \left[e^{\sum_{n=1}^N \beta \omega(n, S_n) - \lambda(\beta)} \mid S_0 = z \right]$$

$H_N(S, \omega)$

Correlated random field over $z \in \mathbb{Z}^2$



Goal: scaling limit of $Z_N^\omega(\lfloor x\sqrt{N} \rfloor) = U_N(1, x)$ for $x \in \mathbb{R}^2$

By construction $E[Z_N^\omega] = 1$. Second moment?

SECOND MOMENT & PHASE TRANSITION

Disorder strength $\beta = \beta_N \sim \frac{\hat{\beta} \sqrt{\pi}}{\sqrt{\log N}}$ $\beta \leftrightarrow \hat{\beta}$

THEOREM 1.

$$\text{VAR} [Z_N(x\sqrt{N})] \begin{cases} \rightarrow \frac{\hat{\beta}^2}{1 - \hat{\beta}^2} < \infty & \hat{\beta} < 1 \\ \sim c \log N \rightarrow \infty & \hat{\beta} = 1 \\ \gg \log N \rightarrow \infty & \hat{\beta} > 1 \end{cases}$$

For $x \neq y \in \mathbb{R}^2$:

$$\text{COV} [Z_N(x\sqrt{N}), Z_N(y\sqrt{N})] \begin{cases} \sim \frac{\hat{\beta}^2}{1 - \hat{\beta}^2} \frac{K(x-y)}{\log N} \rightarrow 0 & \hat{\beta} < 1 \\ \rightarrow K(x-y) < \infty & \hat{\beta} = 1 \\ \rightarrow \infty & \hat{\beta} > 1 \end{cases}$$

$\sim \log \frac{1}{|x-y|^2}$

II. TOOLS & PROOFS

POLYNOMIAL CHAOS EXPANSION

$$\frac{e^{\lambda(2\beta) - 2\lambda(\beta)} - 1}{\beta^2} \sim \frac{e^{\beta^2} - 1}{\beta^2}$$

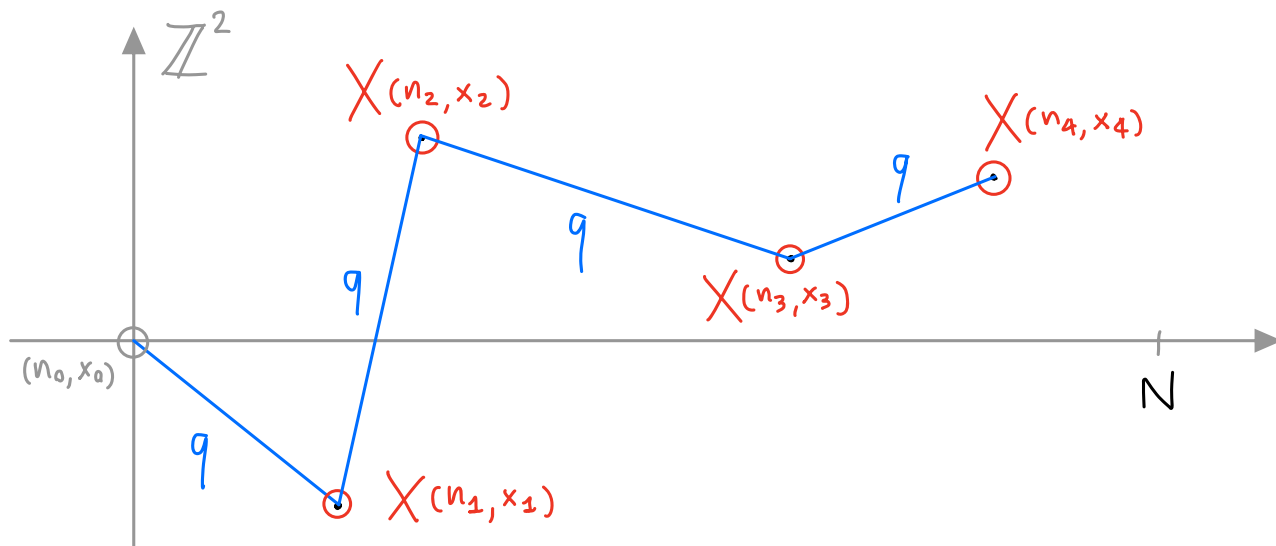
$$\bullet X(n, x) := \frac{e^{\beta \omega(n, x) - \lambda(\beta)} - 1}{\beta} \quad \mathbb{E}[X] = 0 \quad \text{VAR}[X] \sim 1$$

$$\bullet q_n(x) := P(S_n = x) \sim \frac{1}{n} g_{\frac{1}{2}}\left(\frac{x}{\sqrt{n}}\right) \cdot 2 \mathbb{1}_{\{(n, x) \in \mathbb{Z}_{\text{EVEN}}^3\}}$$

$$g_t(z) := \frac{1}{2\pi t} e^{-\frac{|z|^2}{2t}} \quad \text{heat Kernel on } \mathbb{R}^2$$

$$Z_N(0) = 1 + \sum_{k=1}^N \beta^k \sum_{\substack{n_0=0 < n_1 < \dots < n_k \leq N \\ x_0=0, x_1, \dots, x_k \in \mathbb{Z}^2}} \left(\prod_{i=1}^k q_{\substack{(x_i - x_{i-1}) \\ n_i - n_{i-1}}} \right) \prod_{i=1}^k X(n_i, x_i)$$

$$\begin{aligned}
 Z_N(0) = 1 &+ \beta \sum_{n=1}^N \sum_{z \in \mathbb{Z}^2} q_n(z) X(n, z) \\
 &+ \beta^2 \sum_{0 < n < n' \leq N} \sum_{z, z' \in \mathbb{Z}^2} q_n(z) q_{n'-n}(z'-z) X(n, z) X(n', z') + \dots
 \end{aligned}$$



PROOF

$$Z_N(0) := E \left[e^{\sum_{n=1}^N \beta \omega(n, S_n) - \lambda(\beta)} \right]$$

$$= E \left[\prod_{n=1}^N \prod_{z \in \mathbb{Z}^d} e^{\{\beta \omega(n, z) - \lambda(\beta)\}} \mathbb{1}_{\{S_n = z\}} \right]$$

$$= E \left[\prod_{n=1}^N \prod_{z \in \mathbb{Z}^d} \left\{ 1 + (e^{\beta \omega(n, z) - \lambda(\beta)} - 1) \mathbb{1}_{\{S_n = z\}} \right\} \right]$$

$$= 1 + \sum_{(n, z)} \underbrace{(e^{\beta \omega(n, z) - \lambda(\beta)} - 1)}_{\beta X(n, z)} \underbrace{P(S_n = z)}_{q_n(z)}$$

$$+ \sum_{(n, z) < (n', z')} (e^{\beta \omega(n, z) - \lambda(\beta)} - 1) (e^{\beta \omega(n', z') - \lambda(\beta)} - 1) P(S_n = z, S_{n'} = z')$$

KEY COMPUTATIONS

$$\bullet \sum_{x \in \mathbb{Z}^2} q_n(x)^2 = \sum_{x \in \mathbb{Z}^2} q_n(x) q_n(-x) = q_{2n}(0) \sim \frac{1}{\pi n} \quad (n \rightarrow \infty)$$

$$\bullet R_N := \sum_{n=1}^N \sum_{x \in \mathbb{Z}^2} q_n(x)^2 \sim \sum_{n=1}^N \frac{1}{\pi n} \sim \frac{\log N}{\pi} + O(1)$$

Expected replica overlap

$\longrightarrow \mathcal{L}_N$

$$R_N = \sum_{n=1}^N \sum_{x \in \mathbb{Z}^2} \underbrace{P(S_n = x)}_{P(S_n = x, S'_n = x)} = \sum_{n=1}^N \underbrace{P(S_n = S'_n)}_{\substack{\downarrow \quad \downarrow \\ \text{INDEPENDENT RW's}}} = E \left[\sum_{n=1}^N \mathbb{1}_{\{S_n = S'_n\}} \right]$$

PROOF OF THEOREM 1: VARIANCE COMPUTATION

Polynomial chaos: terms of different degrees ORTHOGONAL IN L^2

$$\begin{aligned}
 \text{VAR}[Z_N] &\sim \sum_{k=1}^N (\beta^2)^k \sum_{\substack{0=n_0 < n_1 < \dots < n_k \leq N \\ x_0=0, x_1, \dots, x_k \in \mathbb{Z}^2}} \prod_{i=1}^k q_{n_i - n_{i-1}}(x_i - x_{i-1})^2 \\
 &= \sum_{k=1}^N (\beta^2)^k \sum_{0=n_0 < n_1 < \dots < n_k \leq N} \prod_{i=1}^k \left(\sum_{x \in \mathbb{Z}^2} q_{n_i - n_{i-1}}(x)^2 \right) \\
 &\sim \sum_{k=1}^N \left(\frac{\beta^2}{\pi} \right)^k \sum_{0=n_0 < n_1 < \dots < n_k \leq N} \prod_{i=1}^k \frac{1}{\pi(n_i - n_{i-1})}
 \end{aligned}$$

We now enlarge the sum, allowing each increment $n_i - n_{i-1}$ to range freely in $\{1, 2, \dots, N\}$ $\longrightarrow$ UPPER BOUND:

$$\begin{aligned} \text{VAR}[Z_N] &\lesssim \sum_{k=1}^N \left(\frac{\beta^2}{\pi} \right)^k \prod_{i=1}^k \left(\sum_{n=1}^N \frac{1}{n} \right) \\ &\sim \sum_{k=1}^N \left(\frac{\beta^2}{\pi} \right)^k (\log N)^k \end{aligned}$$

$\swarrow = n_i - n_{i-1}$

$$\lesssim \sum_{k=1}^N (\hat{\beta}^2)^k \sim \frac{\hat{\beta}^2}{1 - \hat{\beta}^2}$$

$$\beta \sim \frac{\hat{\beta} \sqrt{\pi}}{\sqrt{\log N}}$$

For a matching LOWER BOUND, we truncate the sum to $k \leq K$,
 then we restrict each increment to $1 \leq n_i - n_{i-1} \leq \frac{N}{K}$.

$$\text{VAR}[Z_N] \gtrsim \sum_{k=1}^K \left(\frac{\beta^2}{\pi} \right)^k \prod_{i=1}^k \left(\sum_{n=1}^{N/K} \frac{1}{n} \right)$$

$\rightarrow = n_i - n_{i-1}$

$$\sim \sum_{k=1}^K \left(\frac{\beta^2}{\pi} \right)^k \left(\log \frac{N}{K} \right)^k$$

$\rightarrow \sim \log N$ as $N \rightarrow \infty$

$$\sim \sum_{k=1}^K (\hat{\beta}^2)^k \sim \frac{\hat{\beta}^2}{1 - \hat{\beta}^2} \quad \text{as } K \rightarrow \infty$$

$$\beta \sim \frac{\hat{\beta} \sqrt{\pi}}{\sqrt{\log N}}$$



KEY SUB-CRITICAL OBSERVATIONS

- EXPONENTIAL TIME SCALES N^α (vs. tN)

$$\sum_{n=1}^{N^\alpha} \frac{1}{n} \sim \log N^\alpha = \alpha \log N \Rightarrow \text{VAR}[Z_{N^\alpha}] \simeq \alpha \text{VAR}[Z_N]$$

- LOCALIZED DEPENDENCE ON DISORDER:

$$Z_N(z) \text{ depends on disorder in } [0, N^{1-o(1)}] \times B(z, \sqrt{N^{1-o(1)}})$$

- Crucial for generalizations, e.g. semi-linear SHE:

$$\partial_t u = \Delta u + f(u) \cdot \xi \quad [\text{Dunlap, Gu, 22}] [\text{Dunlap, Graham 23+}]$$

III. SUB-CRITICAL RESULTS

SUB-CRITICAL REGIME

There is a phase transition for $\beta = \frac{\hat{\beta} \sqrt{\pi}}{\sqrt{\log N}}$ with critical value $\hat{\beta}_c = 1$.

We look at the sub-critical regime $\hat{\beta} < 1$.

- One point: $\mathbb{E}[Z_N(x\sqrt{N})] = 1$ $\text{VAR}[Z_N(x\sqrt{N})] \rightarrow \frac{\hat{\beta}^2}{1-\hat{\beta}^2} < \infty$

Convergence in distribution $Z_N \xrightarrow{d} ?$

- Average: $\langle Z_N(\cdot\sqrt{N}), \varphi \rangle = \int Z_N(x\sqrt{N}) \varphi(x) dx \xrightarrow{d} \int \varphi(x) dx = \text{const.}$

$$\text{VAR} \left[\int Z_N(x\sqrt{N}) \varphi(x) dx \right] = \iint \text{COV}[Z_N(x\sqrt{N}), Z_N(y\sqrt{N})] \varphi(x) \varphi(y) dx dy \rightarrow 0$$

Fluctuations of $\langle Z_N(\cdot\sqrt{N}), \varphi \rangle$?

LOG - NORMALITY

Rescale disorder strength $\beta = \frac{\hat{\beta} \sqrt{\pi}}{\sqrt{\log N}}$ with $\hat{\beta} < 1$. Fix $x \in \mathbb{R}^2$.

THEOREM 2.

$$\bullet \quad Z_N(x\sqrt{N}) \xrightarrow[N \rightarrow \infty]{d} e^{\sigma Y - \frac{1}{2} \sigma^2} \quad Y \sim N(0, 1) \quad \sigma^2 = \log \frac{\hat{\beta}^2}{1 - \hat{\beta}^2}$$

• Limiting Y 's are independent (!) for distinct $x \in \mathbb{R}^2$

$$\text{Cov}[\cdot, \cdot] \rightarrow 0$$

• If $\hat{\beta} \geq 1$ then $Z_N(x\sqrt{N}) \xrightarrow{d} 0$

SKETCH OF THE PROOF

[Cosco, Donadini, 24+]

[CSZ 24+]

Basic observation: $\sum_{n=1}^{N^\alpha} \frac{1}{n} \sim \log N^\alpha = \alpha \log N$

For $M \in \mathbb{N}$ define the scales $t_i = N^{\frac{i}{M}}$ for $i=0, 1, \dots, M$

$$1 = t_0 \ll t_1 \ll \dots \ll t_M = N \quad \text{as } N \rightarrow \infty$$

Then
$$Z_N^\omega = \prod_{i=1}^M \frac{Z_{t_i}^\omega}{Z_{t_{i-1}}^\omega} \stackrel{d}{\approx} \prod_{i=1}^M Y_i^\omega = e^{\sum_{i=1}^M \log Y_i^\omega}$$

Finally
$$\log Y_i^\omega = \log \{1 + (Y_i^\omega - 1)\} \simeq (Y_i^\omega - 1) - \frac{1}{2} (Y_i^\omega - 1)^2$$

$$\sum_{i=1}^M \log Y_i^\omega \xrightarrow{d} \mathcal{N}(0, \sigma^2) - \frac{1}{2} \sigma^2$$



EDWARDS - WILKINSON FLUCTUATIONS FOR Z

Rescale disorder strength $\beta = \frac{\hat{\beta} \sqrt{\pi}}{\sqrt{\log N}}$ with $\hat{\beta} < 1$. Fix $\varphi \in C_c(\mathbb{R}^2)$.

THEOREM 3.

$$\int v(x) \varphi(x) dx$$

$$\sqrt{\log N} \left\{ \underbrace{\langle Z_N(\cdot \sqrt{N}), \varphi \rangle}_{\int Z_N(x \sqrt{N}) \varphi(x) dx} - \mathbb{E}[\dots] \right\} \xrightarrow{d} \underbrace{\langle v, \varphi \rangle}_{\iint \varphi(x) \varphi(y) K_{\hat{\beta}}(x-y) dx dy} \sim N\left(0, \frac{\hat{\beta}^2}{1-\hat{\beta}^2} \sigma_{\varphi}^2\right)$$

Limiting field $v(x) = v(1, x)$ solves the Edwards-Wilkinson equation:

$$\partial_t v(t, x) = \Delta v(t, x) + \frac{\hat{\beta}^2}{1-\hat{\beta}^2} \xi(t, x) \quad (\text{ADDITIVE NOISE!})$$

EDWARDS - WILKINSON FLUCTUATIONS FOR $\log Z$

Rescale disorder strength $\beta = \frac{\hat{\beta} \sqrt{\pi}}{\sqrt{\log N}}$ with $\hat{\beta} < 1$. Fix $\varphi \in C_c(\mathbb{R}^2)$.

THEOREM 4.

$$\int v(x) \varphi(x) dx$$

$$\sqrt{\log N} \left\{ \int \log Z_N(x\sqrt{N}) \varphi(x) dx - \mathbb{E}[\dots] \right\} \xrightarrow{d} \langle v, \varphi \rangle \sim \mathcal{N}\left(0, \frac{\hat{\beta}^2}{1-\hat{\beta}^2} \sigma_\varphi^2\right)$$

$$\int \log Z_N(x\sqrt{N}) \varphi(x) dx$$

$$\iint \varphi(x) \varphi(y) K_{\hat{\beta}}(x-y) dx dy$$

Limiting field $v(x) = v(1, x)$ solves the Edwards-Wilkinson equation:

$$\partial_t v(t, x) = \Delta v(t, x) + \frac{\hat{\beta}^2}{1-\hat{\beta}^2} \xi(t, x) \quad (\text{ADDITIVE NOISE!})$$

SOME REFERENCES

- THEOREM 2 (LOG-NORMALITY) [CSZ, AAP 17]
[C., Cottini, EJP 22] [Cosco, Donadini, 23+]
- THEOREM 3 (E-W FLUCTUATIONS FOR Z) [CSZ, AAP 17]
- THEOREM 4 (E-W FLUCTUATIONS FOR $\log Z$) [CSZ, AOP 20]
[Chatterjee, Dunlop, AOP 20] [Gu, SPDE 20]
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[Tao, 23+] [Dunlap, Graham, 23+]

Thanks