Gaussian Limits for Subcritical Chaos

Francesco Caravenna
Università di Milano-Bicocca

CIRM (Marseille) ~ 7 JUNE 2021

Joint work with Francesca Cottini

OVERVIEW

1. CLT FOR POLYNOMIAL OR WIENER CHAOS

2. SUB-CRITICAL DIRECTED POLYMERS IN $d=2$

- LOG-NORMALITY OF Z_N
- EDWARDS-WILKINSON FLUCTUATIONS

} REVISITED

3. SKETCH OF THE PROOFS

REFERENCES

[CC 21] F. Caravenna, F. Cottini (in preparation)
GAUSSIAN LIMITS FOR SUBCRITICAL CHAOS

[CSZ 17] F. Caravenna, R. Sun, N. Zygouras (AAP 2017)
UNIVERSALITY IN MARGINALLY RELEVANT DISORDERED SYSTEMS

[CSZ 20] F. Caravenna, R. Sun, N. Zygouras (AOP 2020)
THE TWO-DIMENSIONAL KPZ EQ. IN THE ENTIRE SUBCRITICAL REGIME

1. CLT FOR POLYNOMIAL OR WIENER CHAOS

- $\mathbb{T}$ countable index set

- $\omega = \omega^N = (\omega_t^N)_{t\in\mathbb{T}}$ array of independent r.v.'s $(N \in \mathbb{N})$

$$\mathbb{E}[\omega_t^N] = 0 \qquad \mathbb{E}[(\omega_t^N)^2] = 1$$

U.I. squares $\qquad \lim_{L\to\infty} \sup_{\substack{N\in\mathbb{N}\\ t\in\mathbb{T}}} \mathbb{E}\left[(\omega_t^N)^2 \mathbb{1}_{\{|\omega_t^N|>L\}}\right] = 0$

- Sequence $(X_N)_{N\in\mathbb{N}}$ of centered polynomial chaos:

$$X_N = \sum_{A\subseteq\mathbb{T}} \underbrace{q_N(A)}_{\text{REAL COEFFICIENTS}} \cdot \prod_{t\in A} \omega_t^N$$

$$= \sum_{k=1}^{\infty} \sum_{\substack{\{t_1,\dots,t_k\}\subseteq\mathbb{T}\\ t_i\neq t_j}} \underbrace{q_N(t_1,\dots,t_k)}_{\text{REAL COEFFICIENTS}} \cdot \omega_{t_1}^N \cdot \ldots \cdot \omega_{t_k}^N$$

$$\rightsquigarrow \qquad \mathbb{E}[X_N] = 0 \qquad \mathbb{E}[X_N^2] = \sum_{A\subseteq\mathbb{T}} q_N(A)^2$$

GOAL: conditions for CLT $\quad X_N \xrightarrow{d} \mathcal{N}(0,\sigma^2)$

Theorem (4th MOMENT) [Nourdin, Peccati, Reinert] [de Jong]

Assume that (X_N) belong to a fixed chaos of order K.

Then $X_N \xrightarrow{d} \mathcal{N}(0, \sigma^2)$ if and only if

$$\mathbb{E}[X_N^2] \to \sigma^2 \qquad \mathbb{E}[X_N^4] \to 3(\sigma^2)^2$$

- Fixed chaos is strong assumption
- 4th moment can be hard combinatorial problem

Theorem: CLT via 2nd moments [CC21]

We have $X_N \to \mathcal{N}(0, \sigma^2)$ if (1) $\mathbb{E}[X_N^2] \to \sigma^2$

(2) SUBCRITICALITY $\lim_{K\to\infty} \lim_{N\to\infty} \sum_{|A|>K} q_N(A)^2 = 0$

(3) SPECTRAL LOCALIZATION $\forall M \in \mathbb{N}$ $\mathbb{T} = \mathbb{B}_1 \sqcup \ldots \sqcup \mathbb{B}_M$

$$\sum_{i=1}^{M} \left\{ \sum_{A \subseteq \mathbb{B}_i} q_N(A)^2 \right\} \to \sigma^2 \quad \& \quad \max_{i=1,\ldots,M} \left\{ \sum_{A \subseteq \mathbb{B}_i} q_N(A)^2 \right\} \to 0$$

as $N \to \infty$, $M \to \infty$

Remark

- We only require **second moment conditions** (easy to check)
- Everything extends to continuum setting: Wiener chaos.

Proof: approximate X_N in L^2 by a sum of independent r.v.'s

$$X_N \simeq \sum_{i=1}^{M} X_{N,i} \quad \text{with} \quad X_{N,i} = \sum_{A \subseteq \mathbb{B}_i} q_N(A) \cdot \prod_{t \in A} \omega_t^N$$

We can take $M = M_N \to \infty$ (slowly) and apply the following:

Theorem (Feller-Lindegerg CLT for triangular arrays)

For $N \in \mathbb{N}$, let $(X_{N,i})_{i=1,\dots,M_N}$ be independent r.v.s with zero mean and finite variance. Assume:

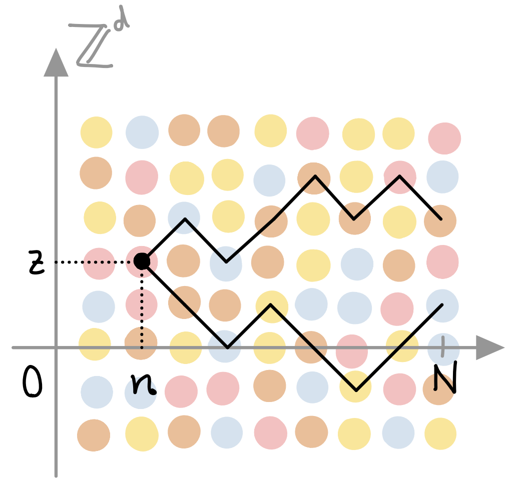
- **VARIANCE CONVERGENCE** $\displaystyle\sum_{i=1}^{M_N} \mathbb{E}[X_{N,i}^2] \to \sigma^2 < \infty$
- **LINDEBERG COND.** $\forall \varepsilon > 0 \quad \displaystyle\sum_{i=1}^{M_N} \mathbb{E}\left[X_{N,i}^2 \, \mathbb{1}_{\{|X_{N,i}| > \varepsilon\}}\right] \to 0$

Then $$X_N := \sum_{i=1}^{M_N} X_{N,i} \xrightarrow{d} N(0, \sigma^2)$$

2. DIRECTED POLYMERS

Two independent ingredients:

- $(S_i)_{i\geq 0}$ simple random walk on $\mathbb{Z}^d$
  POLYMER



- $(\omega(i,y))_{i\geq 0,\, y\in\mathbb{Z}^d}$ i.i.d. $\mathcal{N}(0,1)$
  ENVIRONMENT or DISORDER

Partition functions : $N\in\mathbb{N}$, $\beta\geq 0$

$$\underbrace{Z_N(n,z)}_{\text{STARTING POINT IN } \mathbb{N}\times\mathbb{Z}^d} := \underset{\text{SRW}}{E}\left[e^{\sum_{i=n}^{N}\left\{\beta\,\omega(i,S_i) - \frac{\beta^2}{2}\right\}} \,\middle|\, S_n = z \right]$$

They solve discretized STOCHASTIC HEAT EQUATION :

$$Z_N(n-1,z) - Z_N(n,z) = \frac{1}{2d}\underbrace{\Delta}_{\text{LATTICE LAPLACIAN}} Z_N(n,z) + \beta\, \tilde{Z}_N(n,z)\, \tilde{\omega}(n,z)$$

SLIGHT MODIFICATIONS

$d = 2$: SUB-CRITICAL REGIME

Fix $\beta = \beta_N = \dfrac{\hat\beta\sqrt{\pi}}{\sqrt{\log N}}$, then

$$\lim_{N\to\infty} \mathbb{V}\mathrm{AR}\left[Z_N(0,0)\right] = c^2(\hat\beta) := \begin{cases} \dfrac{\hat\beta^2}{1-\hat\beta^2} & \text{if } \hat\beta < 1 \\ \infty & \text{if } \hat\beta \geq 1 \end{cases}$$

Henceforth we set $d = 2$ and $\hat\beta < 1$.

Theorem: (1) log-normality & (2) E-W fluctuations

$\log \dfrac{1}{1-\hat\beta^2}$

$$(1) \qquad Z_N(0,0) \xrightarrow{d} \begin{cases} e^{\mathcal{N}\left(-\frac{1}{2}\sigma^2(\hat\beta),\ \sigma^2(\hat\beta)\right)} & \text{if } \hat\beta < 1 \\ 0 & \text{if } \hat\beta \geq 1 \end{cases}$$

$$(2) \qquad \sqrt{\frac{\log N}{\pi}} \left\{ Z_N(tN, x\sqrt{N}) - 1 \right\} \xrightarrow{d} v(1-t, x)$$

$$\int \varphi(t,x) \underbrace{\qquad\qquad}\, dt\, dx \xrightarrow{d} \int \varphi(t,x) \underbrace{\quad}\, dt\, dx$$

$$(\text{E-W}) \qquad \partial_t v = \frac{1}{4}\Delta v + c(\hat\beta)\, \dot{W} \qquad (\text{Gaussian})$$

SPACE-TIME WHITE NOISE

Remark:

- log-normality (1) is specific to $d=2$
- E-W fluctuations (2) also hold for $\log Z_N$ (KPZ eq.) and they extend to $d>2$

Purpose: revisit and improve these results (1) & (2) exploiting our general CLT for polynomial chaos

- More elementary proof, strengthened versions
- Goal: try to approach the critical regime

E-W FLUCTUATIONS FOR Z_N

Fix $d=2$ and sub-critical $\beta = \beta_N = \frac{\hat{\beta}\sqrt{\pi}}{\sqrt{\log N}}$ with $\hat{\beta} < 1$

Rescaled field $v_N(1-t,x) := \sqrt{\frac{\log N}{\pi}}\left\{ Z_N(tN, x\sqrt{N}) - 1 \right\}$

Theorem (E-W fluctuations revisited) [CC 21]

v_N satisfies conditions (1), (2), (3) of our CLT

$$\rightsquigarrow \quad v_N \xrightarrow{d} v \quad \text{Gaussian}$$

Why does the limit v solve E-W equation ?

Recall the difference eq. satisfied by $Z_N(n,z)$:

$$\partial_t Z_N = \frac{1}{4}\Delta Z_N + \frac{\hat{\beta}\sqrt{\pi}}{\sqrt{\log N}} Z_N \cdot \tilde{\omega}$$

Then $$\partial_t v_N = \frac{1}{4}\Delta v_N + \hat{\beta}\cdot\Big(1 + \underbrace{\frac{\sqrt{\pi}\, v_N}{\sqrt{\log N}}}_{\to\, 0}\Big)\cdot \underbrace{\left\{N\tilde{\omega}(tN, x\sqrt{N})\right\}}_{\to\, \dot{W}(t,x) \text{ WHITE NOISE}}$$

formally $\downarrow$

$$\partial_t v = \frac{1}{4}\Delta v + \hat{\beta}\cdot \dot{W}$$

WRONG!

What is going wrong?

- It is true that $\{N\tilde{\omega}(\cdot N, \cdot\sqrt{N})\} \xrightarrow{d} \dot{W}$
- It is true that $\dfrac{V_N}{\sqrt{\log N}} \xrightarrow{d} 0$
- But their product $\dfrac{V_N}{\sqrt{\log N}} \cdot N\tilde{\omega}(\cdot N, \cdot\sqrt{N}) \cancel{\xrightarrow{d}} 0$!

Theorem (SINGULAR PRODUCT) [CC21]

$$\underbrace{\frac{\sqrt{\pi}\, V_N}{\sqrt{\log N}} \cdot N\tilde{\omega}(\cdot N, \cdot\sqrt{N})}_{\text{satisfies conditions (1), (2), (3) of our CLT}} \xrightarrow{d} C(\hat{\beta})\, \tilde{\dot{W}}$$

WHITE NOISE INDEP. OF $\dot{W}$

Thus, instead of ⊛, we obtain the correct E-W eq.

$$\partial_t v = \frac{1}{4}\Delta v + \hat{\beta}\left(\dot{W} + \underbrace{C(\hat{\beta})}_{\frac{\hat{\beta}}{\sqrt{1-\hat{\beta}^2}}}\tilde{\dot{W}}\right)$$

$$\stackrel{d}{=} \frac{1}{4}\Delta v + c(\hat{\beta})\,\dot{W}$$

(The same approach can be applied to $\log Z_N \rightsquigarrow$ KPZ)

POLYNOMIAL CHAOS EXPANSION OF Z_N

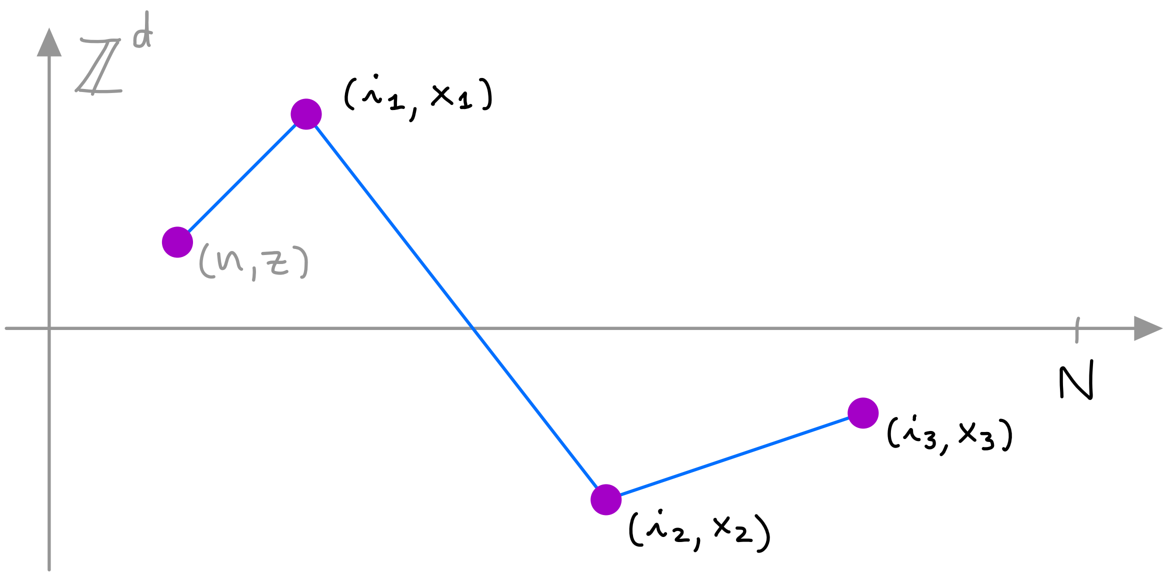
RANDOM WALK TRANSITION KERNEL

MODIFIED DISORDER

$$Z_N(n,z) = 1 + \beta \sum_{(i,x)} q(n,z;i,x)\, \tilde{\omega}(i,x)$$

$$+ \beta^2 \sum_{\substack{(i,x)\\(j,y)}} q(n,z;i,x)\, q(i,x;j,y)\, \tilde{\omega}(i,x)\, \tilde{\omega}(j,y)$$

$$+ \dots$$



Compact notation:

$$Z_N(n,z) = \sum_{A \subseteq \{n+1,\dots,N\} \times \mathbb{Z}^d} q^{\beta}_{(n,z)}(A) \cdot \prod_{(i,x)\in A} \tilde{\omega}(i,x)$$

- Multi-linear polynomial in the $\tilde{\omega}$'s

$$\tilde{\omega}(i,x) := \frac{e^{\beta\omega(i,x) - \frac{\beta^2}{2}} - 1}{\beta} \simeq \omega(i,x) \text{ as } \beta \to 0$$

Random walk transition Kernel:

$$q(n,z\,;\,i,x) := P(S_i = x \mid S_n = z) \simeq \frac{e^{-\frac{|z-x|^2}{n-i}}}{n-i}$$

- Discrete analogue of Wiener chaos

- Ideal for L^2 approximations:

$$\mathbb{E}\left[Z_N(n,z)^2\right] = \sum_{A \subseteq \{n,\dots,N\}\times\mathbb{Z}^d} q^\beta(A)^2$$

" for MOMENT COMPUTATIONS ($\rightsquigarrow$ HYPERCONTRACTIVITY)

" for PROVING CLTs (decoupling RW & disorder)

- NOT ideal for POSITIVITY, CONCENTRATION, FKG, ...

LOG - NORMALITY OF Z_N

Fix $d=2$ and sub-critical $\beta = \beta_N = \frac{\hat\beta\sqrt{\pi}}{\sqrt{\log N}}$ with $\hat\beta < 1$

How to prove a CLT for $\log Z_N(0,0)$?

The original proof in [CSZ17] is subtle and intricate: it proves that $Z_N(0,0) \xrightarrow{d} \exp\{\mathcal{N}\}$

Can we have a polynomial chaos expansion for $\log Z_N$?

Theorem (CHAOS EXPANSION OF $\log Z_N$) [CC21]

$$\lim_{N\to\infty} \left\| \log Z_N(0,0) - \left\{ X_N^{dom} - \frac{1}{2}\mathbb{E}\left[(X_N^{dom})^2\right]\right\}\right\|_{L^2} = 0$$

where

$$X_N^{dom} = \sum_{k=1}^{\infty} \left(\frac{\hat\beta}{\sqrt{\log N}}\right)^k \sum_{\substack{0=n_0<n_1<\dots<n_k\le N \\ \max\{n_i-n_{i-1}:\, i\ge 2\} \le n_1-n_0}} \sum_{\substack{x_1,\dots,x_k\in\mathbb{Z}^2 \\ x_0:=0}} \prod_{i=1}^{k} q(n_{i-1},x_{i-1};n_i,x_i)\cdot \tilde\omega(n_i,x_i)$$

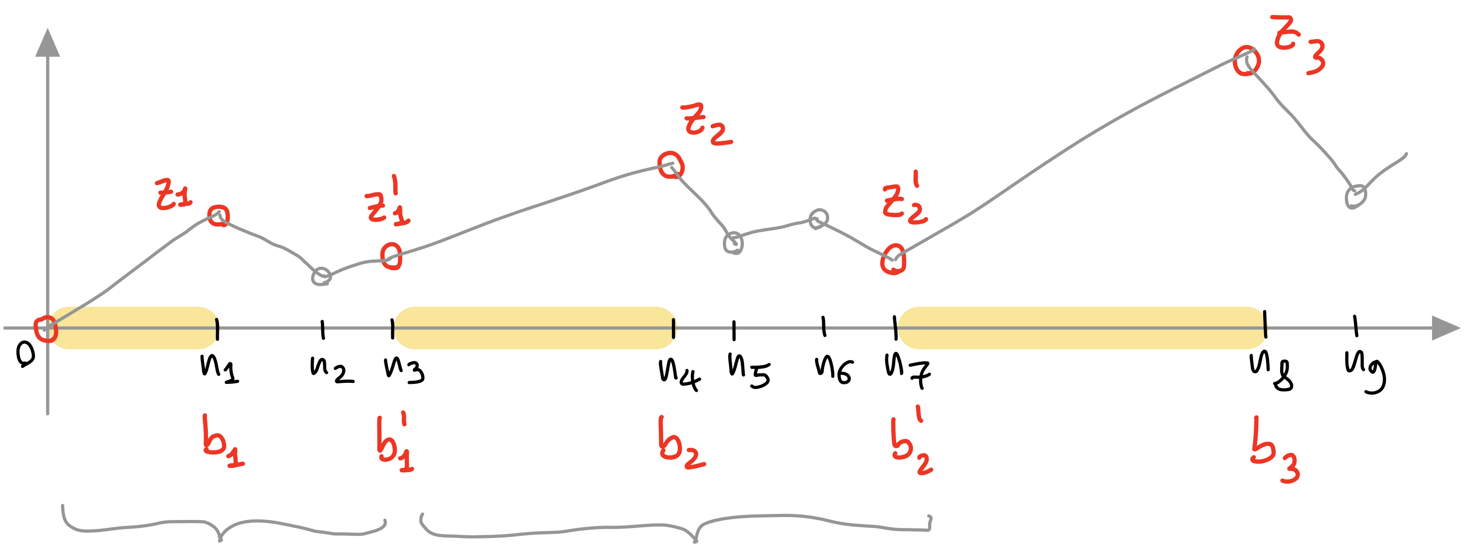
We then obtain the log-normality of Z_N through X_N^{dom}

Theorem (CLT FOR X_N^{dom}) [CC 21]

X_N^{dom} satisfies conditions (1), (2), (3) of our CLT

$$\rightsquigarrow \quad X_N^{dom} \xrightarrow{d} \mathcal{N}(0, \sigma(\hat{\beta})^2) = \mathcal{N}\left(0, \log \frac{1}{1-\hat{\beta}^2}\right)$$

How does X_N^{dom} emerges? RUNNING MAXIMA among $n_i - n_{i-1}$



$$X^{dom}_{N,[0,b_1,b_1']}(0, z_1, z_1') \cdot X^{dom}_{N,[b_1',b_2,b_2']}(z_1', z_2, z_2') \cdot \quad \cdots$$

$$X^{dom}_{N,[0,b_2,b_2']}(0, z_2, z_2') \cdot$$

$$Z_N(0,0) \simeq 1 + \sum_{\ell=1}^{\infty} \sum_{\substack{0<b_1<b_1'<\ldots<b_\ell<b_\ell'\leq N \\ z_1, z_1', \ldots, z_\ell, z_\ell'}} \prod_{i=1}^{\ell} X^{\mathrm{dom}}_{N,[0,b_i,b_i']}(0, z_i, z_i')$$

Assume that we can freely sum over b_i'. Summing also over z_i, z_i' we obtain

$$Z_N(0,0) \simeq 1 + \sum_{\ell=1}^{\infty} \sum_{0<b_1<\ldots<b_\ell\leq N} \prod_{i=1}^{\ell} X^{\mathrm{dom}}_{N,[0,b_i]}$$

$$= \prod_{b=1}^{N} \left(1 + X^{\mathrm{dom}}_{N,[0,b]}\right)$$

Finally, the Taylor expansion $\log(1+x) \simeq x - \frac{x^2}{2}$ gives

$$\log Z_N(0,0) \simeq \sum_{b=1}^{N} X^{\mathrm{dom}}_{N,[0,b]} - \frac{1}{2} \sum_{b=1}^{N} \left(X^{\mathrm{dom}}_{N,[0,b]}\right)^2$$

$$\simeq X_N^{\mathrm{dom}} - \frac{1}{2} \mathbb{E}\left[\left(X_N^{\mathrm{dom}}\right)^2\right]$$

which completes the proof.

CONCLUSIONS

- We presented a CLT for polynomial or Wiener chaos based on 2^{nd} moment assumptions

- Applications to log-normality of Z_N in $d=2$ exploiting a polynomial chaos repr. of $\log Z_N$.

- Applications to Edwards-Wilkinson fluctuations and related singular products.

  These can in principle be applied to $d > 2$

  [Chatterjee, Dunlap] [Dunlap, Gu, Ryzhik, Zeitouni] [Comets, Cosco, Mukerjee] [Magnen, Unterberger] [Lygkonis, Zygouras] [Cosco, Nakajima, Nakashima] ...

- Next challenges:
  - log-normality and E-W fluctuations as $\hat{\beta} \uparrow 1$ (slowly)
  - Robust "pathwise" analysis of SHE/KPZ for $\hat{\beta} < 1$

Thanks!