The critical 2d Stochastic Heat Flow (is not a G.M.C.)

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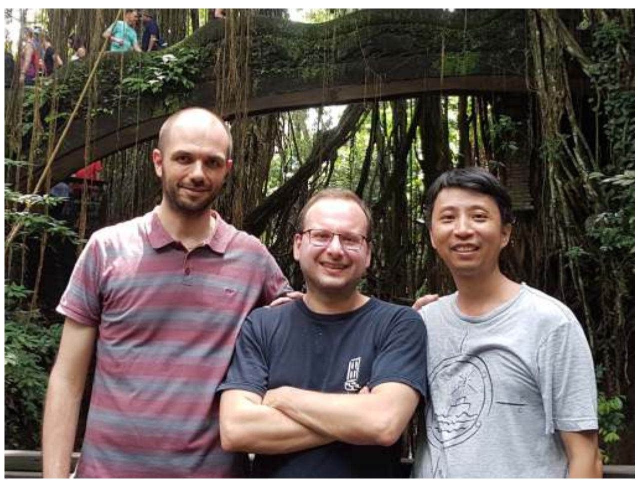


Based on joint works with
Nikos Zygouras and Rongfeng Sun

Overview

I. The critical 2D Stochastic Heat Flow

Motivations, construction, Key features

II. Main results

Gaussian Multiplicative Chaos, moment bounds

III. Ideas from the proof

References

- [CSZ 21] F. Caravenna, R. Sun, N. Zygouras
  THE CRITICAL 2D STOCHASTIC HEAT FLOW
  arXiv (2021)

- [CSZ 22] F. Caravenna, R. Sun, N. Zygouras
  ... IS NOT A GAUSSIAN MULTIPLICATIVE CHAOS
  In preparation

1. WHAT IS THE STOCHASTIC HEAT FLOW ?

It is a "universal" stochastic process of measures on $\mathbb{R}^2$

$$\mathscr{Z}^{\vartheta}_{s,t}(dy, dx) \qquad 0 \leqslant s < t < \infty \qquad x, y \in \mathbb{R}^2 \qquad \vartheta \in \mathbb{R}$$

Heuristics: "natural candidate" solution of the (ill-defined) "critical" 2d Stochastic Heat Equation:

$$\partial_t u(t,x) = \frac{1}{4} \Delta u(t,x) + \beta \underbrace{\xi(t,x)}_{\text{SPACE-TIME WHITE NOISE}} u(t,x) \qquad \text{(SHE)}$$

Fix starting time s, initial condition $u(s,x) = \psi(x)$:

$$\rightsquigarrow \int_{\mathbb{R}^2} \psi(y)\, \mathscr{Z}^{\vartheta}_{s,t}(dy, \cdot) = \text{"natural candidate" for } u(t,\cdot)$$

($u(t,\cdot)$ expected not to be a function, but a measure on $\mathbb{R}^2$)

What do we mean by "natural candidate"? We will see by REGULARIZATION of the eq. & RENORMALIZATION of $\beta \leftrightarrow \vartheta$

Henceforth we fix $s=0$, $\psi \equiv 1$ and call

$$\int_{y \in \mathbb{R}^2} \mathscr{Z}^{\vartheta}_{s,t}(dy, dx) = \boxed{\mathscr{Z}^{\vartheta}_t(dx) = \text{STOCHASTIC HEAT FLOW (SHF)}}$$

2. A LINK WITH DIRECTED POLYMERS

We can REGULARIZE the Stochastic Heat Equation by discretizing time and space (or by mollification ⇝ later)

$$\underbrace{\partial_t^N u(t,x)}_{N\cdot\left\{u(t,x)-u(t-\frac{1}{N},x)\right\}} = \frac{1}{4}\underbrace{\Delta^N}_{\text{SCALED LATTICE LAPLACIAN}} u(t,x) + \beta N \underbrace{\langle \overbrace{\eta(t,x)}^{\text{I.I.D. ZERO MEAN, UNIT VARIANCE}} u(t,x) \rangle}_{\text{SPACE AVERAGE } \frac{1}{4}\sum_{x'\sim x}}$$

For $(t,x) = \left(\frac{n}{N}, \frac{z}{\sqrt{N}}\right) \in \frac{\mathbb{N}}{N} \times \frac{\mathbb{Z}^2}{\sqrt{N}}$

Special choice of noise: $\eta(t,x) = \dfrac{e^{\beta \overbrace{\omega(n,z)}^{\text{I.I.D. } N(0,1)} - \frac{\beta^2}{2}} - 1}{\beta}$

actually $1-t$

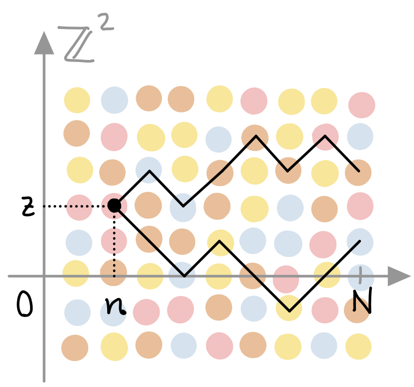
$$u(t,x) = Z_N(n,z) = \mathrm{E}\left[e^{\sum_{i=n+1}^{N} \beta\,\omega(i,S_i) - \frac{\beta^2}{2}} \,\middle|\, S_n = z \right]$$

(S_i) SIMPLE RANDOM WALK ON $\mathbb{Z}^2$

Partition Function of the

DIRECTED POLYMER

IN RANDOM ENVIRONMENT



3. MAIN RESULTS: EXISTENCE OF THE STOCHASTIC HEAT FLOW

The solution of the discretized Stochastic Heat Equation is

$$u(t,x) = Z_N(\underbrace{tN}, \underbrace{x\sqrt{N}}_{\text{INTEGER PARTS}}) \qquad (t,x) = \left(\frac{n}{N}, \frac{z}{\sqrt{N}}\right)$$

A "natural candidate" solution of the original equation (ill-defined) is obtained removing the discretization: do we have

$$\int \varphi(x)\, Z_N(tN, x\sqrt{N})\, dx \xrightarrow[N\to\infty]{d} \;?$$

Not yet! As we remove the discretization, we need to RENORMALIZE the noise strength β (intermediate disorder regime):

$$\beta \sim \hat{\beta} \frac{\sqrt{\pi}}{\sqrt{\log N}} \qquad \text{for fixed } 0 < \hat{\beta} < \infty$$

Indeed, for $x \neq y \in \mathbb{R}^2$

$$\mathbb{E}\left[Z_N(tN, x\sqrt{N}),\, Z_N(tN, y\sqrt{N})\right] \xrightarrow[N\to\infty]{} \begin{cases} 0 & \text{if } \hat{\beta} < 1 \\ K(x,y) \in (0,\infty) & \text{if } \hat{\beta} = 1 \\ \infty & \text{if } \hat{\beta} > 1 \end{cases}$$

$\left(\mathbb{E}\left[Z_N(tN, x\sqrt{N})\right] \equiv 1\right)$

[Bertini-Cancrini '98]

The 2d directed polymer model undergoes a phase transition from weak to strong disorder. (see [CSZ, AAP 2017])

We fix the critical point $\hat{\beta} = 1$, in fact a critical window:

$$\beta = \left(1 + \frac{\vartheta + o(1)}{\log N}\right) \frac{\sqrt{\pi}}{\sqrt{\log N}} \qquad \text{for} \quad -\infty < \vartheta < +\infty \tag{$\star$}$$

Our first main result:

Theorem 1. [CSZ 21]

Fix $\vartheta \in \mathbb{R}$. As $N \to \infty$ we have the convergence in f.d.d. of

$$\left(Z_N(tN, x\sqrt{N})\, dx \right)_{t \in [0,1]}$$

(as random measures on $\mathbb{R}^2$) to a unique limit

$$\left(\mathscr{Z}_t(dx) \right)_{t \in [0,1]}$$

which we call the CRITICAL 2D STOCHASTIC HEAT FLOW.

The SHF encodes asymptotic properties of directed polymers, and it gives a meaning to the solution of the Stochastic Heat Equation.

The rest of the talk is devoted to investigating the SHF.

4. GAUSSIAN MULTIPLICATIVE CHAOS

Let $X = (X(x))_{x\in\mathbb{R}^2}$ be a (generalized) Gaussian field on $\mathbb{R}^2$.
Fix a reference measure $\mu(dx)$ on $\mathbb{R}^2$.

Gaussian Multiplicative Chaos (GMC) is a random measure on $\mathbb{R}^2$:

$$\text{formally} \quad \mathcal{M}(dx) = e^{X(x) - \frac{1}{2}\mathbb{V}\mathrm{AR}[X(x)]} \mu(dx)$$

Often X is specified by (cf. GFF)

$$\mathrm{Cov}[X(x), X(y)] =: K(x,y) \to \infty \quad \text{as } |y-x| \to 0$$

$X(x)$ is not defined pointwise: $X(\varphi) = \text{"}\int \varphi(x)\, X(x)\, dx\text{"} \sim N(0, \sigma_\varphi^2)$

$$\sigma_\varphi^2 = \mathbb{V}\mathrm{AR}[X(\varphi)] = \iint \varphi(x)\, \varphi(y)\, K(x,y)\, dx\, dy < \infty$$

Approximate $K(x,y) = \lim_{\varepsilon\downarrow 0} K_\varepsilon(x,y)$ by smooth covariance functions.

Then $X_\varepsilon \sim N(0, K_\varepsilon)$ is defined pointwise, $X_\varepsilon(x)$, and we set

$$\mathcal{M}(dx) := \lim_{\varepsilon\downarrow 0} e^{X_\varepsilon(x) - \frac{1}{2}\mathbb{V}\mathrm{AR}[X_\varepsilon(x)]} \mu(dx)$$

5. MAIN RESULTS: SHF & GMC

For $\underbrace{\hat\beta < 1}_{\text{SUB-CRITICAL REGIME}}$, it is known that $\log Z_N$ has $\underbrace{\text{Gaussian fluctuations}}_{\text{EDWARDS-WILKINSON}}$:

$$\int \varphi(x) \sqrt{\log N} \left(\log Z_N(tN, x\sqrt{N}) - \mathbb{E}[\log Z_N] \right) dx \xrightarrow[N\to\infty]{d} \text{GAUSSIAN}$$

This can be roughly rephrased as:

$$Z_N(tN, x\sqrt{N}) \approx \exp(\text{GAUSSIAN})$$

Does this picture apply at criticality, i.e. for $\hat\beta = 1$?

We can state our second main result:

Theorem 2. [CSZ 22]

The critical 2d SHF is not a GMC.

This suggests that $\underbrace{\log Z_N(tN, x\sqrt{N})}_{\text{KPZ EQUATION}}$ is not asymptotically Gaussian at criticality $\hat\beta = 1$.

Also: the SHF $\mathscr{Z}_t^\vartheta(dx)$ is a new class of random measures on $\mathbb{R}^2$.

6. MAIN RESULTS: MOMENT BOUNDS

Consider a GMC $\mathcal{M}(dx) = e^{X(x) - \frac{1}{2}\mathbb{V}\mathrm{AR}[X(x)]}\,\mu(dx)$.

COVARIANCE $\kappa(x,y)$

We can compute the moments

$$\mathbb{E}\left[\mathcal{M}(dx)\right] = \mu(dx)$$

$$\mathbb{E}\left[\mathcal{M}(dx)\,\mathcal{M}(dy)\right] = e^{\kappa(x,y)}\,\mu(dx)\,\mu(dy)$$

Henceforth we fix $t>0$, $\vartheta \in \mathbb{R}$. For the SHF $\mathscr{Z}_t^\vartheta(dx)$ we have

$$\mathbb{E}\left[\mathscr{Z}_t^\vartheta(dx)\right] = dx$$

$$\mathbb{E}\left[\mathscr{Z}_t^\vartheta(dx)\,\mathscr{Z}_t^\vartheta(dy)\right] = \underbrace{K(x,y)}\,dx\,dy$$

$$\lim_{N\to\infty} \mathbb{E}\left[Z_N(tN, x\sqrt{N}),\, Z_N(tN, y\sqrt{N})\right] \qquad \sim C \log \frac{1}{|y-x|} \quad \text{as } |y-x| \to 0$$

We can match 1^{st} & 2^{nd} moments of $\mathcal{M}$ and $\mathscr{Z}_t^\vartheta$ by choosing

$$\mu(dx) = dx \qquad \kappa(x,y) = \log K(x,y) \sim \log\log\frac{1}{|y-x|}$$

The GMC $\mathcal{M}$ is now completely determined.

Set $\quad \mathcal{Z}_t^{\vartheta}(\varphi) := \int \varphi(x)\, \mathcal{Z}_t^{\vartheta}(dx), \quad \mathcal{M}(\varphi) := \int \varphi(x)\, \mathcal{M}(dx)$

We can prove that $\mathcal{Z}_t^{\vartheta} \neq \mathcal{M}$ by comparing third moments.

Theorem 3. (3rd moment bound) [CSZ 22]

If φ is the indicator of a ball, or the heat kernel, then

$$\mathbb{E}\left[\mathcal{Z}_t^{\vartheta}(\varphi)^3\right] > \mathbb{E}\left[\mathcal{M}(\varphi)^3\right].$$

For the heat kernel $g_\delta(x) := \dfrac{e^{-\frac{|x|^2}{2\delta}}}{2\pi\delta}$ we bound higher moments.

Theorem 4. (higher moments bound) [CSZ 22]

There is $\eta = \eta_{t,\vartheta} > 0$ such that $\forall h \in \mathbb{N}$

$$\mathbb{E}\left[\mathcal{Z}_t^{\vartheta}(g_\delta)^h\right] \geqslant (1+\eta)\, \mathbb{E}\left[\mathcal{Z}_t^{\vartheta}(g_\delta)^2\right]^{\binom{h}{2}} \qquad \forall \delta \in (0,1),$$

while $$\mathbb{E}\left[\mathcal{M}(g_\delta)^h\right] \sim \mathbb{E}\left[\mathcal{M}(g_\delta)^2\right]^{\binom{h}{2}} \qquad \text{as } \delta \downarrow 0.$$

Moments upper bounds for $\hat\beta < 1$ by [Cosco, Zeitouni], [Lygkonis, Zygouras]

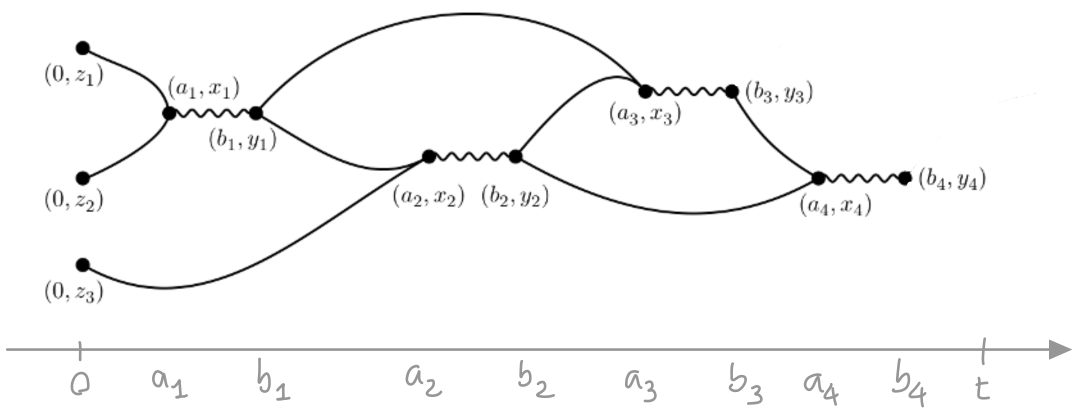
7. IDEAS FROM THE PROOF (1): MOMENT FORMULAS

We prove Theorem 3 exploiting an exact formula for the third moment, derived in [CSZ 19]:

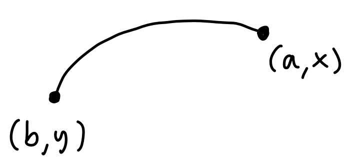
$$\mathbb{E}\left[\mathcal{Z}_t^{\vartheta}(dx)\, \mathcal{Z}_t^{\vartheta}(dy)\, \mathcal{Z}_t^{\vartheta}(dz) \right] = K^{(3)}(x,y,z)\, dx\, dy\, dz$$

$$K^{(3)}(z_1,z_2,z_3) = \sum_{m \geq 2} 3 \cdot 2^{m-1} (2\pi)^m \underset{\substack{0<a_1<b_1<\dots<a_m<b_m<t \\ x_1,\, y_1,\, \dots,\, x_m,\, y_m \in \mathbb{R}^2}}{\int \cdots \int} d\vec{a}\, d\vec{b}\, d\vec{x}\, d\vec{y}\;\; g^{(m)}_{\vec{z}}(\vec{a},\vec{b},\vec{x},\vec{y})$$

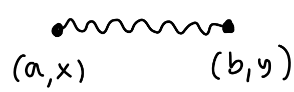
$g^{(4)} =$



where:



$g_{\frac{a-b}{2}}(x-y)$



$g_{\frac{b-a}{4}}(y-x) \cdot G_{\vartheta}(b-a)$

By an explicit Gaussian integration, we can integrate out all space variables $\vec{x}, \vec{y}$ to get the equality

$$\int \cdots \int g_{\vec{z}}^{(m)}(\vec{a}, \vec{b}, \vec{x}, \vec{y}) \, d\vec{x} \, d\vec{y} = \prod_{i=1}^{m} G_\vartheta(b_i - a_i) \cdot$$

$$\cdot \, g_{a_1}(z_1 - z_2) \cdot g_{\bar{a}_2}\left(z_3 - \frac{z_1 + z_2}{2}\right) \cdot \prod_{i=3}^{m} g_{\overline{a_i - b_{i-2}}}(0)$$

for suitable $\bar{a}_2 < a_2$ and $\overline{a_i - b_{i-2}} < a_i - b_{i-2}$.

Exploiting monotonicity of the heat kernel, we replace the quantities $\bar{a}_2$ by a_2 and $\overline{a_i - b_{i-2}}$ by $a_i - b_{i-2}$.

This yields the bound $\mathbb{E}[\mathscr{Z}_t^\vartheta(\varphi)^3] > \mathbb{E}[\mathcal{M}(\varphi)^3]$. $\square$

Remark. Exact formulas are available for higher moments:

$$\mathbb{E}\left[\prod_{i=1}^{h} \mathscr{Z}_t^\vartheta(dz_i)\right] = K^{(h)}(z_1, \ldots, z_h) \, dz_1 \cdots dz_h$$

$$K^{(h)}(z_1, \ldots, z_h) = \sum_m \text{EXPLICIT (COMPLICATED) INTEGRALS ON } (\text{TIME} \times \text{SPACE}^h)^{2m}$$

This was proved by [Gu, Quastel, Tsai, PMP '21]

8. IDEAS FROM THE PROOF (II): CORRELATION INEQUALITY

Recall that

$$Z_N(\cancel{0}, z) = \mathrm{E}\left[e^{\sum_{\ell=1}^{N} \beta\,\omega(\ell, S_\ell) - \frac{\beta^2}{2}} \,\middle|\, S_0 = z \right]$$

Then

$$\boxed{\star} \quad \mathrm{E}\left[\prod_{i=1}^{h} Z_N(z_i) \right] = \mathrm{E}\left[\prod_{1 \le i < j \le h} e^{\beta^2 L_N^{(i,j)}} \,\middle|\, S_0^{(i)} = z_i \ \forall i \right]$$

$$L_N^{(i,j)} = \sum_{\ell=1}^{N} \mathbb{1}_{\{S_\ell^{(i)} = S_\ell^{(j)}\}}$$ OVERLAP OF $S^{(i)}$ AND $S^{(j)}$

Our goal is $\quad \mathrm{E}\left[\mathcal{Z}_t^{\vartheta}(g_\delta)^{h} \right] \geq (1+\eta)\, \mathrm{E}\left[\mathcal{Z}_t^{\vartheta}(g_\delta)^{2} \right]^{\binom{h}{2}}$

Set $\eta = 0$ (for simplicity) and replace $\mathcal{Z}_t^{\vartheta}$ by Z_N.

We need to exchange E and $\prod_{1 \le i < j \le h}$ in $\boxed{\star}$:

are $\exp(\beta^2 L_N^{(i,j)})$ positively correlated? (at least as $N \to \infty$)

We prove that this holds in a continuum setting, replacing

random walks $S^{(i)}$ $\rightsquigarrow$ Brownian motions $B^{(i)}$

Based on the GAUSSIAN CORRELATION INEQUALITY [Royen '14].

CONCLUSIONS

We introduced the CRITICAL 2D STOCHASTIC HEAT FLOW as a scaling limit of directed polymer partition functions

It is a universal process of random measures on $\mathbb{R}^2$, different from GMC, which is a natural candidate for the solution of the critical 2d Stochastic Heat Equation.

It has some explicit (but challenging!) features.
Many interesting questions are open:

- SINGULARITY W.R.T. LEBESGUE MEASURE
- FLOW PROPERTY
- CHARACTERIZING PROPERTIES
- TAKING LOG $\rightsquigarrow$ KPZ

Thanks!