From Disordered Systems to the Critical 2D Stochastic Heat Flow

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A guiding question

How does randomness reshape mathematical structures

and when does it create new universal objects?

We will look at this question through a novel example:

the **Critical 2D Stochastic Heat Flow (SHF)**

The Critical 2D Stochastic Heat Flow (SHF)

1. SHF : Singular Stochastic PDEs

2. SHF : Disordered Systems

3. SHF : Present and Future

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2D Stochastic Heat Equation

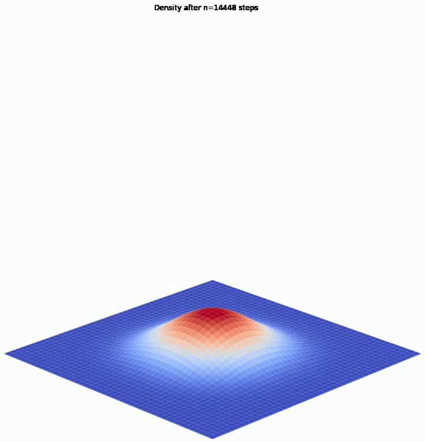
$$\partial_t u(t,x) = \underbrace{\Delta_x}_{\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}} u(t,x) + \beta \underbrace{\xi(t,x)}_{\substack{\text{“space-time white noise”} \\ \text{cov. } \delta(t-t')\,\delta(x-x')}} \cdot u(t,x) \qquad \text{(SHE)}$$

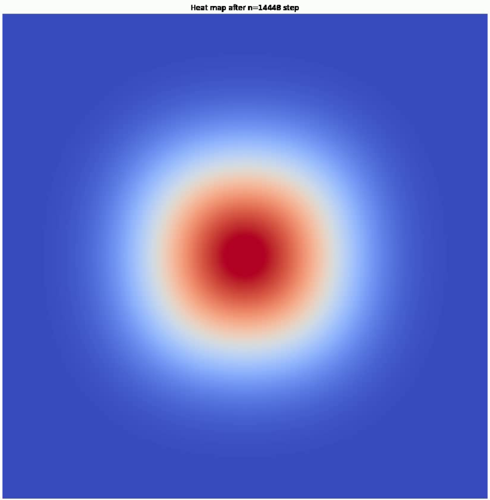
A sigular stochastic PDE (SPDE) in critical dimension $d = 2$ $\rightsquigarrow$ ill-defined

Goal

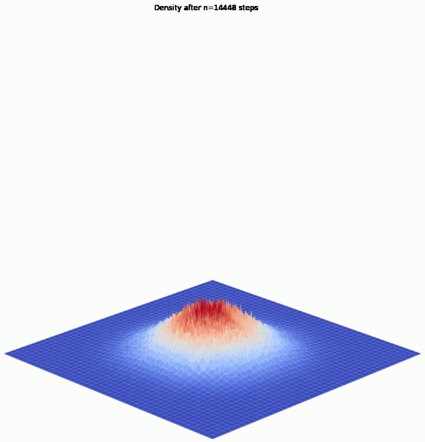
Make sense of solution $u(t,x)$ $\rightsquigarrow$ Critical 2D Stochastic Heat Flow

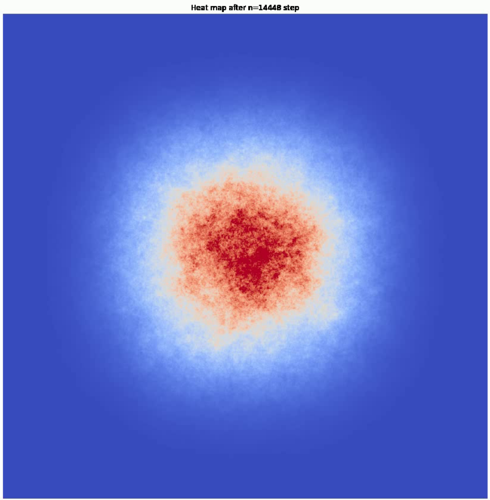
Lattice discretization: $\beta = 0$ (by Q. Berger)





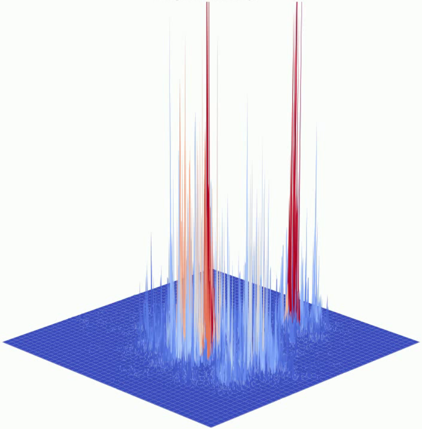
Lattice discretization: β very small (by Q. Berger)



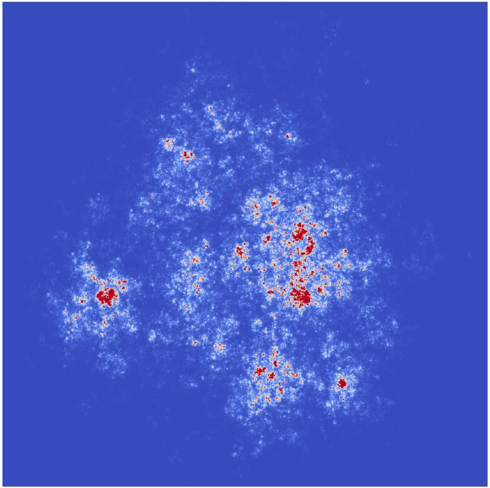


Lattice discretization: β "critical" (by Q. Berger)

Density after n=14448 steps



Heat map after n=14448 step



A link to KPZ universality

Kardar-Parisi-Zhang (KPZ) equation [KPZ 86]

$$\partial_t h(t,x) = \Delta_x h(t,x) + |\nabla_x h(t,x)|^2 + \beta\, \xi(t,x)$$

KPZ $\leftrightsquigarrow$ SHE

$$h(t,x) = \log u(t,x)$$

Tremendous progress in $d = 1$ [J. Quastel talk]

Still open in dimensions $d \geq 2$

The notion of (sub / super) criticality

Space-time zoom-in: $\tilde{u}(t,x) := u(\varepsilon^2 t, \varepsilon x)$

$$\partial_t \tilde{u} = \Delta \tilde{u} + \varepsilon^{(2-d)/2} \beta \tilde{\xi} \cdot \tilde{u}$$

$d < 2$	sub-critical	$d = 2$	critical	$d > 2$	supercritical

Different singular SPDEs have their own critical dimensions

$$\partial_t h = \Delta h + |\nabla h|^2 + \beta \xi \qquad d_c = 2 \qquad \text{(KPZ)}$$

$$\partial_t \Phi = \Delta \Phi \pm m^2 \Phi - \Phi^3 + \beta \xi \qquad d_c = 4 \qquad (\text{Allen-Cahn} \ / \ \Phi^4_d)$$

$$\ldots$$

Breakthroughs in sub-critical singular SPDEs

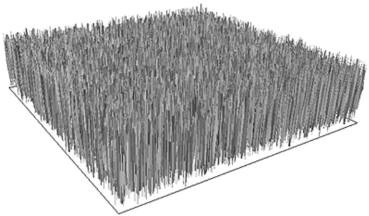
Revolution in 2010s: Pathwise Stochastic Analysis

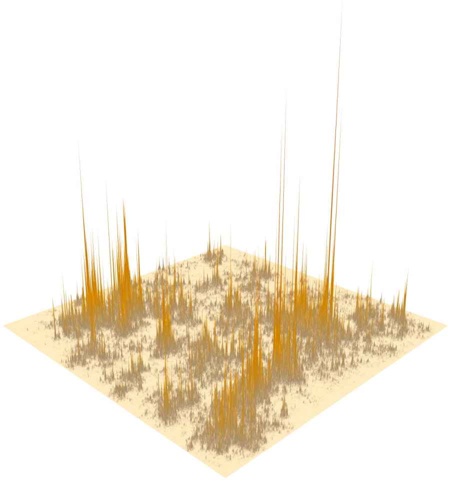
- Regularity Structures [Hairer 14]
- Paracontrolled Distributions [Gubinelli-Imkeller-Perkowski 15]
- Energy Solutions [Gonçalves-Jara 14] [Gubinelli-Perkowski 18]
- Renormalization Group Approach [Kupiainen 16] [Duch 24]

These approaches do not apply in critical dimensions

Singular SPDEs

White noise $\xi(t, x)$ is rough $\quad \rightsquigarrow \quad$ Solution $u(t, x)$ distribution valued in $d \geq 2$





Singular SPDEs

Ill-defined products

for SHE $\xi(t,x) \cdot u(t,x)$

$\rightsquigarrow$

Singular SPDEs

Q. How to make sense?

Regularize the noise ξ_ε on scale ε

(discretize, mollify, Fourier, ...)

Q. Does solution $u_\varepsilon(t,x)$ converge as regularization is removed $\varepsilon \downarrow 0$?

The need for renormalization

Naive approximation has no interesting limit

$$u_\varepsilon \xrightarrow[\varepsilon\downarrow 0]{} 0\,,\ \infty\,,\ \text{or trivial}$$

Renormalization

Modify the equation while the cutoff is removed

Tune parameters (to 0 or ∞), add / subtract counterterms, ...

"Tame infinities"

SHE in the critical dimension $d = 2$

ξ_ε = suitable regularization of white noise ξ

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + \beta_\varepsilon \, \xi_\varepsilon \cdot u_\varepsilon$$

Renormalized disorder strength: "critical window"

[Bertini-Cancrini 98]

[Albeverio-Gesztesy-Hoegh-Krohn-Holden 88]

$$\beta_\varepsilon = \frac{\hat{\beta}_{\mathrm{crit}}}{\sqrt{|\log \varepsilon|}} \left(1 + \frac{\vartheta}{|\log \varepsilon|} \right) \qquad \vartheta \in \mathbb{R}$$

Renormalized parameter ϑ survives in the limit

The critical 2D Stochastic Heat Flow

Main result

[C.S.Z. 23]

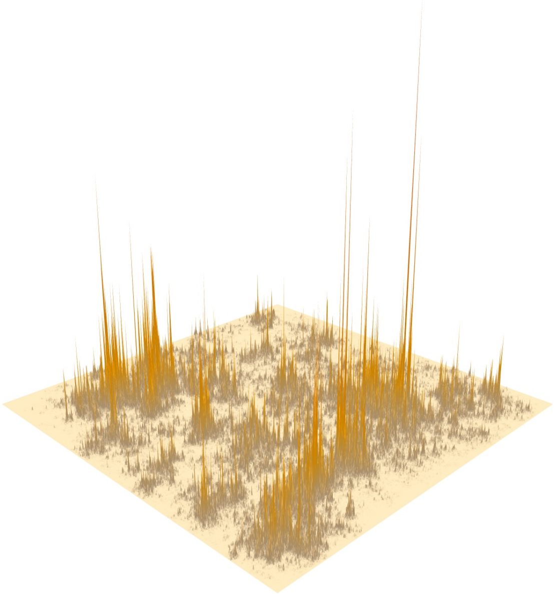
For β_ε in the critical window

$$\Big(u_\varepsilon(t,x)\,\mathrm{d}x \Big)_{t\ge 0} \xrightarrow[\varepsilon\downarrow 0]{d} \Big(\mathrm{SHF}^{\vartheta}(t,\mathrm{d}x) \Big)_{t\ge 0}$$

Critical 2D Stochastic Heat Flow

A stochastic process of random measures on $\mathbb{R}^2$

The critical 2D Stochastic Heat Flow



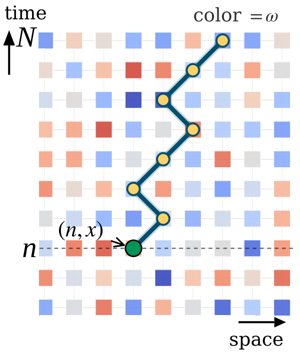
The Critical 2D Stochastic Heat Flow (SHF)

1. SHF: Singular Stochastic PDEs

2. SHF: Disordered Systems

3. SHF: Present and Future

Disordered systems perspective: Directed polymer



- $S = (S_i)$ simple random walk on $\mathbb{Z}^2$
- $\omega(i,y)$ i.i.d. standard normal disorder

Hamiltonian $H(S,\omega) := \sum_{i=n}^{N} \omega(i,S_i)$

Partition Function $Z^{\omega}_{N,\beta}(n,x) := \mathbf{E}_{(n,x)}\left[e^{\beta H(S,\omega)}\right]$

Solves a space-time discretized 2D SHE

$$Z^{\omega}_{N,\beta}(n,x) - Z^{\omega}_{N,\beta}(n+1,x) \approx \Delta_{\mathrm{d}} Z^{\omega}_{N,\beta}(n+1,x) + \beta\, \omega(n,x)\, Z^{\omega}_{N,\beta}(n+1,x)$$

Disordered systems perspective: Directed polymer

Space-time rescaled partition functions = regularized SHE

$$Z^{\omega}_{N,\beta}(Nt, \sqrt{N}x) \quad \longleftrightarrow \quad u_{\varepsilon}(t,x) \qquad \text{with} \quad \varepsilon = N^{-1/2}$$

2D polymer continuum limit $\Longleftrightarrow$ making sense of 2D SHE solution

Phase transition of directed polymer

Gibbs measure for the polymer S in disorder ω

$$\mathbf{P}^{\omega}_{N,\beta}(S) = \frac{1}{Z^{\omega}_{N,\beta}} e^{\beta H(S,\omega)}\, \mathbf{P}(S) \qquad H(S,\omega) := \sum_{i=n}^{N} \omega(i,S_i)$$

$\beta = 0$: simple random walk

$\beta \gg 1$: near optimal paths

Weak disorder

$$\beta \leq \beta_c(d)$$

diffusive paths $\quad Z^{\omega}_{N,\beta} \to Z^{\omega}_{\infty,\beta} > 0$

Strong disorder (KPZ Universality)

$$\beta > \beta_c(d)$$

super-diffusive paths $\quad Z^{\omega}_{N,\beta} \to 0$

[Imbrie–Spencer, Bolthausen, Derrida–Spohn, Sinai, Comets–Shiga–Yoshida, Carmona–Hu, Vargas, Lacoin, Berger, Nakashima, Chatterjee–Bates, Junk, . . .]

Disorder relevance vs. irrelevance

Directed Polymer (DP) is a disorder perturbation of random walk

SHE is a noise perturbation of the heat equation

Disorder Relevance vs Irrelevance

Relevant if small $\beta > 0$ still changes the qualitative large-scale behavior

Effective disorder strength grows when we zoom out

DP in $d = 1$	DP in $d = 2$	DP in $d \geq 3$
$\beta_c = 0$	$\beta_c = 0$	$\beta_c > 0$
relevant	**marginally relevant**	irrelevant

Matching terminologies across fields

Disordered systems	Singular SPDEs	QFT
Relevance	Subcriticality	Super-renormalizable
Irrelevance	Supercriticality	Non-renormalizable
Marginal relevance/irrelevance	Criticality	Renormalizable

2D SHE / 2D polymer are exactly in the critical / marginally relevant case

Disorder relevance and continuum limits

Scaling-limit philosophy

Disorder is relevant $\rightsquigarrow$ we can tune $\beta_N \downarrow 0$ as lattice spacing $\varepsilon = N^{-1/2} \downarrow 0$ to get a non-trivial continuum limit with disorder

Continuum limits for disorder relevant models

Alberts–Khanin–Quastel 14 : 1D Directed Polymer, $\beta_N = \hat{\beta}\, N^{-1/4}$

CSZ 17 : General framework for disorder relevant models

CSZ 17 does not cover marginally relevant models

New ideas needed!

Phase transition of 2D polymer

Phase transition on the intermediate disorder scale

$$\beta_N := \frac{\hat{\beta}}{\sqrt{\log N}} \qquad \text{with} \quad \hat{\beta}_c = \sqrt{\pi}$$

One-point transition [CSZ 17]

$$Z^{\omega}_{N,\beta}(0,0) \xrightarrow[N\to\infty]{d} \begin{cases} \exp(\text{Gaussian}) > 0 & \hat{\beta} < \hat{\beta}_c \\ 0 & \hat{\beta} \geq \hat{\beta}_c \end{cases}$$

Random field transition [CSZ 17]

Fluctuation of $Z^{\omega}_N(Nt, \sqrt{N}x)\,\mathrm{d}x \xrightarrow[N\to\infty]{d}$ Gaussian Free Field $\quad \hat{\beta} < \hat{\beta}_c$

SHF from directed polymer

Main result [CSZ 23]

In the critical window

$$\beta_N = \frac{\hat{\beta}}{\sqrt{\log N}} \quad \text{with} \quad \hat{\beta} = \hat{\beta}_c \Big(1 + \frac{\vartheta}{\log N}\Big)$$

the random field (as random measures on $\mathbb{R}^2$)

$$\Big(Z^{\omega}_{N,\beta_N}(Nt, \sqrt{N}x)\mathrm{d}x\Big)_{t\geq 0} \xrightarrow[\varepsilon\downarrow 0]{d} \Big(\mathrm{SHF}^{\vartheta}(t, \mathrm{d}x)\Big)_{t\geq 0}$$

A non-Gaussian limit in critical dimension AND at phase transition point

Proof strategy

Main challenge: uniqueness of subsequential limits for

$$u_N(t, \mathrm{d}x) := Z^{\omega}_{N,\beta_N}(Nt, \sqrt{N}\, x)\, \mathrm{d}x$$

$(u_N)_{N \in \mathbb{N}}$ is Cauchy $\qquad u_N \approx u_{N'} \qquad$ for large N, N'

Key ideas

- Self-similarity on different scales — Coarse-graining
- Robustness to perturbations — Lindeberg principle

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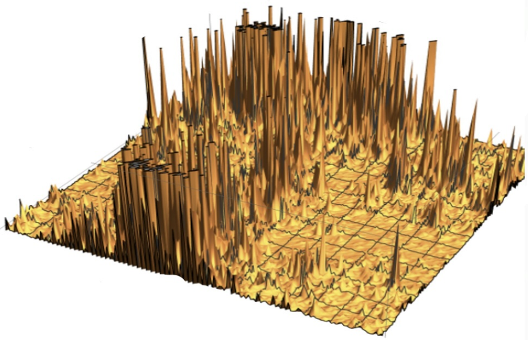
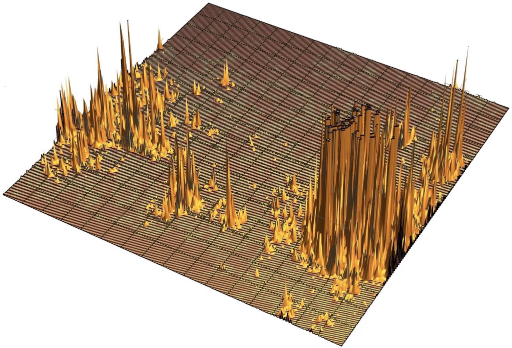
The picture of Critical 2D SHF



Critical 2D SHF with truncated peaks

Bryce Canyon, Utah

The picture of Critical 2D SHF

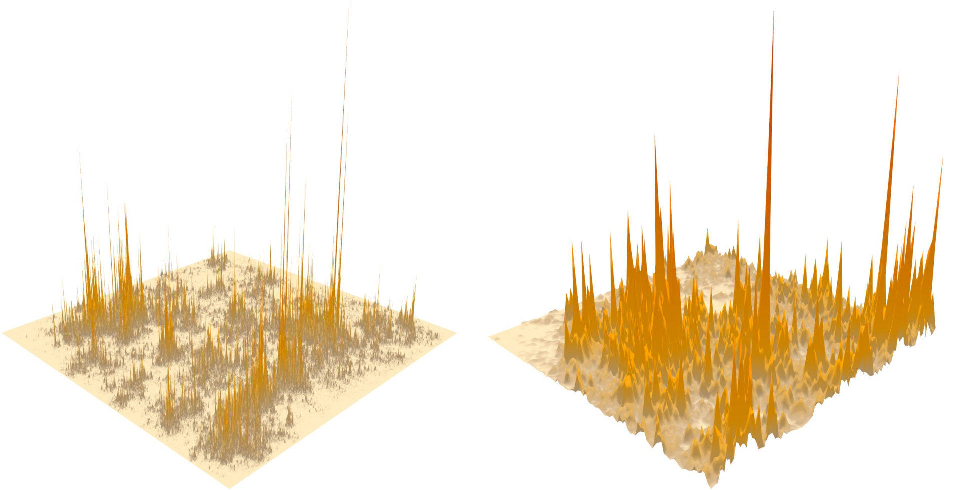


Critical 2D SHF with truncated peaks

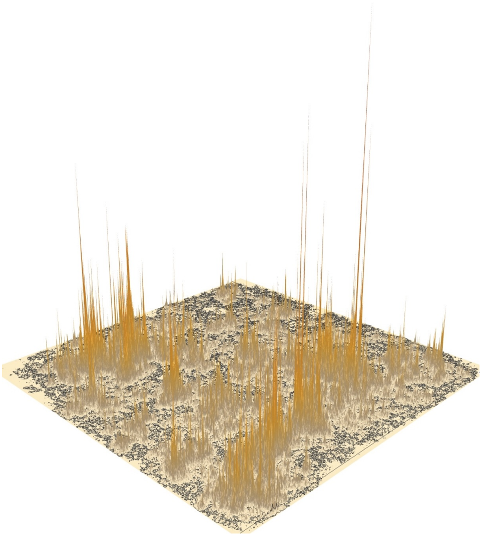
Meteora, Greece

Self-similarity

$$\mathrm{SHF}^{\vartheta}(\alpha t, d\sqrt{\alpha}x) = \alpha\, \mathrm{SHF}^{\vartheta+\log\alpha}(t, dx)$$



Extremes and multi-fractal structure



- The Critical 2D SHF has full support

  [Clark–Tsai 25] [Nakashima 25]

- Lebesgue-a.e. point x [Gu–Tsai 26]

  $$\mathrm{SHF}^{\vartheta}\big(B_\varepsilon(x)\big) \approx \big|\log\varepsilon\big|^{-1/2}$$

- Conjecture: height of peaks

  $$\sup_x \mathrm{SHF}^{\vartheta}\big(B_\varepsilon(x)\big) \approx \exp\big(\sqrt{|\log\varepsilon|}\big)$$

- Conjecture: mass supported at points x

  $$\mathrm{SHF}^{\vartheta}\big(B_\varepsilon(x)\big) \approx \big|\log\varepsilon\big|^{1/2}$$

- Conjecture (multi-fractality): non-trivial (log-)Hausdorff dim. for heights $\big|\log\varepsilon\big|^{\alpha}$

More features of SHF

A chracterization [Tsai 24]

The Critical 2D SHF$^{\vartheta}$ is the unique continuous measure valued process with

- Independent "increments" $\mathscr{Z}_{s,t}(\cdot,\cdot)$ and $\mathscr{Z}_{t,u}(\cdot,\cdot)$ for $0 < s < t < u$
- Chapman-Kolmogorov [Clark-Mian 24]

$$\int \mathscr{Z}_{s,t}(dx,dy)\,\mathscr{Z}_{s,t}(y,dz) \;=\; \mathscr{Z}_{s,t}(dx,dz)$$

- Same moments up to order 4 as SHF$^{\vartheta}$

More features of SHF

- Martingale formulation [Nakashima 2025, Chen 2025]
- Not an *exp*(Gaussian) i.e. not a GMC [CSZ 23]
- Singularity w.r.t. Lebesgue but C^{0-} [CSZ 25]
- Noise sensitivity & black noise [Caravenna-Donadini 25, Gu-Tsai 25]
- Continuum polymer measure & path space GMC [Clark-Mian 24, Clark-Tsai 25]
- Moments and delta-Bose gas [Liu-Zygouras 25, Ganguly-Nam 25, Chen 24-25]
- Extrema at sub-criticality [Cosco-Nakajima-Zeitouni 25]
- Local extinction [CSZ 25, Gu-Tsai 25, Berger-Caravenna-Turchi 25]

Progress in critical SPDEs

- Subcritical 2D KPZ
$$\partial_t h_\varepsilon = \Delta h_\varepsilon + \frac{\hat{\beta}}{\sqrt{|\log \varepsilon|}} \left(\partial_x^2 h_\varepsilon + \partial_y^2 h_\varepsilon \right) + \xi_\varepsilon$$
[Chatterjee-Dunlap 25] [CSZ 25] [Gu 25]

- Anisotropic 2D KPZ
$$\partial_t h_\varepsilon = \Delta h_\varepsilon + \frac{\hat{\beta}}{\sqrt{|\log \varepsilon|}} \left(\partial_x^2 h_\varepsilon - \partial_y^2 h_\varepsilon \right) + \xi_\varepsilon$$
[Cannizzaro-Erhard-Toninelli 23]

- Semi-linear 2D SHE
$$\partial_t u_\varepsilon = \Delta u_\varepsilon + \frac{1}{\sqrt{|\log \varepsilon|}} \sigma(u_\varepsilon) \cdot \xi_\varepsilon$$
[Dunlap-Gu 22] [Dunlap-Graham 23] [R.Tao 24]

- Burgers, Diffusion in the curl of GFF, Quasi-Geostrophic

[Cannizzaro-Gubinelli-Toninelli 24] [Chatzigeorgiou-Morfe-Otto-Wang 25]
[Armstrong-Bou Rabee-Kuusi 24+] [Cannizzaro-Moulard-Toninelli 25+] [Grafner-Perkowski 26+]

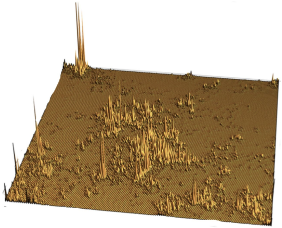
- 4D Parabolic Anderson Model

[Gabriel-Rosati 26, Deng-Shen 26]

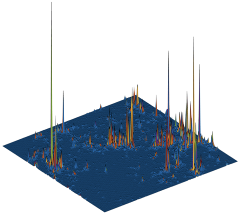
Outlook

- Build a Framework for Critical SPDEs
  - Examples include: 2D KPZ, Φ^4_4, 4D Yang-Mills, 4D PAM
  - Phase transitions? Universal features?
- Disordered Systems
  - Scaling limits of marginally relevant (critical) disordered systems?
  - Connections between SHF and CFT / QFT?
- Connections between Universality Classes in 2D Geometries

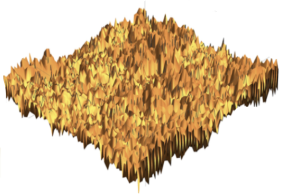
Universalities in 2D Geometries



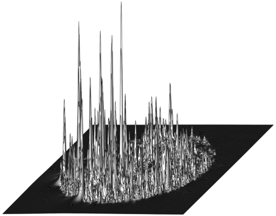
Critical 2d SHF



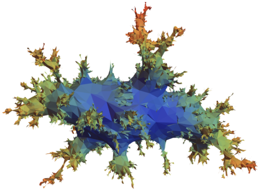
GMC ©Rhodes-Vargas



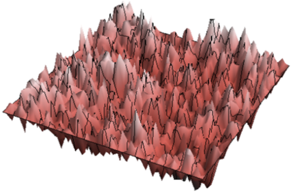
Critical 2d KPZ



Ginibre ©Pain



LQG ©Rhodes-Vargas



Gaussian Free Field