The Critical 2D Stochastic Heat Flow in the strong disorder limit

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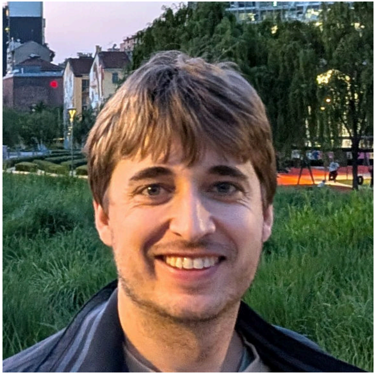

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Outline

Critical 2D Stochastic Heat Flow

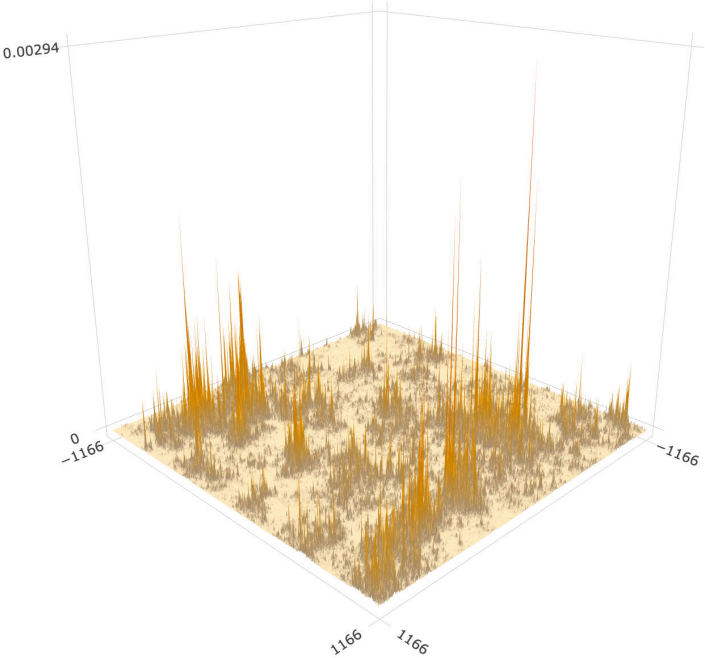
One-time marginals of SHF $\qquad \mathscr{Z}_t^{\vartheta}(\mathrm{d}x) = \int_{y \in \mathbb{R}^2} \mathscr{Z}_{0,t}^{\vartheta}(\mathrm{d}y, \mathrm{d}x)$

SHF = "solution" to 2D Stochastic Heat Equation (ill-defined)

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, \xi(t,x)\, u(t,x) \qquad u(0,x) = 1 \qquad \text{(SHE)}$$

We investigate the regimes of long time $t \to \infty$ and strong disorder $\vartheta \to \infty$

Scaling covariance property: $\qquad \dfrac{\mathscr{Z}_{at}^{\vartheta}(\sqrt{a}\,\mathrm{d}x)}{a} \stackrel{d}{=} \mathscr{Z}_t^{\vartheta + \log a}(\mathrm{d}x)$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Local extinction

Average mass is constant $\mathbb{E}[\mathscr{Z}_t^\vartheta(\mathrm{d}x)] = \mathrm{d}x$ but intermittent behavior

- SHF vanishes for long time $t \to \infty$ [C.S.Z. 25]

$$\forall \varphi \in C_c(\mathbb{R}^2): \qquad \mathscr{Z}_t^\vartheta(\varphi) = \int_{\mathbb{R}^2} \varphi(x)\, \mathscr{Z}_t^\vartheta(\mathrm{d}x) \xrightarrow[t\to\infty]{d} 0$$

- SHF vanishes for strong disorder $\vartheta \to \infty$ [Clark Tsai 25]

$$\forall \varphi \in C_c(\mathbb{R}^2): \qquad \mathscr{Z}_t^\vartheta(\varphi) \xrightarrow[\vartheta\to\infty]{d} 0$$

We obtain quantitative bounds: fractional moments or truncated mean

$$\mathbb{E}\left[\mathscr{Z}_t^\vartheta(\varphi)^\gamma\right] \quad \text{with } \gamma \in (0,1) \qquad\qquad \mathbb{E}\left[\mathscr{Z}_t^\vartheta(\varphi) \wedge 1\right] = \mathbb{P}\left(\mathscr{Z}_t^\vartheta(\varphi) > U(0,1)\right)$$

Quantitative bounds

Theorem [Berger C. Turchi 25]

There are c, C such that

$$c \exp\left(-C\, t\, e^{\vartheta}\right) \;\leq\; \sup_{\varphi \in \mathscr{M}_1\left(e^{c\,t\,e^{\vartheta}}\sqrt{t}\right)} \mathbb{E}\left[\mathscr{Z}_t^{\vartheta}(\varphi) \wedge 1\right] \;\leq\; C \exp\left(-c\, t\, e^{\vartheta}\right)$$

$\forall \varepsilon \in (0,1)$ there are $c_\varepsilon, C_\varepsilon$ such that

$$c_\varepsilon \exp\left(-C\, t\, e^{\vartheta}\right) \;\leq\; \sup_{\varphi \in \mathscr{M}_1\left(e^{c\,t\,e^{\vartheta}}\sqrt{t}\right)} \mathbb{P}\left(\mathscr{Z}_t^{\vartheta}(\varphi) \geq \varepsilon\right) \;\leq\; C_\varepsilon \exp\left(-c\, t\, e^{\vartheta}\right)$$

LB: 2nd moment method   UB: coarse-graining + change of measure

Spatial scale for mass escape

SHF mass escapes to infinity at spatial scale $\exp(c\, t\, \mathrm{e}^{\vartheta})$

Rescaled SHF $$\mathscr{Z}_t^{\vartheta,c}(\mathrm{d}x) := \frac{\mathscr{Z}_t^{\vartheta}(\mathrm{e}^{c\,t\,\mathrm{e}^{\vartheta}}\,\mathrm{d}x)}{(\mathrm{e}^{c\,t\,\mathrm{e}^{\vartheta}})^2}$$ $\mathbb{E}[\dots] = \mathrm{d}x$

Theorem [Berger C. Turchi 25]

There exist $0 < a < b < \infty$ such that

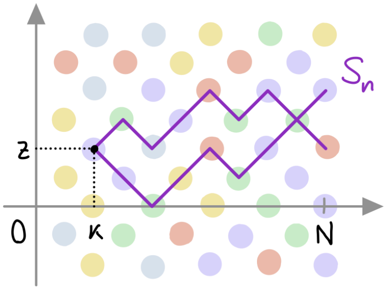
$$\forall \varphi \in C_c(\mathbb{R}^2): \qquad \mathscr{Z}_t^{\vartheta,c}(\varphi) \xrightarrow[\substack{t\to\infty \\ \text{or } \vartheta\to\infty}]{d} \begin{cases} 0 & \text{if } c < a \\ \int \varphi(x)\,\mathrm{d}x & \text{if } c > b \end{cases}$$

Conjecture: non trivial limit $\mathscr{Z}_t^{\vartheta,\hat{c}}(\varphi) \xrightarrow[\vartheta\to\infty]{d} \mathscr{U}_t(\mathrm{d}x)$ for some $\hat{c} \in (a,b)$

Outline

Directed polymers partition functions

- $S = (S_n)_{n\geq 0}$ simple random walk on $\mathbb{Z}^d$
- Independent Gaussians $\omega(n,x) \sim \mathcal{N}(0,1)$
- $H(S,\omega) := \sum_{n=k+1}^{N} \omega(n, S_n)$



Partition Functions $(k \in \mathbb{N},\ z \in \mathbb{Z}^d)$

$$Z^{\omega}_{N,\beta}(k,z) = \mathrm{E}\left[e^{\beta H(S,\omega) - \frac{1}{2}\beta^2 (N-k)} \,\middle|\, S_k = z \right] \qquad \mathbb{E}\left[Z^{\omega}_{N,\beta}\right] = 1$$

SHF from directed polymers

SHF is the scaling limit of partition functions [C.S.Z. 23]

$$Z^{\omega}_{N,\beta}(\varphi_N) = \sum_{z\in\mathbb{Z}^2} \varphi_N(z)\, Z^{\omega}_{N,\beta}(0,z) \xrightarrow[N\to\infty]{d} \mathscr{Z}^{\vartheta}_t(\varphi)$$

in the critical regime $\beta^2 = \frac{\pi}{\log N}\left(1 + -\frac{\vartheta}{\log N}\right)^{-1}$ with $\varphi_N(z) = \frac{1}{\sqrt{N}}\varphi\left(\frac{z}{\sqrt{N}}\right)$

Super-critical regime: any $\vartheta = \vartheta_N \to \infty$ such that

$$\beta \le \beta_0 \in (0,\infty) \qquad \text{i.e.} \qquad \vartheta = \log N - \frac{\pi}{\beta^2} \le \log N - \frac{\pi}{\beta_0^2}$$

Fixed $\beta = \beta_0 > 0$ is also allowed

Local extinction and free energy

Quantitative bounds for $Z^{\omega}_{N,\beta}(f) \to 0$ uniformly over $N \in \mathbb{N}$, $\beta \in (0,\beta_0)$

Theorem [Berger C. Turchi 25]

$$c\,\exp(-C\,t\,\mathrm{e}^{\vartheta}) \;\le\; \sup_{f\in\mathscr{M}_1^{\mathrm{disc}}\left(\mathrm{e}^{c\,t\,\mathrm{e}^{\vartheta}}\sqrt{N\,t}\right)} \mathbb{E}\left[Z^{\omega}_{Nt,\beta}(f)\wedge 1\right] \;\le\; C\,\exp(-c\,t\,\mathrm{e}^{\vartheta})$$

Free energy: $Z^{\omega}_{N,\beta}(0) = \mathrm{e}^{\mathrm{F}(\beta)\,N + o(N)} \xrightarrow[N\to\infty]{} 0$ [Lacoin 10, Berger Lacoin 17]

Theorem [Berger C. Turchi 25]

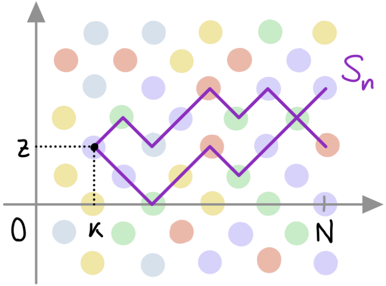
$$-\frac{c'}{\beta^8}\,\exp\left(-\frac{\pi}{\beta^2}\right) \le \mathrm{F}(\beta) \le -c\,\exp\left(-\frac{\pi}{\beta^2}\right)$$

Outline

Partition functions and SHE

Partition functions $Z^{\omega}_{N,\beta}(k,z)$ are solutions of

discretised Stochastic Heat Equation



Diffusive rescaling (+ time reversal)

$$u_N(t,x) := Z^{\omega}_{N,\beta}\big(N(1-t), \sqrt{N}x\big)$$

$$\begin{cases} \partial_t u_N(t,x) = \Delta_x u_N(t,x) + \beta \, \underbrace{\xi_N(t,x)}_{\text{white noise regularized on scale } \varepsilon = \frac{1}{\sqrt{N}}} \, u_N(t,x) \\ u_N(0,x) \equiv 1 \end{cases} \qquad \text{(disc-SHE)}$$

Spatial scale of mass escape for SHE

Spatial scale $e^{c\,t\,e^{\vartheta}} = e^{c\,t\,f_\beta\,N}$ with $f_\beta = e^{-\frac{\pi}{\beta^2}}$ $\quad (\vartheta = \log N - \frac{\pi}{\beta^2})$

$\rightsquigarrow$ Rescaled solution $\quad u_N^{\beta,c}(t,x) := u_N\big(t, e^{c\,t\,f_\beta\,N}\,x\big)$

Theorem [Berger C. Turchi 25]

There are $0 < a < b < \infty$ such that

$$\forall \varphi \in C_c(\mathbb{R}^2): \qquad \int_{\mathbb{R}^2} \varphi(x)\, u_N^{\beta,c}(t,x)\,\mathrm{d}x \xrightarrow[N\to\infty]{d} \begin{cases} 0 & \text{if } c < a \\ \int \varphi(x)\,\mathrm{d}x & \text{if } c > b \end{cases}$$

Special case $\beta^2 = \frac{\pi\hat{\beta}^2}{\log N}$ with $\hat{\beta} > 1$ $\quad\rightsquigarrow\quad e^{c\,t\,f_\beta\,N} = e^{c\,t\,N^\gamma}$ with $\gamma = 1 - \frac{1}{\hat{\beta}^2}$

Outline

Proof of the UB: coarse-graining

$$\sup_{f \in \mathscr{M}_1^{\mathrm{disc}}\left(\mathrm{e}^{c\,\mathrm{e}^{\vartheta}}\sqrt{N}\right)} \mathbb{E}\left[Z_{N,\beta}^{\omega}(f) \wedge 1\right] \le C \exp\left(-c\,\mathrm{e}^{\vartheta}\right) \qquad \text{for any } \vartheta = \vartheta_N \to \infty$$

- Change of scale argument: reduce $\mathscr{M}_1^{\mathrm{disc}}\left(\mathrm{e}^{c\,\mathrm{e}^{\vartheta}}\sqrt{N}\right)$ to $\mathscr{M}_1^{\mathrm{disc}}(\sqrt{N})$
- Coarse-graining argument: $\begin{cases} \text{reduce } \vartheta = \vartheta_N \to \infty \text{ to fixed } \vartheta \in \mathbb{R} \\ \text{replace } \exp\left(-c\,\mathrm{e}^{\vartheta}\right) \text{ by any } f(\vartheta) \to 0 \end{cases}$

Key bound

$$\sup_{f \in \mathscr{M}_1^{\mathrm{disc}}(\sqrt{N})} \mathbb{E}\left[Z_{N,\beta}^{\omega}(f) \wedge 1\right] \le \frac{C}{\vartheta} \qquad \text{for fixed } \vartheta \in \mathbb{R}$$

Proof of the UB: change of measure

Change of scale: $\sup_{f \in \mathscr{M}_1^{\mathrm{disc}}(\sqrt{N})} \mathbb{E}\left[Z_{N,\beta}^{\omega}(f) \wedge 1\right] \leq \frac{2}{\varepsilon} \sup_{f \in \mathscr{M}_1^{\mathrm{disc}}(\sqrt{\varepsilon N})} \mathbb{E}\left[Z_{N,\beta}^{\omega}(f) \wedge 1\right]$

Idea: for f on scale $\sqrt{\varepsilon N}$, partition function $Z_{N,\beta}^{\omega}(f)$ is almost point-to-plane

Size-biased law $\tilde{\mathbb{P}}(\mathrm{d}\omega) := Z(\omega)\, \mathbb{P}(\mathrm{d}\omega)$ for $Z(\omega) = Z_{N,\beta}^{\omega}(f)$

Change of measure

$$\mathbb{E}[Z \wedge 1] \leq \mathbb{P}(A) + \tilde{\mathbb{P}}(A^c) \qquad \text{for any event } A$$

Optimal with $A = \{Z > 1\}$ (but we don't know Z ...)

Proof of the UB: choice of a proxy

Take X with $\mathbb{E}[X] = 0$ and set $A = \{X > \frac{1}{2}\tilde{\mathbb{E}}[X]\}$

By Chebychev $\quad \mathbb{P}(A) \le 4\,\frac{\mathbb{V}\mathrm{ar}[X]}{\tilde{\mathbb{E}}[X]^2} \qquad \tilde{\mathbb{P}}(A^c) \le 4\,\frac{\tilde{\mathbb{V}}\mathrm{ar}[X]}{\tilde{\mathbb{E}}[X]^2}$

We take X as a manageable proxy of Z: restrict the chaos expansion of Z to

$$I = \{(n_1, x_1), \ldots, (n_k, x_k)\} \qquad \text{with} \quad \begin{cases} \mathrm{width}(I) = n_k - n_1 \le \varepsilon N\,, \\ |I| = k \le \log(\varepsilon N) \end{cases}$$

We finally estimate $\mathbb{V}\mathrm{ar}[X]$, $\tilde{\mathbb{E}}[X]$ (2nd moment) and $\tilde{\mathbb{V}}\mathrm{ar}[X]$ (3rd moment)

Outline

Conclusions

Quantitative bounds for local extinction of SHF and directed polymers

large time and/or strong disorder

Mass escapes to infinity at spatial scale $\exp\big(c\,t\,\mathrm{e}^{\vartheta}\big) = \exp\big(c\,t\,\mathrm{e}^{-\pi/\beta^2}\,N\big)$

Application to discretized SHE in the super-critical regime up to fixed β

We expect analogous result for the mollified SHE (in progress)

Robust proof based on coarse-graining + change of scale + change of measure

Q. Berger, F.C., N. Turchi. arXiv: 2508.02478 (to be updated soon!)

Thanks