Strong disorder for 2D directed polymers and stochastic heat flow

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Localization, Delocalization, and Diffusion in Disordered Systems

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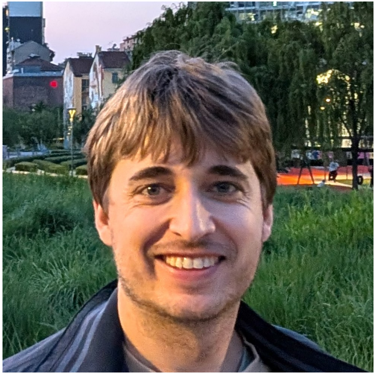

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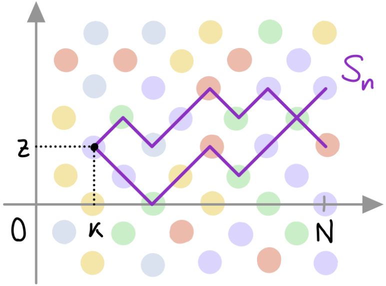
Outline

1. Directed Polymer and Stochastic Heat Flow
2. Supercritical Directed Polymer
3. Supercritical Stochastic Heat Flow
4. Sketch of the proof
5. Conclusions

Directed polymer in random environment

- $S = (S_n)_{n \geq 0}$ simple random walk on $\mathbb{Z}^d$
- Independent Gaussians $\omega(n,x) \sim \mathcal{N}(0,1)$
- $H(S,\omega) := \sum_{n=k+1}^{N} \omega(n,S_n) \sim \mathcal{N}(0,N-k)$



Partition Functions

$(k \in \mathbb{N},\ z \in \mathbb{Z}^d)$

$$Z^{\omega}_{N,\beta}(k,z) = \mathrm{E}\left[e^{\beta H(S,\omega) - \underset{\text{(constant)}}{\frac{1}{2}\beta^2 (N-k)}} \,\middle|\, S_k = z \right]$$

$$\mathbb{E}\left[Z^{\omega}_{N,\beta}\right] = 1$$

Partition functions and SHE

Diffusive rescaling (+ time rev.) $\qquad u_N(t,x) := Z_{N,\beta}\big(N(1-t), \sqrt{N}x\big)$

Discretised Stochastic Heat Equation (SHE)

$$\partial_t u_N(t,x) = \Delta_x u_N(t,x) + \beta\, N^{\frac{2-d}{4}}\, N^{\frac{2-d}{4}} \underbrace{\xi_N(t,x)}_{\text{approx. of white noise on scale } \varepsilon = 1/\sqrt{N}} u_N(t,x) \qquad \text{(disc-SHE)}$$

Critical dimension $d = 2$: approximation to ill-defined 2D SHE

Critical 2D Stochastic Heat Flow

[C.S.Z. 23]

$$\lim_{N\to\infty} u_N(t,x)\,\mathrm{d}x = \mathscr{Z}_t^{\vartheta}(\mathrm{d}x) \qquad \text{renomalizing } \beta = \beta_N^{\text{crit}} \to 0$$

Deterministic Heat Equation

(by Q. Berger)

(Video)

Stochastic Heat Equation: subcritical

(by Q. Berger)

(Video)

Stochastic Heat Equation: critical

(by Q. Berger)

(Video)

Stochastic Heat Equation: supercritical (by Q. Berger)

(Video)

From directed polymer to SHF

Averaged partition functions on balls

$$Z_{N,\beta}(f) := \sum_{z\in\mathbb{Z}^2} f(z)\, Z_{N,\beta}(k=0,z) \qquad f(\cdot) = B_R(\cdot) = \tfrac{1}{\pi R^2}\mathbb{1}_{|\cdot|\le R}$$

- Diffusive scale $R \propto \sqrt{N}$
- Critical regime $(\beta_N^{\mathrm{crit}})^2 = \frac{\pi}{\log N}\left(1 + \frac{\vartheta + o(1)}{\log N}\right)$ for fixed $\vartheta \in \mathbb{R}$

Convergence to Stochastic Heat Flow

$$Z_{N,\beta_N^{\mathrm{crit}}}(B_{\sqrt{N}}) \xrightarrow[N\to\infty]{d} \mathscr{Z}_t^{\vartheta}(B_1)$$

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Beyond the critical regime

Critical regime $\beta = \beta_N^{\mathrm{crit}}$: rewrite as $\beta^2 = \frac{\pi}{\log N}\left(1 + - \frac{\vartheta + o(1)}{\log N}\right)^{-1}$

$$\boxed{\boxed{\beta^2 = \frac{\pi}{\log N - \vartheta}}}$$

One-to-one correspondence (given N)

$$(0,1) \ni \beta \;\leftrightarrow\; \vartheta \in (-\infty, \log N - \pi)$$

$$\begin{cases} \text{subcritical regime} & \beta \text{ ``}\ll\text{'' } \beta_N^{\mathrm{crit}} & \vartheta \to -\infty \\ \text{critical regime} & \beta = \beta_N^{\mathrm{crit}} & \vartheta = O(1) \\ \text{supercritical regime} & \beta \text{ ``}\gg\text{'' } \beta_N^{\mathrm{crit}} & \vartheta \to +\infty \end{cases}$$

Supercritical directed polymer

We prove that $Z_{N,\beta}(B_{\sqrt{N}}) \to 0$ in the entire supercritical regime

$$\beta \text{ ``}\gg\text{'' } \beta_N^{\mathrm{crit}} \qquad \text{i.e.} \qquad \vartheta \to \infty$$

Quantitative bounds via
$$\begin{cases} \text{fractional moments} & \mathbb{E}\Big[Z_{N,\beta}(B_{\sqrt{N}})^{\gamma}\Big] \text{ for } \gamma \in (0,1) \\ \text{truncated mean} & \mathbb{E}\Big[Z_{N,\beta}(B_{\sqrt{N}}) \wedge 1\Big] \end{cases}$$

Main Theorem

[Berger C. Turchi 25+]

$$c\,\exp(-C\,\mathrm{e}^{\vartheta}) \;\le\; \mathbb{E}\Big[Z_{N,\beta}\big(B_{\sqrt{N}}\big) \wedge 1\Big] \;\le\; C\,\exp(-c\,\mathrm{e}^{\vartheta}) \qquad \forall N, \beta$$

LB: 2nd moment method UB: coarse-graining + change of measure

Tail bounds and free energy

Right tail bounds

for $\varepsilon \in (0,1)$

$$c_\varepsilon \exp\left(-C\,\mathrm{e}^{\vartheta}\right) \;\le\; \mathbb{P}\Big(Z_{N,\beta}\big(B_{\sqrt{N}}\big) \ge \varepsilon\Big) \;\le\; C_\varepsilon \exp\left(-c\,\mathrm{e}^{\vartheta}\right)$$

Free energy (point-to-plane)

[Berger Lacoin 17]

$$\mathrm{F}(\beta) := \lim_{N\to\infty} \frac{1}{N} \log Z_{N,\beta}(0) = \exp\Big(-\frac{\pi + o(1)}{\beta^2}\Big) \qquad \text{as } \beta \downarrow 0$$

Free energy asymptotics

[Berger C. Turchi 25+]

$$-\frac{C}{\beta^8}\exp\Big(-\frac{\pi}{\beta^2}\Big) \;\le\; \mathrm{F}(\beta) \;\le\; -c\exp\Big(-\frac{\pi}{\beta^2}\Big) \qquad \Big[\mathrm{e}^{\vartheta} = N\,\mathrm{e}^{-\pi/\beta^2}\Big]$$

Back to discretized SHE

Solution of discretized SHE $\varphi \in C_c(\mathbb{R}^2)$

$$u_{N,\beta}(t,x) = Z_{N,\beta}\left(N(1-t), \sqrt{N}x\right)$$

$$\int \varphi(x)\, u_{N,\beta}(t,x)\, \mathrm{d}x \xrightarrow[N\to\infty]{d} \begin{cases} \int \varphi(x)\, \mathrm{d}x & \text{for } \beta \text{ ``}\ll\text{''} \beta_N^{\mathrm{crit}} \\ \int \varphi(x)\, \mathscr{Z}_t^{\vartheta}(\mathrm{d}x) & \text{for } \beta = \beta_N^{\mathrm{crit}} \\ 0 & \text{for } \beta \text{ ``}\gg\text{''} \beta_N^{\mathrm{crit}} \end{cases}$$

Local extinction for β “$\gg$” β_N^{crit} (e.g. for fixed $\beta > 0$)

Superdiffusivity of the supercritical SHE

Mass “conserved on average”: $\int B_R(x)\, u_{N,\beta}(t,x)\,\mathrm{d}x \xrightarrow[R\to\infty]{d} 1$ (fixed N, β, t)

Mass escape to infinity in the supercritical regime β “$\gg$” β_N^{crit}

Theorem

[Berger C. Turchi 25+]

Rate of mass escape $\exp\left(A\,t\,\mathrm{e}^{\vartheta}\right) = \exp\left(A\,t\,N\,\mathrm{e}^{-\pi/\beta^2}\right)$ $\left(\beta^2 = \frac{\pi}{\log N - \vartheta}\right)$

$\exists 0 < A_- < A_+ < \infty$:

$$\int \varphi(x)\, u_{N,\beta}\left(t,\, e^{A\,t\,N\,\mathrm{e}^{-\pi/\beta^2}} x\right)\mathrm{d}x \xrightarrow[N\to\infty]{d} \begin{cases} \int \varphi(x)\,\mathrm{d}x & \text{if } A > A_+ \\ 0 & \text{if } A < A_- \end{cases}$$

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Local extinction

Analogous results for the SHF $\mathscr{Z}_t^{\vartheta}$

- SHF vanishes for long time $t \to \infty$ [C.S.Z. 25+]

$$\forall \varphi \in C_c(\mathbb{R}^2): \qquad \mathscr{Z}_t^{\vartheta}(\varphi) = \int_{\mathbb{R}^2} \varphi(x)\, \mathscr{Z}_t^{\vartheta}(\mathrm{d}x) \xrightarrow[t\to\infty]{d} 0$$

- SHF vanishes for strong disorder $\vartheta \to \infty$ [Clark Tsai 25+]

$$\mathscr{Z}_t^{\vartheta}(\varphi) \xrightarrow[\vartheta\to\infty]{d} 0$$

We obtain quantitative bounds. For simplicity $\varphi(\cdot) = B_r(\cdot) = \frac{1}{\pi r^2} \mathbb{1}_{|\cdot| \le r}$

Quantitative bounds

Theorem [Berger C. Turchi 25+]

$$c \exp(-C\, t\, e^{\vartheta}) \;\le\; \mathbb{E}\big[\mathscr{Z}_t^{\vartheta}(B_{\sqrt{t}}) \wedge 1\big] \;\le\; C \exp(-c\, t\, e^{\vartheta}) \qquad \forall t, \vartheta$$

In ongoing work, we extend this bound to B_r uniformly for $0 < r \le \sqrt{t}$

Theorem $(\gamma \in (0,1))$ [Berger C. Sun Turchi Zygouras 26+]

$$c\, \mathbb{E}\Big[\mathscr{Z}_t^{\vartheta}(B_r)^2\Big]^{-C\gamma(1-\gamma)} \;\le\; \mathbb{E}\Big[\mathscr{Z}_t^{\vartheta}(B_r)^{\gamma}\Big] \;\le\; C\, \mathbb{E}\Big[\mathscr{Z}_t^{\vartheta}(B_r)^2\Big]^{-c\gamma(1-\gamma)} \qquad \forall t, \vartheta, r$$

LB agrees with log-log fluctuations of $\mathscr{Z}_t^{\vartheta}(B_r)$ as $r \downarrow 0$ [Gu Tsai 26+]

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Proof of the UB: coarse-graining

$$\mathbb{E}\Big[Z_{N,\beta}\big(B_{\mathrm{e}^{\hat{c}\,\mathrm{e}^{\vartheta}}\sqrt{N}}\big)\wedge 1\Big] \;\le\; C\,\exp(-c\,\mathrm{e}^{\vartheta})$$

- Change of scale argument: reduce $B_{\mathrm{e}^{\hat{c}\,\mathrm{e}^{\vartheta}}\sqrt{N}}$ to $B_{\sqrt{N}}$
- Coarse-graining argument: $\begin{cases} \text{reduce } \vartheta \to \infty \text{ to fixed (large) } \vartheta > 0 \\ \text{replace } \exp(-c\,\mathrm{e}^{\vartheta}) \text{ by any } f(\vartheta) \to 0 \end{cases}$

Key bound

$$\mathbb{E}\Big[Z_{N,\beta}\big(B_{\sqrt{N}}\big)\wedge 1\Big] \;\le\; \frac{C}{\vartheta}$$

for fixed $\vartheta \in \mathbb{R}$

Proof of the UB: change of measure

Change of scale: $\mathbb{E}\Big[Z_{N,\beta}\big(B_{\sqrt{N}}\big) \wedge 1\Big] \le \frac{C}{\varepsilon}\,\mathbb{E}\Big[Z_{N,\beta}\big(B_{\sqrt{\varepsilon N}}\big) \wedge 1\Big] \qquad \forall \varepsilon \in (0,1)$

For small $\varepsilon > 0$ the partition function $Z_{N,\beta}\big(B_{\sqrt{\varepsilon N}}\big)$ is almost point-to-plane

Size-biased law $\qquad \tilde{\mathbb{P}}(\mathrm{d}\omega) := Z(\omega)\,\mathbb{P}(\mathrm{d}\omega) \qquad Z(\omega) := Z_{N,\beta}\big(B_{\sqrt{\varepsilon N}}\big)$

Change of measure

$$\mathbb{E}[Z \wedge 1] \le \mathbb{P}(A) + \tilde{\mathbb{P}}(A^c) \qquad \text{for any event } A$$

Goal: find some event A which is typical for $\mathbb{P}$ and atypical for $\tilde{\mathbb{P}}$

Optimal choice $A = \{Z > 1\}$ (but we don't know Z)

Proof of the UB: choice of a proxy

Take $A = \left\{X > \frac{1}{2}\tilde{\mathbb{E}}[X]\right\}$ for some r.v. X with $\mathbb{E}[X] = 0$ $(X \approx Z - 1)$

By Markov and Chebychev $\mathbb{P}(A) \leq 4\,\frac{\mathbb{V}\mathrm{ar}[X]}{\tilde{\mathbb{E}}[X]^2}$ $\tilde{\mathbb{P}}(A^c) \leq 4\,\frac{\tilde{\mathbb{V}}\mathrm{ar}[X]}{\tilde{\mathbb{E}}[X]^2}$

We take X as a manageable proxy of Z restricting its chaos expansion to

$$I = \{(n_1, x_1), \dots, (n_k, x_k)\} \qquad \text{with} \quad \begin{cases} \mathrm{width}(I) = n_k - n_1 \leq \varepsilon N, \\ |I| = k \leq \log(\varepsilon N) \end{cases}$$

We finally estimate $\mathbb{V}\mathrm{ar}[X]$, $\tilde{\mathbb{E}}[X]$ (2nd moment) and $\tilde{\mathbb{V}}\mathrm{ar}[X]$ (3rd moment)

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Conclusions

Quantitative bounds for local extinction of SHF and directed polymers

large time and/or strong disorder

Mass escape to infinity on the spatial scale $\exp\big(c\, t\, \mathrm{e}^{\vartheta}\big) = \exp\big(c\, t\, N\, \mathrm{e}^{-\pi/\beta^2}\big)$

Application: discretized SHE in the supercritical regime (including fixed $\beta > 0$)
We expect analogous result for the mollified SHE (in progress)

Robust proof based on coarse-graining + change of scale + change of measure

Thanks