Singolarità e Regolarità Aleatorie

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I seminari del Centenario

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Obiettivo

Raccontare in modo non tecnico un po' di storia e alcuni progressi fondamentali nelle

$$\text{Equazioni Differenziali Stocastiche} \begin{cases} \text{ODE} & \text{Ordinarie} \\ \text{PDE} & \text{alle derivate Parziali} \end{cases}$$

Raccontare alcune idee profonde alla base di questi progressi alla frontiera tra

Probabilità, Analisi e Fisica Matematica

(interazioni con Algebra, Analisi Numerica, Geometria, ...)

Avvertenza: non sono un esperto di equazioni differenziali stocastiche!

Ci sono "andato a sbattere"...

Sommario

ODE e PDE

ODE controllata

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = \sigma\big(X(t)\big)\,\xi(t)$$

$X(t)$ incognita $\qquad X(0),\ \sigma(\cdot),\ \xi(t)$ assegnati

$X : [0,\infty) \to \mathbb{R}^k \qquad \xi : [0,\infty) \to \mathbb{R}^d \qquad \sigma : \mathbb{R}^k \to \mathbb{R}^k \otimes \mathbb{R}^d$

PDE controllata

$$\frac{\partial}{\partial t}u(t,x) = \underbrace{\Delta_x u(t,x)}_{\sum_{i=1}^{d} \frac{\partial^2}{\partial {x_i}^2} u(t,x)} + f\big(u(t,x), \nabla u(t,x)\big) + \xi(t,x)$$

$u(t,x)$ incognita $\qquad u(0,x),\ f(\cdot),\ \xi(t,x)$ assegnati

$u,\ \xi : [0,\infty) \times \mathbb{R}^d \to \mathbb{R} \qquad f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$

Che cos'è una equazione differenziale stocastica?

ODE o PDE dove il "controllo" ξ è stocastico (aleatorio, random) e irregolare:

l'equazione non si può formulare in modo puramente analitico

- $(\Omega, \mathcal{A}, \mathbb{P})$ spazio di probabilità $\qquad \Omega \ni \omega \longmapsto \xi^\omega(t,x)$
- $(t,x) \longmapsto \xi^\omega(t,x)$ non è una funzione ~~regolare~~
  ma solo una distribuzione (Schwartz)

"Perché farsi del male?" Perché ξ irregolari emergono nei casi più interessanti

Rumore bianco: "i valori di $\xi(t,x)$ sono indipendenti in ogni punto (t,x)"

ODE stocastiche (SDE): la teoria classica

ODE $\quad \frac{\mathrm{d}}{\mathrm{d}t} X_t = \sigma(X_t)\, \xi(t) \quad$ in forma integrale: $\qquad [\, X_t := X(t)\,]$

$$X_t \;=\; X_0 + \int_0^t \sigma(X_s)\, \xi(s)\, \mathrm{d}s \;=\; X_0 + \int_0^t \sigma(X_s)\, \mathrm{d}B_s \qquad (\star)$$

$B_t := \int_0^t \xi(s)\, \mathrm{d}s$ è il moto browniano $\qquad$ ($t \mapsto B_t$ funzione continua, non derivabile)

Teorema (Ito 1944)

- Integrale stocastico $\int_0^t Y_s\, \mathrm{d}B_s$ per un'ampia classe di funzioni aleatorie $(Y_s^\omega)_{s \geq 0}$
- Esistenza e unicità per l'equazione $(\star)$ mediante punto fisso $\qquad [\, \sigma(\cdot) \in C^1\,]$

Ito (1944)

No. 8.] 519

109. *Stochastic Integral.**

By Kiyosi Itô.

Mathematical Institute, Nagoya Imperial University.

(Comm. by S. Kakeya, M.I.A., Oct. 12, 1944.)

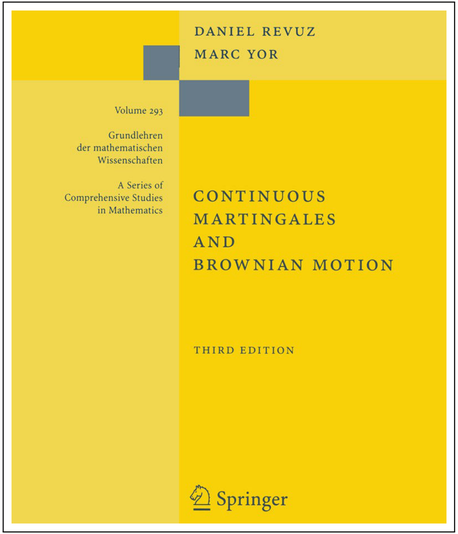
1. *Introduction.* Let (Ω, P) be any probability field, and $g(t, \omega)$, $0 \leqq t \leqq 1$, $\omega \in \Omega$, be any *brownian motion*[1] on (Ω, P) i. e. a (real) stochastic differential process with no moving discontinuity such that $\mathscr{E}\big(g(s, \omega)-g(t, \omega)\big)=0$[2] and $\mathscr{E}\big(g(s, \omega)-g(t, \omega)\big)^2=|s-t|$. In this note we shall investigate an integral $\int_0^t f(\tau, \omega)\; d_\tau g(\tau, \omega)$ for any element $f(t, \omega)$ in a functional class S^* which will be defined in § 2; the particular case in which $f(t, \omega)$ does not depend upon ω has already been treated by Paley and Wiener[3].

In § 2 we shall give the definition and prove fundamental properties concerning this integral. In § 3 we shall establish three theorems which give sufficient conditions for integrability. In § 4 we give an example, which will show a somewhat singular property of our integral.

2. *Definition and Properties.* For brevity we define the classes of measurable functions defined on $[0, 1]\times\Omega$: $\boldsymbol{G}$, $\boldsymbol{S}(t_0, t_1, \dots, t_n)$, $\boldsymbol{S}$ and $\boldsymbol{S}^*$ respectively as the classes of $f(t, \omega)$ satisfying the corresponding conditions, as follows,

$\boldsymbol{G}$: $f(\tau, \omega)$, $g(\tau, \omega)$, $0 \leqq \tau \leqq t$, are independent of $g(\sigma, \omega)-g(t, \omega)$, $t \leqq \sigma \leqq 1$, for any t, $g(\tau, \omega)$ being the above mentioned brownian motion,

Revuz, Yor (1999)



PDE stocastiche (SPDE): la teoria classica

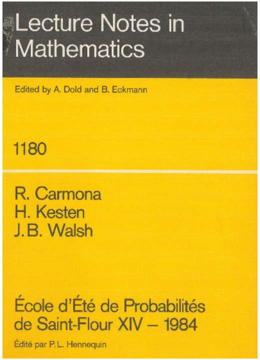
PDE $\quad \partial_t u - \Delta_x u = f(u, \nabla u) + \xi \quad$ in forma integrale ("mild"):

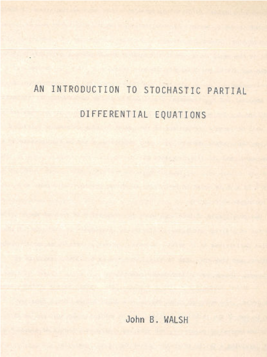
$$
\begin{aligned}
u(t,x) &= \text{``}(\partial_t - \Delta_x)^{-1}\big(f(u,\nabla u) + \xi\big)\text{''} \\
&= u(0,x) + \int_0^t \int_{\mathbb{R}^d} \underbrace{g_{t-s}(x-y)}_{\text{nucleo del calore } \propto \exp\left(-\frac{|x-y|^2}{4(t-s)}\right)} \big\{ f(u(s,y), \nabla u(s,y)) + \xi(s,y) \big\} \, \mathrm{d}s \, \mathrm{d}y \qquad (\star)
\end{aligned}
$$

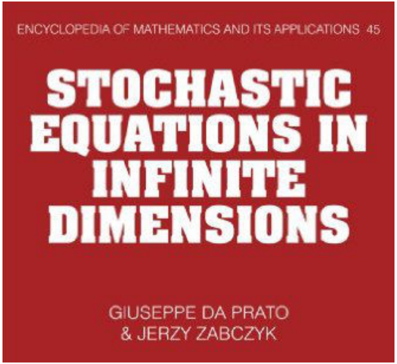
Teorema (AA. VV. '70-'80)

- Integrale stocastico per un'ampia classe di funzioni aleatorie
- Esistenza e unicità per $f(\cdot)$ non lineari solo in bassa dimensione $d \leq d_0$

Walsh (1986) St.Flour Lecture Notes   Da Prato, Zabczyk (1992)







PDE stocastiche singolari

Alcune importanti PDE stocastiche singolari sfuggono alla teoria classica

in particolare quando $f(u, \nabla u)$ *è non lineare*

Due esempi fondamentali:

- Quantizzazione stocastica (Φ^4_d) per $d \geq 2$ [Parisi, Wu (1981)]

$$\partial_t u = \Delta_x u - u^3 + \xi$$

- Dinamica di interfacce (KPZ_d) per $d \geq 1$ [Kardar, Parisi, Zhang (1986)]

$$\partial_t u = \Delta_x u + |\nabla u|^2 + \xi$$

Parisi, Wu (1981)

Vol. XXIV No. 4 SCIENTIA SINICA April 1981

PERTURBATION THEORY WITHOUT GAUGE FIXING

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AND WU YONGSHI (吴詠时)
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Received July 7, 1980.

ABSTRACT

We propose to formulate the perturbative expansion for field theory starting from the Langevin equation which describes the approach to equilibrium. We show that this formulation can be applied to gauge theories to compute gauge invariant quantities without fixing the gauge. A very simple example is worked out in detail. We also discuss the speed of approaching to equilibrium of the solution of the Langevin equation in the framework of perturbation theory.

Kardar, Parisi, Zhang (1986)

VOLUME 56, NUMBER 9 PHYSICAL REVIEW LETTERS 3 MARCH 1986

Dynamic Scaling of Growing Interfaces

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(Received 12 November 1985)

A model is proposed for the evolution of the profile of a growing interface. The deterministic growth is solved exactly, and exhibits nontrivial relaxation patterns. The stochastic version is studied by dynamic renormalization-group techniques and by mappings to Burgers's equation and to a random directed-polymer problem. The exact dynamic scaling form obtained for a one-dimensional interface is in excellent agreement with previous numerical simulations. Predictions are made for more dimensions.

PACS numbers: 05.70.Ln, 64.60.Ht, 68.35.Fx, 81.15.Jj

Many challenging problems are associated with growth patterns in clusters[1] and solidification fronts.[2] Several models have been proposed recently to describe the growth of smoke and colloid aggregates, flame fronts, tumors, etc.[1] It is generally recognized that the growth process occurs mainly at an "active" zone on the surface of the cluster, with interesting scaling properties.[3] However, a systematic *analytic* treatment of the static and dynamic fluctuations of the growing interface has been lacking so far.

The interface profile, suitably coarse-grained, is described by a height $h(\mathbf{x},t)$. As usual, it is convenient to ignore overhangs so that h is a single-valued function of $\mathbf{x}$. The simplest nonlinear Langevin equation for a local growth of the profile is given by[12]

$$\frac{\partial h}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(\mathbf{x},t). \qquad (1)$$

The first term on the right-hand side describes relaxation of the interface by a surface tension ν. The

PDE stocastiche e distribuzioni

Il rumore bianco $\xi = \xi^\omega$ non è una funzione ma solo una distribuzione (aleatoria)

Non è definito puntualmente $\xi(t,x)$ ma solo "integrato" $\xi(A) = $ "$\displaystyle\int_A \xi(t,x)\,\mathrm{d}t\,\mathrm{d}x$"

- $\xi(A)$ e $\xi(B)$ indipendenti per $A \cap B = \emptyset$
- $\xi(A) \sim \mathcal{N}(0, |A|)$

Non sono ben definite operazioni non lineari sulle distribuzioni (ad es. ξ^2, ξ^3)

Ci aspettiamo la soluzione u *un po' più regolare* di ξ ... ma non abbastanza:

- u distribuzione per $d \geq 2$ $\rightsquigarrow$ u^3 non è definita $\rightsquigarrow$ Φ^4_d è singolare
- ∇u distribuzione per $d \geq 1$ $\rightsquigarrow$ $|\nabla u|^2$ non è definita $\rightsquigarrow$ KPZ_d è singolare

Jona-Lasinio, Mitter (1985) Da Prato, Debussche (2003)

Φ^4_d per $d = 2$: soluzioni deboli e soluzioni forti

Commun. Math. Phys. 101, 409–436 (1985)

Communications in Mathematical Physics

On the Stochastic Quantization of Field Theory

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Abstract. We give a rigorous construction of a stochastic continuum $P(\phi)_2$ model in finite Euclidean space-time volume. It is obtained by a weak solution of a non-linear stochastic differential equation in a space of distributions. The resulting Markov process has continuous sample paths, and is ergodic with the finite volume Euclidean $P(\phi)_2$ measure as its unique invariant measure. The procedure may be called stochastic field quantization.

The Annals of Probability
2003, Vol. 31, No. 4, 1900–1916

STRONG SOLUTIONS TO THE STOCHASTIC QUANTIZATION EQUATIONS

BY GIUSEPPE DA PRATO AND ARNAUD DEBUSSCHE

Scuola Normale Superiore di Pisa and Ecole Normale Supérieure de Cachan

We prove the existence and uniqueness of a strong solution of the stochastic quantization equation in dimension 2 for almost all initial data with respect to the invariant measure. The method is based on a fixed point result in suitable Besov spaces.

1. Introduction. In this article, we consider stochastic quantization equations in space dimension 2 with periodic boundary conditions. These are reaction–diffusion equations driven by a space–time white noise. It is well known that the solution is not expected to be a smooth process and the nonlinear term is modified thanks to a renormalization.

More precisely, let $G = [0, 2\pi]^2$ and $H = L^2(G)$. We are concerned with the equation set

$$\begin{aligned} dX &= \big(AX + :p(X):\big)\,dt + dW(t), \\ X(0) &= x, \end{aligned} \tag{1.1}$$

La mappa soluzione

ODE stocastica (SDE) $\qquad B_t =$ moto browniano in $\mathbb{R}^d$

$$\frac{\mathrm{d}}{\mathrm{d}t} X_t = \sigma(X_t) \frac{\mathrm{d}}{\mathrm{d}t} B_t = \sigma(X_t)\, \xi(t)$$

Forma integrale

$$X_t = X_0 + \underbrace{\int_0^t \sigma(X_u)\, \mathrm{d}B_u}_{\text{integrale stocastico}}$$

- ($d > 1$) La mappa soluzione $B \longmapsto X$ tipicamente è solo misurabile
- ($d = 1$) La mappa soluzione $B \longmapsto X$ è continua $\qquad$ Convergenza loc. uniforme

[Doss (1977), Sussmann (1978)]

Uno sviluppo di Taylor aleatorio

$$X_t = X_0 + \underbrace{\int_0^t \sigma(X_u)\, \mathrm{d}B_u}_{\text{integrale stocastico}} \qquad (\star)$$

Se $\sigma(\cdot)$ è regolare (C^3) possiamo caratterizzare le traiettorie della soluzione X

Teorema (Lyons 1998, Davie 2007)

Per q.o. B, la soluzione di $(\star)$ è l'unica funzione $X : [0, \infty) \to \mathbb{R}^k$ tale che

$$X_t - X_s = \sigma(X_s)(B_t - B_s) + \nabla\sigma(X_s)\,\sigma(X_s) \int_s^t (B_u - B_s) \otimes \mathrm{d}B_u + o(t-s)$$

loc. uniformemente per $0 \leq s < t < \infty$.

Probabilità e Analisi

Riscriviamo la ODE stocastica $(\star)$ come

$$X_t - X_s = \sigma(X_s)\,\mathbb{B}^1_{st} + \nabla\sigma(X_s)\,\sigma(X_s)\,\mathbb{B}^2_{st} + o(t-s) \qquad (\star')$$

dove

$$\mathbb{B}^1_{st} := B_t - B_s \qquad \mathbb{B}^2_{st} := \underbrace{\int_s^t (B_u - B_s) \otimes \mathrm{d}B_u}_{\text{integrale stocastico}}$$

- $\mathbb{B}^2 = (\mathbb{B}^2_{st})$ è un oggetto probabilistico (non canonico: Ito, Stratonovich, ...)
- Assegnato $\mathbb{B}^2$, l'equazione $(\star')$ diventa puramente analitica
- La mappa soluzione $(\mathbb{B}^1, \mathbb{B}^2) \longmapsto X$ è ora continua (topologia Hölder)

Rough paths

Abbiamo fattorizzato la mappa soluzione $B \longmapsto X$ isolandone la parte “singolare”

$$\begin{tikzcd} & (\mathbb{B}^1, \mathbb{B}^2) \arrow[dr, "\substack{\text{(analisi)}\\ \text{continua}}"] & \\ B \arrow[ur, "\substack{\text{(probabiiltà)}\\ \text{misurabile}}"] \arrow[rr, "\text{misurabile}"] & & X \end{tikzcd}$$

La coppia $(\mathbb{B}^1, \mathbb{B}^2)$ è un esempio di rough path (Lyons 1998) (analisi + algebra)

La soluzione X è un esempio di path controllato da B (Gubinelli 2004)

$(\mathbb{B}^1_\varepsilon, \mathbb{B}^2_\varepsilon) \to (\mathbb{B}^1, \mathbb{B}^2)$ convergenza del rough path	$\Longrightarrow$	$X_\varepsilon \to X$ convergenza della soluzione

Lyons (1998)	Gubinelli (2004)

REVISTA MATEMÁTICA IBEROAMERICANA
VOL. 14, N.º 2, 1998

Differential equations driven by rough signals

Terry J. Lyons

1. Preliminaries.

1.1. Introduction.

1.1.1. Inhomogeneous differential equations.

Time inhomogeneous (or non-autonomous) systems of differential equations are often treated rather formally as extensions of the homogeneous (or autonomous) case by adding an extra parameter to the system; however this can be a travesty. Consider an equation of the kind

$$dy_t = \sum_i f^i(y_t)\, dx_t^i \,, \tag{1.1}$$

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Journal of Functional Analysis 216 (2004) 86–140

JOURNAL OF Functional Analysis

http://www.elsevier.com/locate/jfa

Controlling rough paths

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Received 1 October 2003; accepted 23 January 2004

Communicated by D. Stroock

Abstract

We formulate indefinite integration with respect to an irregular function as an algebraic problem which has a unique solution under some analytic constraints. This allows us to define a good notion of integral with respect to irregular paths with Hölder exponent greater than 1/3 (e.g. samples of Brownian motion) and study the problem of the existence, uniqueness and continuity of solution of differential equations driven by such paths. We recover Young's theory of integration and the main results of Lyons' theory of rough paths in Hölder topology.

Rough paths e PDE singolari

Si possono applicare le tecniche dei rough paths alle PDE stocastiche singolari?

SÌ : esistenza, unicità, continuità per KPZ_1 (Hairer 2013)

Annals of Mathematics **178** (2013), 559–664
http://dx.doi.org/10.4007/annals.2013.178.2.4

Solving the KPZ equation

By Martin Hairer

Abstract

We introduce a new concept of solution to the KPZ equation which is shown to extend the classical Cole-Hopf solution. This notion provides a factorisation of the Cole-Hopf solution map into a "universal" measurable map from the probability space into an explicitly described auxiliary metric space, composed with a new solution map that has very good continuity properties. The advantage of such a formulation is that it essentially provides a pathwise notion of a solution, together with a very detailed approximation theory. In particular, our construction completely bypasses the Cole-Hopf transform, thus laying the groundwork for proving that the KPZ equation describes the fluctuations of systems in the KPZ universality class.

$$\big(u(t,x)\big)_{(t,x)\in[0,\infty)\times\mathbb{R}^d}$$

$$\downarrow$$

$$\big(U_t\big)_{t\geq 0} \quad \text{dove} \quad U_t = \big(u(t,x)\big)_{x\in\mathbb{R}^d}$$

Le Strutture di Regolarità

Teoria genuinamente multi-dimensionale: Strutture di Regolarità (Hairer 2014)

Permette di definire un'ampia classe di PDE stocastiche singolari "sotto-critiche"

Φ^4_3 $\quad$ KPZ_1 $\quad$ e molte altre

Per questi contributi Martin Hairer ha ricevuto la Medaglia Fields 2014

Parallelamente sono stati sviluppati approcci alternativi, in particolare (non solo)

- Calcolo Paracontrollato (Gubinelli, Imkeller, Perkowski 2015)

Descriverò alcune delle idee cruciali alla base delle Strutture di Regolarità

Hairer (2014) Gubinelli, Imkeller, Perkowski (2015)

Invent. math. (2014) 198:269–504
DOI 10.1007/s00222-014-0505-4

A theory of regularity structures

M. Hairer

Received: 1 October 2013 / Accepted: 21 January 2014 / Published online: 14 March 2014

Abstract We introduce a new notion of "regularity structure" that provides an algebraic framework allowing to describe functions and/or distributions via a kind of "jet" or local Taylor expansion around each point. The main novel idea is to replace the classical polynomial model which is suitable for describing smooth functions by arbitrary models that are purpose-built for the problem at hand. In particular, this allows to describe the local behaviour not only of functions but also of large classes of distributions. We then build a calculus allowing to perform the various operations (multiplication, composition with smooth functions, integration against singular kernels) necessary to formulate fixed point equations for a very large class of semilinear PDEs driven by some very singular (typically random) input. This allows, for the first time, to give a

Forum of Mathematics, Pi (2015), Vol. 3, e6, 75 pages
doi:10.1017/fmp.2015.2

1

PARACONTROLLED DISTRIBUTIONS AND SINGULAR PDES

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Abstract

We introduce an approach to study certain singular partial differential equations (PDEs) which is based on techniques from paradifferential calculus and on ideas from the theory of controlled rough paths. We illustrate its applicability on some model problems such as differential equations driven by fractional Brownian motion, a fractional Burgers-type stochastic PDE (SPDE) driven by space-time white noise, and a nonlinear version of the parabolic Anderson model with a white noise potential.

1. Arricchire il rumore

Messaggio dei rough paths: "arricchire il rumore" (per guadagnare continuità)

$$B \quad \rightsquigarrow \quad \mathbb{B} = (\mathbb{B}^1_{st}\,,\, \mathbb{B}^2_{st}) = \left(B_t - B_s \,,\, \int_s^t (B_u - B_s)\, \mathrm{d} B_u \right)$$

PDE stocastiche singolari sotto-critiche: numero finito di arricchimenti

$$\text{rumore bianco } \xi \quad \rightsquigarrow \quad \text{“modello” } \boldsymbol{\Xi} = \Big(\underbrace{\boldsymbol{\Xi}^1_{(s,y)}(\cdot)}_{\xi(\cdot)}, \, \boldsymbol{\Xi}^2_{(s,y)}(\cdot) \,,\, \ldots \,,\, \boldsymbol{\Xi}^M_{(s,y)}(\cdot) \Big)$$

$\boldsymbol{\Xi}^i_{(s,y)}(\cdot)$: distribuzioni che descrivono funzioni non-lineari di $\xi(\cdot)$ in prossimità di (s, y)

(costruzione probabilistica non banale)

2. Distribuzioni modellate

Supponiamo di avere il "modello" $\boldsymbol{\Xi} = \big(\boldsymbol{\Xi}^i_{(s,y)}(\cdot)\big)_{1 \le i \le M}$ per la nostra PDE stocastica

Descriviamo la soluzione $u(\cdot)$ con uno "sviluppo di Taylor" rispetto a $\boldsymbol{\Xi}$

$$u(\cdot) \approx \boldsymbol{U}_{(s,y)}(\cdot) = \sum_{i=1}^{M} c_i(s,y)\, \boldsymbol{\Xi}^i_{(s,y)}(\cdot) + \text{polinomio} \qquad \text{vicino a } (s,y)$$

Teorema di Ricostruzione (Hairer 2014)

Imponendo condizioni di "coerenza" sui coefficienti, la famiglia

$$\boldsymbol{U} = \big(\boldsymbol{U}_{(s,y)}(\cdot)\big)_{(s,y)} \qquad \text{“distribuzione modellata”}$$

determina un'unica distribuzione $u(\cdot)$ su $[0,\infty) \times \mathbb{R}^d$.

3. Sollevare e risolvere la PDE

Operazioni non lineari su ξ (ad es. prodotti): descritte dal modello $\boldsymbol{\Xi} = \big(\boldsymbol{\Xi}^i_{(s,y)}(\cdot)\big)$

$\rightsquigarrow$ ben definite per distribuzioni modellate $\boldsymbol{U} = \big(\boldsymbol{U}_{(s,y)}(\cdot)\big)$

Stime di Schauder multi-livello (Hairer 2014)

L'integrazione è ben definita e regolarizzante per distribuzioni modellate $\boldsymbol{U}$

La PDE si "solleva" ed è ben posta

in uno spazio di distribuzioni modellate $\boldsymbol{U}$

$$\boldsymbol{U} = \mathcal{K}\big(\, f(\boldsymbol{U}, \nabla \boldsymbol{U}) + \boldsymbol{\Xi}^1 \,\big)$$

$$\begin{tikzcd}
\boldsymbol{\Xi} \arrow[r, "\substack{\text{(PDE sollevata)} \\ \text{continua}}"] & \boldsymbol{U} \arrow[d, "\substack{\text{(ricostruzione)} \\ \text{continua}}"] \\
\xi \arrow[u, dashed] & u
\end{tikzcd}$$

4. La rinormalizzazione

La parte probabilistica è confinata alla costruzione del modello $\mathbf{\Xi} = \left(\mathbf{\Xi}^i_{(s,y)}(\cdot)\right)_{1\le i\le M}$

Operazioni non lineari su ξ richiedono spesso forme di rinormalizzazione

In concreto: regolarizzando $\xi = \lim_{\varepsilon\downarrow 0} \xi_\varepsilon$ le soluzioni corrispondenti u_ε *non convergono*

Per aver convergenza $u_\varepsilon \to u$ occorre modificare le equazioni

$$\partial_t u_\varepsilon = \Delta_x u_\varepsilon - (u_\varepsilon^3 - c_\varepsilon\, u_\varepsilon) + \xi_\varepsilon \qquad\qquad \partial_t u_\varepsilon = \Delta_x u_\varepsilon + |\nabla u|_\varepsilon^2 - \tilde{c}_\varepsilon + \xi_\varepsilon$$

per opportune costanti $c_\varepsilon, \tilde{c}_\varepsilon \to \infty$

Solo in questo modo il modello converge: $\mathbf{\Xi}_\varepsilon \to \mathbf{\Xi}$

Conclusione

Le Strutture di Regolarità permettono di formulare PDE stocastiche singolari

Per domare le singolarità aleatorie si definisce una nuova nozione di regolarità aleatoria:

sviluppi di Taylor con "monomi aleatori" costruiti sul rumore ξ

La teoria generale, molto bella e complessa, si intreccia con diverse aree matematiche

(algebre di Hopf, analisi numerica, . . .)

I risultati cruciali (Ricostruzione, Schauder) si possono formulare indipendentemente

[C., Zambotti, EMS Survey (2020)] [Broux, C., Zambotti, preprint (2023)]

Le idee profonde possono essere utili in altri ambiti della matematica

Grazie