On the 2D Stochastic Heat Equation

Lecture 1

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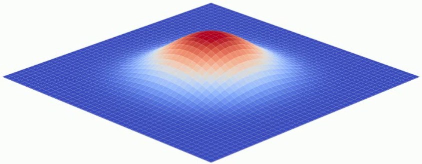
Stochastic dynamics: foundations and applications

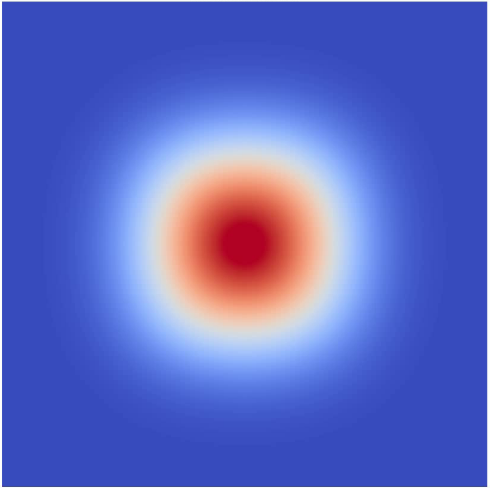
June 29 – July 3, 2026

Heat Equation $\partial_t u(t,x) = \Delta_x u(t,x)$ (video by Q. Berger)

Density after n=14448 steps



Heat map after n=14448 step



This course in a nutshell

Stochastic Heat Equation: ill-defined! multiplicative noise (or "disorder")

$$\partial_t u(t,x) = \underbrace{\Delta_x u(t,x)}_{\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} u(t,x)} + \beta \underbrace{\xi(t,x)}_{\substack{\text{Gaussian "space-time white noise"} \\ \text{covariance } \delta(t-t')\,\delta(x-x')}} \cdot u(t,x)$$

Construct and study a "non-trivial solution" in critical space dimension $d = 2$

Our goal: Critical 2D Stochastic Heat Flow (SHF)

Singular SPDEs $\leftrightsquigarrow$ Statistical Mechanics

Plan of the course

This lecture: **general overview**

- Stochastic Heat Equation
- Construction and properties of SHF

Next lectures:

- Link with Statistical Mechanics (directed polymers)
- Selected topics on the SHF

Lecture notes

THE CRITICAL $2d$ STOCHASTIC HEAT FLOW AND RELATED MODELS

FRANCESCO CARAVENNA, RONGFENG SUN, AND NIKOS ZYGOURAS

ABSTRACT. In these lecture notes, we review recent progress in the study of the stochastic heat equation and its discrete analogue, the directed polymer model, in spatial dimension 2. It was discovered that a phase transition emerges on an intermediate disorder scale, with Edwards-Wilkinson (Gaussian) fluctuations in the sub-critical regime. In the critical window, a unique scaling limit has been identified and named the *critical* $2d$ *stochastic heat flow*. This gives a meaning to the solution of the stochastic heat equation in the critical dimension 2, which lies beyond existing solution theories for singular SPDEs. We outline the proof ideas, introduce the key ingredients, and discuss related literature on disordered systems and singular SPDEs. A list of open questions is also provided.

`https://arxiv.org/abs/2412.10311`

Nikos Zygouras (Warwick) Rongfeng Sun (NUS)

Why is it interesting?

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, \xi(t,x) \cdot u(t,x)$$

- Fundamental PDE $+$ universal random potential

- KPZ equation [Kardar–Parisi–Zhang 86]

$$\partial_t h(t,x) = \Delta_x h(t,x) + |\nabla_x h(t,x)|^2 + \beta\, \xi(t,x)$$

Cole–Hopf: $h(t,x) = \log u(t,x)$

Why is it difficult?

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, \xi(t,x) \cdot u(t,x)$$

$\xi(t,x)$ is a distribution $\rightsquigarrow$ u(t,x) expected (additive noise) $\begin{cases} \text{non-smooth funct.} & d = 1 \\ \text{distribution} & d \geq 2 \end{cases}$

Ill-defined product $\xi(t,x) \cdot u(t,x)$ $\rightsquigarrow$ Singular SPDEs

How to make sense of the equation?

Stochastic integral

Duhamel's principle: integral formulation ("mild") $\quad g_t(x) = \text{heat kernel}$

$$u(t,x) = \int_{\mathbb{R}^d} g_t(x-z)\, \underbrace{\psi_0(z)}_{u(0,\cdot)=\psi_0(\cdot)}\, \mathrm{d}z + \iint_{(0,t)\times\mathbb{R}^d} g_{t-s}(x-y)\, u(s,y)\, \beta\, \xi(s,y)\, \mathrm{d}s\, \mathrm{d}y$$

Works for $d = 1$

- stochastic integral [Ito/Walsh] [Da Prato–Zabczyk]
- well-posedness, positivity [Chen–Dalang 15] [Müller 91]

Breaks down for $d \geq 2$ $\quad g_t(x) \notin L^2(\mathrm{d}t, \mathrm{d}x)$

Pathwise stochastic analysis

Revolution in 2010s: robust solution theories for **subcritical singular SPDEs**

- Regularity Structures [Hairer]
- Paracontrolled Distributions [Gubinelli–Imkeller–Perkowski]
- Energy Solutions [Gonçalves–Jara] [Gubinelli–Perkowski]
- Renormalization Group Approach [Kupiainen, Duch]

Stochastic Heat Equation and KPZ are subcritical only for $d = 1$

These approaches do not apply for $d \geq 2$

The role of dimension

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, \xi(t,x)\, u(t,x)$$

Space-time blow-up $\tilde{u}(t,x) := u(\delta^2 t, \delta x)$

$$\partial_t \tilde{u}(t,x) = \Delta_x \tilde{u}(t,x) + \delta^{\frac{2-d}{2}}\, \beta\, \tilde{\xi}(t,x)\, \tilde{u}(t,x)$$

As $\delta \downarrow 0$ the noise strength $\begin{cases} \text{vanishes} & d < 2 \quad \text{sub-critical} \\ \text{unchanged} & d = 2 \quad \textbf{critical} \\ \text{diverges} & d > 2 \quad \text{super-critical} \end{cases}$

We will focus on the **critical dimension** $d = 2$

What can we do?

Regularized noise $\xi_\varepsilon(t,x)$ on scale $\varepsilon > 0$ (discretization, mollification, ...)

Well-defined equation
$$\begin{cases} \partial_t u_\varepsilon(t,x) = \Delta_x u_\varepsilon(t,x) + \beta\, \xi_\varepsilon(t,x)\, u_\varepsilon(t,x) \\ u_\varepsilon(0,x) = \psi_0(x) \geq 0 \in C_b \qquad (\text{e.g. } \psi_0 \equiv 1) \end{cases}$$

Fix $\xi_\varepsilon(t,x) \xrightarrow[\varepsilon \downarrow 0]{d} \xi(t,x)$ Does the solution u_ε converge?

The answer depends subtly on the noise strength β

Which notion of convergence?

Space-average $$u_\varepsilon(t,\varphi) := \int_{\mathbb{R}^2} \varphi(x)\, u_\varepsilon(t,x)\,\mathrm{d}x \xrightarrow[\varepsilon\downarrow 0]{d} \mathscr{U}(t,\varphi) \qquad \varphi \in C_c$$

$$\Longleftrightarrow \qquad u_\varepsilon(t,x)\,\mathrm{d}x \xrightarrow[\varepsilon\downarrow 0]{d} \mathscr{U}(t,\mathrm{d}x)$$

vague convergence of random measures

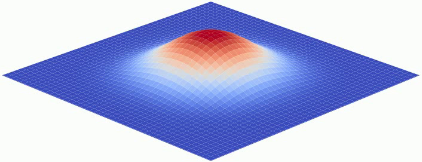
Theorem [Berger–C.–Turchi 25]

Any fixed noise strength β: trivial limit $\mathscr{U} =$ *non random*

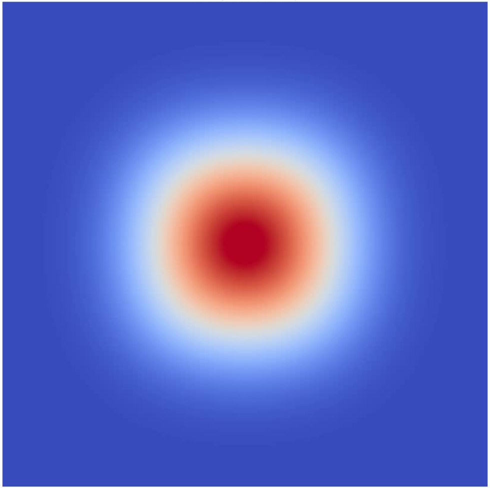
$$u_\varepsilon(t,\cdot) \xrightarrow[\varepsilon\downarrow 0]{} \begin{cases} \text{solution of heat equation} & \text{if } \beta = 0 \\ 0 & \text{if } \beta > 0 \end{cases}$$

Discretization: $\beta = 0$ (by Q. Berger)

Density after n=14448 steps

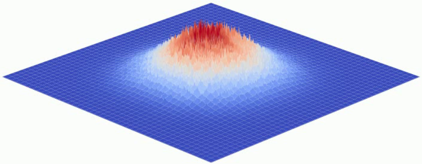


Heat map after n=14448 step

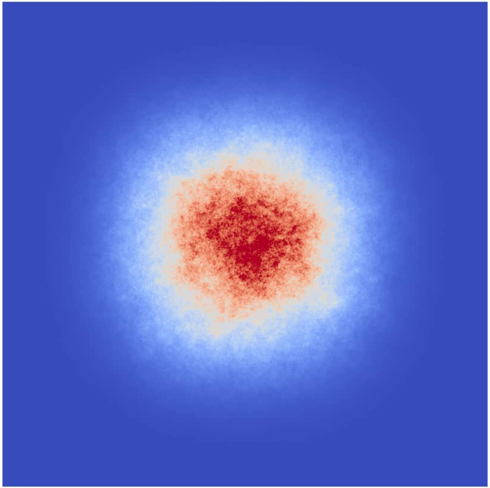


Discretization: β very small (by Q. Berger)

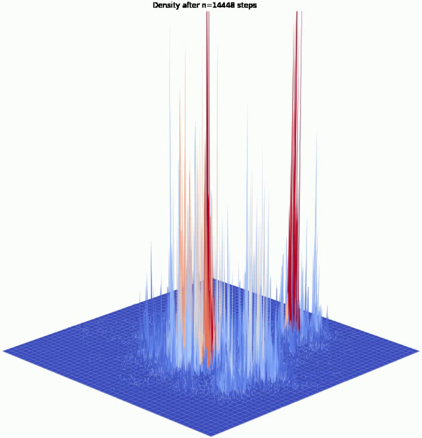
Density after n=14448 steps

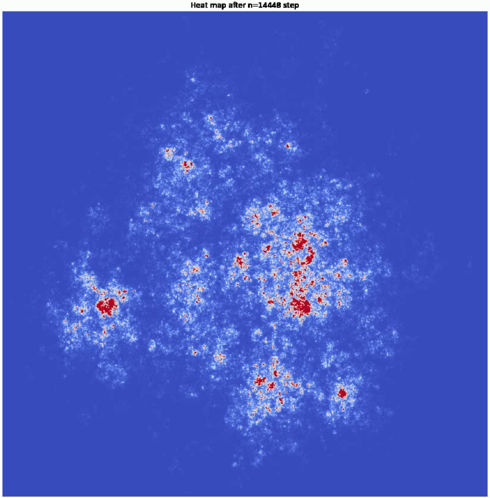


Heat map after n=14448 step



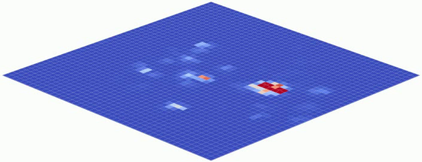
Discretization: β “critical” (by Q. Berger)





Discretization: β "supercritical" (by Q. Berger)

Density after n=14448 steps



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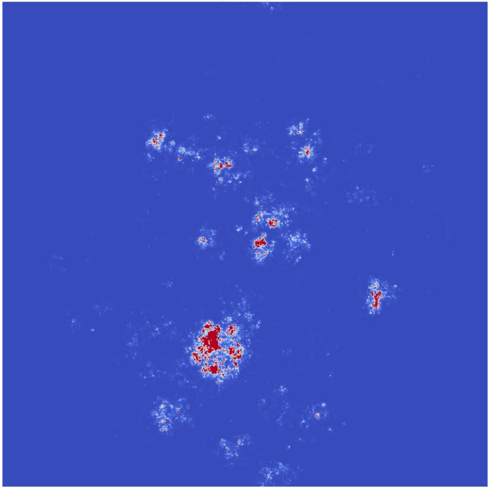


The need for renormalization

Critical scaling

($\rightsquigarrow$ phase transition)

$$\beta = \beta_\varepsilon = \frac{\hat\beta_{\mathrm{crit}}}{\sqrt{|\log \varepsilon|}}$$

our discretization

$$\hat\beta_{\mathrm{crit}} = \sqrt{\pi}$$

Not at all obvious why we should take $\beta = \beta_\varepsilon$ $\rightsquigarrow$ Lecture 2

- First moment $\mathbb{E}[u_\varepsilon(t,\cdot)] = \int_{\mathbb{R}^2} g_t(\cdot - z)\, \psi_0(z)\, \mathrm{d}z \equiv 1$ if $\psi_0 \equiv 1$

- Second moment diverges for any fixed $\beta > 0$ $\mathbb{E}\big[u_\varepsilon(t,\varphi)^2\big] \xrightarrow[\varepsilon\downarrow 0]{} \infty$

  It converges for $\beta = \beta_\varepsilon$ with $\hat\beta_{\mathrm{crit}} = \sqrt{\pi}$

The need for renormalization

Critical window

($\rightsquigarrow$ phase transition)

$$\beta = \beta_\varepsilon = \frac{\hat{\beta}_{\text{crit}}}{\sqrt{|\log \varepsilon|}} \left(1 + \frac{\vartheta}{|\log \varepsilon|}\right)$$

our discretization

$$\hat{\beta}_{\text{crit}} = \sqrt{\pi}$$

Noise strength vanishes as $\varepsilon \downarrow 0$. **The effect of noise persists** (tuned by ϑ)

- $\mathbb{V}\mathrm{ar}\big[u_\varepsilon(t,\varphi) \big] \xrightarrow[\varepsilon \downarrow 0]{} K_t^{\vartheta}(\varphi,\varphi) > 0$ [Bertini–Cancrini 98] [CSZ 19]
- All moments converge [CSZ 19] [Gu–Quastel–Tsai 21]

$$\mathbb{E}\big[u_\varepsilon(t,\varphi)^q \big] \geq \exp(\exp(c\, q))$$

[Ganguly–Nam 25]

Main result

Theorem [CSZ 23]

Discretize the Stochastic Heat Equation on the lattice $\varepsilon\mathbb{N} \times \sqrt{\varepsilon}\,\mathbb{Z}^2$

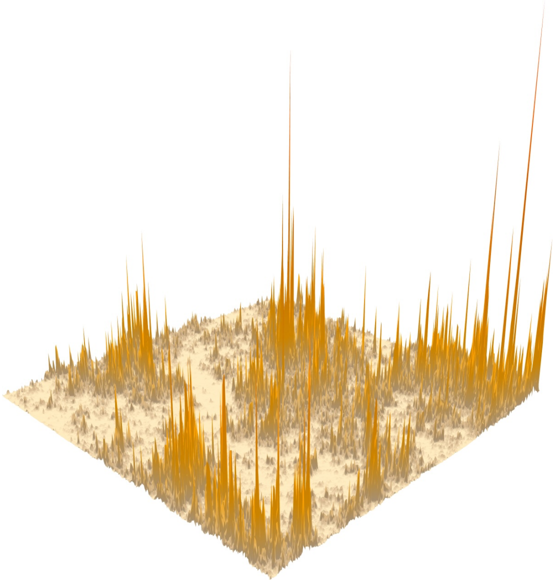
Rescale $$\beta_\varepsilon = \frac{\sqrt{\pi}}{\sqrt{|\log \varepsilon|}} \left(1 + \frac{\vartheta}{|\log \varepsilon|} \right) \qquad \vartheta \in \mathbb{R}$$

The solution u_ε converges to a unique and non-trivial limit SHF^ϑ

$$u_\varepsilon = \big(u_\varepsilon(t,x)\,\mathrm{d}x \big)_{t \ge 0} \xrightarrow[\varepsilon \downarrow 0]{d} \mathrm{SHF}^\vartheta = \big(\mathrm{SHF}^\vartheta(t, \mathrm{d}x) \big)_{t \ge 0}$$

Measure-valued process on $\mathbb{R}^2$ Critical 2D **Stochastic Heat Flow**

The critical 2D Stochastic Heat Flow



Techniques and Ideas

Subsequential limits of $(u_\varepsilon)_{\varepsilon>0}$ exist by tightness   via moments bounds

Challenge is uniqueness   lacking characterization of the limit

What we proved

$(u_\varepsilon)_{\varepsilon>0}$ is Cauchy $u_\varepsilon \approx u_{\varepsilon'}$ for small $\varepsilon, \varepsilon' > 0$

- Coarse-graining   approximate self-similarity
- Lindeberg priciple   robustness to microscopic details

These ideas have played a key role $\rightsquigarrow$ Lectures 2 and 3

Plan of the course

We have introduced the main character $\mathrm{SHF}^{\vartheta}(t, \mathrm{d}x)$ of this course

In the next lectures we will try to understand it

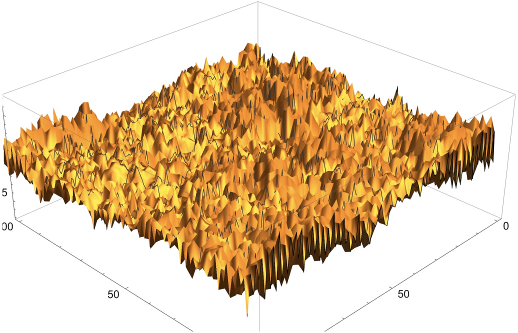
- Lecture 2. Link to directed polymers + critical scaling of $\beta = \beta_\varepsilon$
- Lecture 3. Universality features of the SHF
- Lecture 4. Refined properties of the SHF

Rest of this lecture: up-to-date account on the SHF

How does the SHF look like?

Some not-so-randomly picked realizations

[M. Mucciconi, N. Zygouras]



KPZ $\approx \log u_\varepsilon(t, x)$

Alps, Italy

(by Mg-k)

Universality of SHF

We constructed SHF$^{\vartheta}$ via discretization of the Stochastic Heat Equation

Universality: other regularizations should also lead to SHF$^{\vartheta}$

Theorem [Tsai 24]

Stochastic Heat Equation with mollified noise $\xi_\varepsilon = \xi * \rho_\varepsilon$

Rescale $$\beta_\varepsilon = \frac{\sqrt{2\pi}}{\sqrt{|\log \varepsilon|}} \left(1 + \frac{\vartheta + c}{|\log \varepsilon|}\right) \qquad \vartheta \in \mathbb{R}$$

The solution u_ε converges to SHF$^{\vartheta}$ as $\varepsilon \downarrow 0$

$\rightsquigarrow$ Far reaching universality [Drillick, Hou, Parekh 26+]

SHF with two space-time parameters

SHF $=$ limit of solutions of renormalized 2D Stochastic Heat Equation

$$\mathrm{SHF}^{\vartheta}(t, \mathrm{d}x) \qquad \mathrm{SHF}^{\vartheta}(0, \psi_0\,;\, t, \mathrm{d}x) = \int_{\mathbb{R}^2} \psi_0(y)\, \mathrm{SHF}^{\vartheta}(0, \mathrm{d}y\,;\, t, \mathrm{d}x)$$

$$\rightsquigarrow \quad \text{2 space-time parameters} \quad \big(\mathrm{SHF}^{\vartheta}(s, \mathrm{d}y\,;\, t, \mathrm{d}x)\big)_{0 \le s \le t < \infty}$$

Flow property/Chapman–Kolmogorov: for $s < t < u$ [Clark–Mian 24]

$$\mathrm{SHF}^{\vartheta}(s, \mathrm{d}y\,;\, u, \mathrm{d}z) = \int_{x \in \mathbb{R}^2} \mathrm{SHF}^{\vartheta}(s, \mathrm{d}y\,;\, \underbrace{t, \mathrm{d}x)\, \mathrm{SHF}^{\vartheta}(t, \mathrm{d}x}_{\text{non-trivial "product" of measures}}\,;\, u, \mathrm{d}z)$$

Characterization of SHF

Theorem [Tsai 24]

Let $\mathscr{Z} = \big(\mathscr{Z}_{s,t}(\mathrm{d}x, \mathrm{d}y)\big)_{s \le t}$ be a stochastic process on $\mathscr{M}_+(\mathbb{R}^2 \times \mathbb{R}^2)$ satisfying

- continuity of $(s,t) \mapsto \mathscr{Z}_{s,t}$
- independence of $\mathscr{Z}_{s_1,t_1}\,,\ \mathscr{Z}_{s_2,t_2}\,,\ \ldots,\ \mathscr{Z}_{s_k,t_k}$ $\qquad \forall s_1 < t_1 < \ldots < s_k < t_k$
- flow property (Chapman-Kolmogorov) $\mathscr{Z}_{s,u} = \mathscr{Z}_{s,t} \bullet \mathscr{Z}_{t,u}$ $\qquad \forall s < t < u$
- same moments $\mathbb{E}\big[\prod_{i=1}^n \mathscr{Z}_{s,t}(\varphi_i, \psi_i)\big]$ as SHF^{ϑ} up to $n = 4$

Then

$$\mathscr{Z} \stackrel{d}{=} \mathrm{SHF}^{\vartheta}$$

SHF and noise

Stochastic Heat Equation with regularized noise (discretized/mollified)

$$\partial_t u_\varepsilon(t,x) = \Delta_x u_\varepsilon(t,x) + \beta_\varepsilon\, u_\varepsilon(t,x)\, \xi_\varepsilon(t,x)$$

As $\varepsilon \downarrow 0$ $\qquad u_\varepsilon \longrightarrow \mathrm{SHF}^\vartheta \qquad \xi_\varepsilon \longrightarrow \xi \qquad \beta_\varepsilon = \beta_\varepsilon(\vartheta) \longrightarrow 0$

Theorem [C.–Donadini 25] [Gu–Tsai 25]

$$(u_\varepsilon\, ,\, \xi_\varepsilon) \xrightarrow[\varepsilon\downarrow 0]{d} (\mathrm{SHF}^\vartheta\, ,\, \xi)$$ with SHF^ϑ and ξ **independent**

$\rightsquigarrow$ **No SPDE** driven by ξ for SHF^ϑ

- Noise sensitivity [C.–Donadini 25]
- Black noise [Gu–Tsai 25]

Basic Properties of SHF

- $\mathrm{SHF}^{\vartheta}(t, \mathrm{d}x) \neq$ Gaussian (positive random measure)

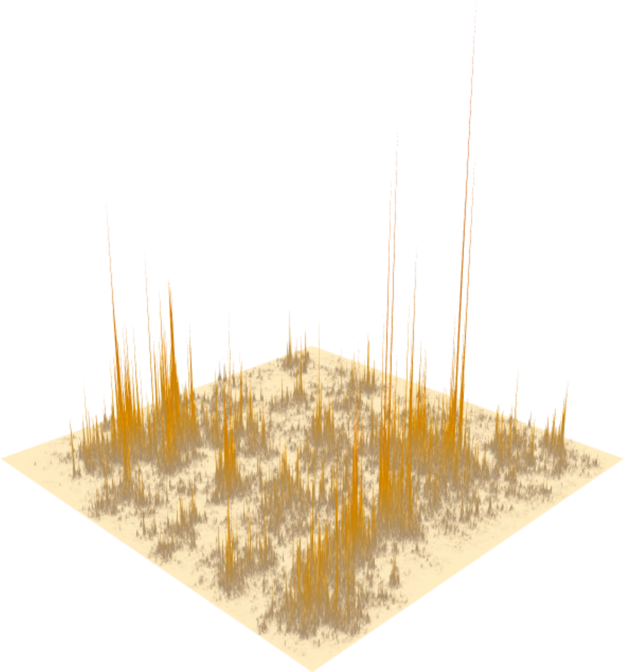
- $\mathrm{SHF}^{\vartheta}(t, \mathrm{d}x) \neq$ GMC $\simeq \exp(\text{Gaussian})$ [CSZ 23]

  Conditional GMC structure on path space [Clark–Tsai 25]

- Moment formulas [CSZ 19] [Gu–Quastel–Tsai 21] [Ganguly–Nam 25]

  $\rightsquigarrow$ [Cosco, Nakajima, Zeitouni] [Lygkonis, Zygouras] [Liu, Zygouras] . . .

- Scaling covariance $$\frac{\mathrm{SHF}^{\vartheta}(\alpha\, t\,,\, \mathrm{d}(\sqrt{\alpha}\, x))}{\alpha} \stackrel{d}{=} \mathrm{SHF}^{\vartheta + \log \alpha}(t\,,\, \mathrm{d}x)$$



Singularity and Regularity

- $\mathrm{SHF}^{\vartheta}(t, \mathrm{d}x)$ singular w.r.t. Lebesgue [CSZ 25+]
  "not a function"

- $\mathrm{SHF}^{\vartheta}(t, \mathrm{d}x) \in \mathscr{C}^{-\kappa}$ for any $\kappa > 0$ $\rightsquigarrow$ non atomic
  "*barely* not a function"

- Small ball $\mathrm{SHF}^{\vartheta}(1, g_{\varepsilon^2}) = |\log \varepsilon|^{-1/2+o(1)}$ [Gu–Tsai 26]
  $+$ Gaussian fluctuations

- Strict positivity $\mathrm{SHF}^{\vartheta}(t, B) > 0 \;\; \forall$ ball B [Clark–Tsai 25] [Nakashima 25]

More

- **Strong disorder** $\quad \mathrm{SHF}^{\vartheta}(t,B) \xrightarrow[\vartheta\to\infty]{d} 0 \quad \forall$ ball B [Clark–Tsai 25]

  $+$ sharp rates [Berger.–C.–Turchi 25]

- **Weak disorder** $\quad \mathrm{SHF}^{\vartheta}(t,\mathrm{d}x) \xrightarrow[\vartheta\to-\infty]{d} \mathrm{d}x$ for $\psi_0 \equiv 1$

  $+$ Gaussian fluctuations (Edwards-Wilkinson) [C.–Cottini–Rossi 25]

- **Martingale problem** [Nakashima 25] [Chen 25]

- **Many questions still open!**

  Fractal properties Superdiffusivity KPZ = "log SHF" ...