On the 2D Stochastic Heat Equation

Lecture 2

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Stochastic dynamics: foundations and applications

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Reminders from Lecture 1

Stochastic Heat Equation: ill-defined *multiplicative* noise

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta \underbrace{\xi(t,x)}_{\text{Gaussian "space-time white noise"}} \cdot u(t,x)$$

Regularized noise $\xi_\varepsilon(t,x)$ on scale $\varepsilon > 0$ (discretization, mollification, ...)

Well-defined equation
$$\begin{cases} \partial_t u_\varepsilon(t,x) = \Delta_x u_\varepsilon(t,x) + \beta\, \xi_\varepsilon(t,x)\, u_\varepsilon(t,x) \\ u_\varepsilon(0,x) = \psi_0(x) \geq 0 \in C_b \qquad (\text{e.g. } \psi_0 \equiv 1) \end{cases}$$

Fix $\xi_\varepsilon(t,x) \xrightarrow[\varepsilon \downarrow 0]{d} \xi(t,x)$ Does the solution u_ε converge?

Reminders from Lecture 1

Space-average $$u_\varepsilon(t,\varphi) := \int_{\mathbb{R}^2} \varphi(x)\, u_\varepsilon(t,x)\, \mathrm{d}x$$ $\varphi \in C_c$

Critical window

($\rightsquigarrow$ phase transition)

$$\beta = \beta_\varepsilon = \frac{\hat\beta_{\mathrm{crit}}}{\sqrt{|\log \varepsilon|}} \left(1 + \frac{\vartheta}{|\log \varepsilon|} \right)$$ $\vartheta \in \mathbb{R}$

our discretization

$$\hat\beta_{\mathrm{crit}} = \sqrt{\pi}$$

Noise strength vanishes as $\varepsilon \downarrow 0$, yet the effect of noise persists (tuned by ϑ)

This lecture: for $\beta = \beta_\varepsilon$ [Bertini–Cancrini 98] [CSZ 19]

$$\mathbb{V}\mathrm{ar}\big[u_\varepsilon(t,\varphi) \big] \xrightarrow[\varepsilon \downarrow 0]{} K_t^\vartheta(\varphi,\varphi) > 0$$

Reminders from Lecture 1

This is the first key step toward our construction of the SHF

Theorem [CSZ 23]

For $\beta = \beta_\varepsilon$ the solution u_ε has a non-trivial limit

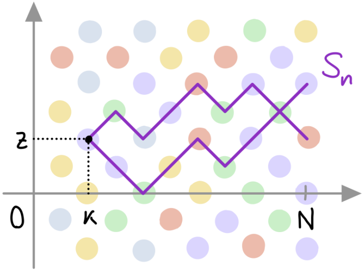
$$u_\varepsilon = \big(u_\varepsilon(t,x)\, \mathrm{d}x \big)_{t \ge 0} \xrightarrow[\varepsilon \downarrow 0]{d} \mathrm{SHF}^{\vartheta} = \big(\mathrm{SHF}^{\vartheta}(t, \mathrm{d}x) \big)_{t \ge 0}$$

Measure-valued process on $\mathbb{R}^2$ Critical 2D **Stochastic Heat Flow**

We are going to exploit a link between $u_\varepsilon(t,\varphi)$ and directed polymers

Directed Polymer in Random Environment

- Simple random walk on $\mathbb{Z}^2$ $\qquad S = (S_n)$
- Independent Gaussians $\qquad \omega(n,x) \sim \mathcal{N}(0,1)$
- Interaction $\qquad H_N(S,\omega) = \sum_{n \le N} \omega(n, S_n)$



Polymer measure $\quad \mathrm{P}_N(S) = \dfrac{\mathrm{e}^{\beta H_N(S,\omega)}}{Z_N} \mathrm{P}^{\mathrm{SRW}}(S)$

Partition Functions $\qquad Z_N(k,z) = \mathrm{E}^{\mathrm{SRW}}_{(k,z)}\left[\mathrm{e}^{\beta H_N(S,\omega)}\right] \,/\, \mathbb{E}[\ldots]$

Stochastic Heat Equation and Feynman–Kac

We show that partition functions solve a discretized Stochastic Heat Equation

Heuristics

Discretized Feynman–Kac $\qquad Z_N = \mathrm{E}\left[\mathrm{e}^{\beta H_N(S,\omega)} \right] \approx \mathrm{E}\left[\mathrm{e}^{\beta \int \xi(s,B_s)\,\mathrm{d}s} \right]$

Morally $\qquad -\partial_k Z_N(k,z) = \Delta_z Z_N(k,z) + \left(\mathrm{e}^{\beta\omega(k,z)} - 1 \right) Z_N(k,z)$

More precisely: discretized Duhamel's formula

Duhamel's formula

Recall Duhamel's formula (mild formulation) for Stochastic Heat Equation

$$g_t(x) = \text{heat kernel on } \mathbb{R}^2 = \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi)\,t} \qquad \text{Take } \psi_0 \equiv 1$$

$$u(t,x) = 1 + \iint_{(0,t)\times\mathbb{R}^d} g_{t-s}(x-y)\,\beta\,\xi(s,y)\,u(s,y)\,\mathrm{d}s\,\mathrm{d}y = 1 + \big(g * (\beta\,\xi\,u)\big)(t,x)$$

Ill-defined in dimensions $d \geq 2$ because $g_t(x) \notin L^2_{\mathrm{loc}}(\mathrm{d}t, \mathrm{d}x)$

Partition functions satisfy a discretized Duhamel's formula + time reversal

Partition functions and Stochastic Heat Equation

$$Z_N(k,z) = \mathrm{E}^{\mathrm{SRW}}_{(k,z)}\Big[e^{\beta H_N(S,\omega)} \Big] = \mathrm{E}^{\mathrm{SRW}}_{(k,z)}\Big[e^{\beta \sum_{n=k+1}^{N} \omega(n,S_n)} \Big]$$

Lemma

Define

$$q_\ell(y) := \mathrm{P}^{\mathrm{SRW}}(S_\ell = y) \qquad X(\ell,y) := e^{\beta\,\omega(\ell,y)} - 1$$

Then

$$Z_N(k,z) = 1 + \sum_{\ell\in(k,N]} \underbrace{\sum_{y\in\mathbb{Z}^2} q_{\ell-k}(y-z)\, X(\ell,y)\, Z_N(\ell,y)}_{\mathrm{E}^{\mathrm{SRW}}_{(k,z)}\left[X(\ell,S_\ell)\, Z_N(\ell,S_\ell) \right]}$$

Proof of the Lemma

$$Z_N(k,z) = \mathrm{E}^{\mathrm{SRW}}_{(k,z)}\left[\mathrm{e}^{\beta \sum_{n=k+1}^{N} \omega(n,S_n)} \right] = \mathrm{E}^{\mathrm{SRW}}_{(k,z)}\left[\prod_{n=k+1}^{N} \underbrace{\mathrm{e}^{\beta \omega(n,S_n)}}_{a_n} \right]$$

Expansion with $\delta_n := a_n - 1$ $\qquad$ $\ell :=$ first index n where we "pick" δ_n

$$\prod_{n=k+1}^{N} a_n = \prod_{n=k+1}^{N} \{1+\delta_n\} = 1 + \sum_{\ell=k+1}^{N} \delta_\ell \prod_{n=\ell+1}^{N} \underbrace{\{1+\delta_n\}}_{a_n}$$

$$Z_N(k,z) = 1 + \sum_{\ell=k+1}^{N} \mathrm{E}^{\mathrm{SRW}}_{(k,z)}\left[\underbrace{\left(\mathrm{e}^{\beta \omega(\ell,S_\ell)} - 1\right)}_{X(\ell,S_\ell)} \underbrace{\prod_{n=\ell+1}^{N} \mathrm{e}^{\beta \omega(n,S_n)}}_{Z_N(\ell,S_\ell)} \right] \qquad \square$$

Discretized Stochastic Heat Equation

Partition functions solve discretized Stochastic Heat Equation (time reversal)

$$Z_N(N-t,x) = u_1(t,x) \qquad (t,x) \in \mathbb{N} \times \mathbb{Z}^2$$

Diffusive rescaling by N $\rightsquigarrow$ Discretization on time scale $\varepsilon = \frac{1}{N}$

$$Z_N\big(N - N\,t,\, \sqrt{N}\,x\big) = u_\varepsilon(t,x) \qquad (t,x) \in \varepsilon\,\mathbb{N} \times \sqrt{\varepsilon}\,\mathbb{Z}^2$$

Henceforth we focus on $u_\varepsilon(t=1,x)$ (polymers starting at time zero)

$$Z_N(\sqrt{N}\,x) = \mathrm{E}^{\mathrm{SRW}}\Big[\mathrm{e}^{\beta \sum_{n=1}^N \omega(n,S_n) - \frac{1}{2}\beta^2 N} \,\Big|\, S_0 = \sqrt{N}\,x\Big] \qquad \mathbb{E}\big[Z_N(\cdot)\big] = 1$$

Switching to polymer language

Recall $$u_\varepsilon(1,\varphi) = \int_{\mathbb{R}^2} u_\varepsilon(1,x)\,\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^2} Z_N(\sqrt{N}\,x)\,\varphi(x)\,\mathrm{d}x$$

$$\mathbb{E}\big[u_\varepsilon(1,\varphi)\big] = \int_{\mathbb{R}^2} \mathbb{E}\big[Z_N(\sqrt{N}\,x)\big]\,\varphi(x)\,\mathrm{d}x = 1 \qquad \varphi \in C_c(\mathbb{R}^2) \text{ density}$$

$$\mathbb{V}\mathrm{ar}\big[u_\varepsilon(1,\varphi)\big] = \iint_{\mathbb{R}^2\times\mathbb{R}^2} \mathbb{C}\mathrm{ov}\big[Z_N(\sqrt{N}\,x)\,,\,Z_N(\sqrt{N}\,y)\big]\,\varphi(x)\,\varphi(y)\,\mathrm{d}x\,\mathrm{d}y = ?$$

Recall our goal:

$$\mathbb{V}\mathrm{ar}\big[u_\varepsilon(1,\varphi)\big] \xrightarrow[\varepsilon\downarrow 0]{} K_\varphi \in (0,\infty)$$ for critical rescaling $\beta = \beta_\varepsilon$

Variance estimates

We first control the variance $\qquad \mathbb{V}\mathrm{ar}\big[Z_{N,\beta}(\sqrt{N}x)\big] = \mathbb{V}\mathrm{ar}\big[Z_{N,\beta}(0)\big]$

Critical rescaling $\qquad \beta_N = \dfrac{\hat{\beta}\sqrt{\pi}}{\sqrt{\log N}} \qquad$ for $\hat{\beta} \in (0,\infty)$

Proposition 1

As $N \to \infty$

$$\mathbb{V}\mathrm{ar}\big[Z_{N,\beta_N}(0)\big] \begin{cases} \to \frac{\hat{\beta}^2}{1-\hat{\beta}^2} & \text{if } \hat{\beta} < 1 \\ \sim c \log N & \text{if } \hat{\beta} = 1 \\ \gg \log N & \text{if } \hat{\beta} > 1 \end{cases}$$

Covariance estimates

We then control covariances on the diffusive scale which yield our goal

Proposition 2

$$\mathbb{C}\mathrm{ov}\big[Z_{N,\beta}(\sqrt{N}x)\,,\,Z_{N,\beta}(\sqrt{N}y)\big] \underset{N\to\infty}{\sim} \Phi(x-y)\,\beta^2\,\big\{1+\mathbb{V}\mathrm{ar}\big[Z_{N,\beta}(0)\big]\big\}$$

$$\underset{\beta=\beta_N}{\sim} \Phi(x-y)\,\frac{\hat{\beta}^2\,\pi}{\log N}\,\big\{1+\mathbb{V}\mathrm{ar}\big[Z_{N,\beta_N}(0)\big]\big\}$$

$$\underset{\hat{\beta}=1}{\sim} \Phi(x-y)\,\pi\,c < \infty \qquad \text{for } x \neq y$$

$$\Phi(x-y) := \int_0^1 g_t(x-y)\,\mathrm{d}t \sim \tfrac{1}{2\pi}\log\tfrac{1}{|x-y|} \quad \text{as } |x-y|\to 0 \qquad \Phi \in L^1_{\mathrm{loc}}$$

Plan for today

In the so-called subctitical regime

$$\beta_N = \frac{\hat{\beta}\,\sqrt{\pi}}{\sqrt{\log N}} \qquad \text{for} \quad \beta \in (0,1)$$

the variance has a finite limit

$$\mathbb{V}\mathrm{ar}\big[\, Z_{N,\beta_N}(0) \,\big] \xrightarrow[N\to\infty]{} \frac{\hat{\beta}^2}{1-\hat{\beta}^2}$$

We give an explanation based on intersections of 2D random walks

Key random walk computations

Random walk kernel $\qquad q_n(z) \,=\, \mathrm{P}^{\mathrm{SRW}}(S_n = z) \sim \frac{1}{n}\, g_{\frac{1}{2}}\!\left(\frac{z}{\sqrt{n}}\right) 2\, \mathbb{1}_{\mathrm{parity}(n,z)}$

- $$\sum_{z\in\mathbb{Z}^2} q_n(z)^2 \,=\, \sum_{z\in\mathbb{Z}^2} q_n(z)\, q_n(-z) \,=\, q_{2n}(0) \,\sim\, \frac{1}{\pi\, n}$$
- $$R_N \,:=\, \sum_{n=1}^{N} \sum_{z\in\mathbb{Z}^2} q_n(z)^2 \,\sim\, \sum_{n=1}^{N} \frac{1}{\pi\, n} \,\sim\, \frac{\log N}{\pi}$$

R_N = "expected replica overlap" of two independent random walks S, S'

$$R_N \,=\, \sum_{n=1}^{N} \sum_{z\in\mathbb{Z}^2} \mathrm{P}(S_n = z\,,\, S'_n = z) \,=\, \sum_{n=1}^{N} \mathrm{P}(S_n = S'_n) \,=\, \mathrm{E}\Bigg[\underbrace{\sum_{n=1}^{N} \mathbb{1}_{S_n = S'_n}}_{\mathscr{L}_N}\Bigg]$$

Overlap of 2D random walks

We saw that $\mathscr{L}_N := \sum_{n=1}^{N} \mathbb{1}_{S_n = S'_n}$ has expectation $\frac{1}{\pi} \log N$

Theorem [Erdös–Taylor 60]

$$\frac{\mathscr{L}_N}{\frac{1}{\pi} \log N} \xrightarrow[N\to\infty]{d} \mathrm{Exp}(1)$$

Intersection / return time

$$T_1 := \min\{n > 0 : S_n = S'_n\} \stackrel{d}{=} \min\{n > 0 : S_{2n} = 0\}$$

Very heavy tail: $\mathrm{P}(T_1 > t) \sim \frac{c}{\log t}$ as $t \to \infty$

Second moment and overlap

Recall $$Z_N(z) = \mathrm{E}_z\left[\mathrm{e}^{\beta H_N(S,\omega)}\right] = \mathrm{E}\left[\mathrm{e}^{\sum_{n=1}^N \{\beta\,\omega(n,S_n) - \frac{1}{2}\beta^2\}} \,\middle|\, S_0 = z\right]$$

Replica trick $$Z_N(0)^2 = \mathrm{E}^{\otimes 2}\left[\mathrm{e}^{\sum_{n=1}^N \{\beta\,\omega(n,S_n) + \beta\,\omega(n,S'_n) - \beta^2\}}\right]$$

Computation $$\mathbb{E}\left[\mathrm{e}^{\beta\,\omega(n,s) + \beta\,\omega(n,s')}\right] = \mathrm{e}^{\beta^2\,\mathbb{1}_{s\neq s'} + 2\beta^2\,\mathbb{1}_{s=s'}}$$

Finally $$\mathbb{E}\left[Z_N(0)^2\right] = \mathrm{E}^{\otimes 2}\left[\mathrm{e}^{\sum_{n=1}^N \beta^2\,\mathbb{1}_{S_n=S'_n}}\right] = \mathrm{E}^{\otimes 2}\left[\mathrm{e}^{\beta^2\,\mathscr{L}_N}\right]$$

(for $\beta = \beta_N$) $$= \mathrm{E}^{\otimes 2}\left[\mathrm{e}^{\hat{\beta}^2\,\frac{\pi}{\log N}\,\mathscr{L}_N}\right] \approx \mathrm{E}^{\otimes 2}\left[\mathrm{e}^{\hat{\beta}^2\,\mathrm{Exp}(1)}\right] = \frac{1}{1-\hat{\beta}^2} \qquad \square$$