On the 2D Stochastic Heat Equation

Lecture 3

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Stochastic dynamics: foundations and applications

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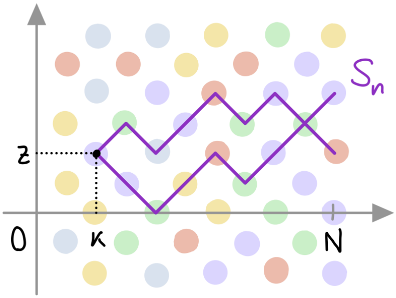
Reminders from Lecture 2

Directed polymer in random environment

- Simple random walk on $\mathbb{Z}^2$ $\quad S = (S_n)$
- Random environment $\quad \omega(n,x) \sim \mathcal{N}(0,1)$



Partition functions (with $k=0$)

$$Z_N(z) \;=\; \mathrm{E}_z^{\mathrm{SRW}}\left[e^{\sum_{n=1}^N \{\beta\, \omega(n,S_n) - \frac{1}{2}\beta^2\}} \right]$$

Diffusive rescaled partition functions $=$ Discretized Stochastic Heat Equation

$$Z_N\left(\sqrt{N}x\right) \;=\; u_\varepsilon(1,x)$$

Reminders from Lecture 2

Critical rescaling $\quad \beta_N = \frac{\hat{\beta}\sqrt{\pi}}{\sqrt{\log N}} \quad$ for $\hat{\beta} \in (0, \infty)$

Proposition 1

$$\text{As } N \to \infty \qquad \mathbb{V}\mathrm{ar}\big[Z_{N,\beta_N}(z)\big] \begin{cases} \to \frac{\hat{\beta}^2}{1-\hat{\beta}^2} & \text{if } \hat{\beta} < 1 \\ \sim c \log N & \text{if } \hat{\beta} = 1 \\ \gg \log N & \text{if } \hat{\beta} > 1 \end{cases}$$

We gave motivations based on overlap of 2D random walks [Erdös–Taylor 60]

Today: independent proof based on polynomial chaos

Covariance estimates

We also control covariances on the diffusive scale

Proposition 2

$$\mathbb{C}\mathrm{ov}\big[Z_{N,\beta}(\sqrt{N}x) \,,\, Z_{N,\beta}(\sqrt{N}y) \big] \underset{N\to\infty}{\sim} \Phi(x-y)\, \beta^2 \left\{ 1 + \mathbb{V}\mathrm{ar}\big[Z_{N,\beta}(0) \big] \right\}$$

$$\text{(for } \beta = \beta_N \text{ with } \hat{\beta} = 1) \qquad \sim \quad \Phi(x-y)\, \pi\, c \qquad\qquad \Phi \in L^1_{\mathrm{loc}}$$

This yields our original goal

$$\mathbb{V}\mathrm{ar}\big[u_\varepsilon(1,\varphi) \big] = \iint_{\mathbb{R}^2\times\mathbb{R}^2} \mathbb{C}\mathrm{ov}\big[Z_N(\sqrt{N}x) \,,\, Z_N(\sqrt{N}y) \big]\, \varphi(x)\, \varphi(y)\, \mathrm{d}x\, \mathrm{d}y \xrightarrow[N\to\infty]{} K \in (0,\infty)$$

Polynomial chaos

- New disorder variables $$X_{n,z} := \mathrm{e}^{\beta\omega(n,z)-\frac{1}{2}\beta^2} - 1$$

  i.i.d. $\quad \mathbb{E}[X] = 0 \qquad \mathbb{V}\mathrm{ar}[X] := \sigma^2 = \mathrm{e}^{\beta^2} - 1 \sim \beta^2$ as $\beta \downarrow 0$

- Random walk kernel $$q_n(z) := \mathrm{P}^{\mathrm{SRW}}(S_n = z)$$

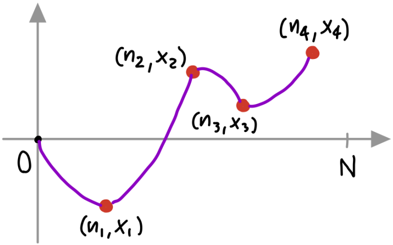
Proposition (polynomial chaos)

$$Z_{N,\beta} = Z_{N,\beta}(0) = 1 + \sum_{k=1}^{N} \sum_{\substack{0<n_1<\ldots<n_k\leq N \\ z_1,\ldots,z_k\in\mathbb{Z}^d}} \underbrace{q_{\vec{n}}(\vec{z})}_{\prod_{i=1}^{k} q_{n_i-n_{i-1}}(z_i-z_{i-1})} \prod_{i=1}^{k} X_{n_i,z_i}$$

Polynomial chaos

Partition function $Z_{N,\beta}$ is a **multi-linear polynomial** in the variables $X_{n,z}$

$$Z_{N,\beta} = 1 + \sum_{k=1}^{N} Z_{N,\beta}^{(k)} = 1 + \sum_{k=1}^{N} \sum_{\substack{I \subseteq \{1,\dots,N\} \times \mathbb{Z}^2 \\ |I| = k}} q(I)\, X^{I}$$



with $I = \{(n_1, z_1), \dots, (n_k, z_k)\}$

- $q(I) = q_{\vec{n}}(\vec{z}) = \prod_{i=1}^{k} q_{n_i - n_{i-1}}(z_i - z_{i-1})$
- $X^{I} = \prod_{i=1}^{k} X_{n_i, z_i}$

Proof: Mayer expansion

$$
\begin{aligned}
e^{\beta H(S,\omega) - \frac{1}{2}\beta^2 N} &= e^{\sum_{n=1}^{N} \{\beta\,\omega(n,S_n) - \frac{1}{2}\beta^2\}} \\
&= \prod_{n=1}^{N} e^{\beta\,\omega(n,S_n) - \frac{1}{2}\beta^2} = \prod_{n=1}^{N} \left\{ 1 + \left(e^{\beta\,\omega(n,S_n) - \frac{1}{2}\beta^2} - 1\right) \right\} \\
&= \prod_{n=1}^{N} \left\{ 1 + X_{n,S_n} \right\} = \prod_{n=1}^{N} \prod_{z \in \mathbb{Z}^d} \left\{ 1 + X_{n,z}\, \mathbb{1}_{S_n = z} \right\} \\
&= 1 + \sum_{k=1}^{N} \sum_{\substack{0 < n_1 < \ldots < n_k \le N \\ z_1, \ldots, z_k \in \mathbb{Z}^d}} \prod_{i=1}^{k} X_{n_i,z_i} \cdot \underbrace{\prod_{i=1}^{k} \mathbb{1}_{S_{n_i} = z_i}}_{\mathrm{E}[\ldots] \,=\, q_{\vec{n}}(\vec{z})} \qquad \square
\end{aligned}
$$

Reminders: random walk computations

Random walk kernel $$q_n(z) \;=\; \mathrm{P}^{\mathrm{SRW}}(S_n = z) \sim \tfrac{1}{n}\, g_{\frac{1}{2}}\big(\tfrac{z}{\sqrt{n}}\big)\, 2\,\mathbb{1}_{\mathrm{parity}(n,z)}$$

- $$\sum_{z\in\mathbb{Z}^2} q_n(z)^2 \;=\; \sum_{z\in\mathbb{Z}^2} q_n(z)\, q_n(-z) \;=\; q_{2n}(0) \sim \frac{1}{\pi\, n}$$
- $$R_N \;:=\; \sum_{n=1}^{N} \sum_{z\in\mathbb{Z}^2} q_n(z)^2 \sim \sum_{n=1}^{N} \frac{1}{\pi\, n} \sim \frac{\log N}{\pi}$$

R_N = "expected replica overlap" of two independent random walks S, S'

$$R_N \;=\; \sum_{n=1}^{N} \sum_{z\in\mathbb{Z}^2} \mathrm{P}(S_n = z\,,\, S'_n = z) \;=\; \sum_{n=1}^{N} \mathrm{P}(S_n = S'_n) \;=\; \mathrm{E}\Bigg[\, \underbrace{\sum_{n=1}^{N} \mathbb{1}_{S_n = S'_n}}_{\mathscr{L}_N} \,\Bigg]$$

Proof of Proposition 1: variance computation

$$Z_N = 1 + \sum_{k=1}^{N} \sum_{\substack{0<n_1<\ldots<n_k\leq N \\ z_1,\ldots,z_k\in\mathbb{Z}^2}} \prod_{i=1}^{k} q_{n_i-n_{i-1}}(z_i - z_{i-1})\, X_{n_i,z_i}$$

$$\begin{aligned}
\mathbb{V}\mathrm{ar}\left[Z_N\right] &= \sum_{k=1}^{N} (\sigma^2)^k \sum_{\substack{0<n_1<\ldots<n_k\leq N \\ z_1,\ldots,z_k\in\mathbb{Z}^2}} \prod_{i=1}^{k} q_{n_i-n_{i-1}}(\underbrace{z_i - z_{i-1}}_{y_i})^2 \\
&= \sum_{k=1}^{N} (\sigma^2)^k \sum_{0<n_1<\ldots<n_k\leq N} \prod_{i=1}^{k} \left(\sum_{y_i\in\mathbb{Z}^2} q_{n_i-n_{i-1}}(y_i)^2 \right) \\
&\approx \sum_{k=1}^{N} (\sigma^2)^k \sum_{0<n_1<\ldots<n_k\leq N} \prod_{i=1}^{k} \frac{1}{\pi(n_i - n_{i-1})}
\end{aligned}$$

Proof of Proposition 1: subcritical variance

We can sum over $m_i := n_i - n_{i-1} > 0$ with constraint $\boxed{m_1 + \ldots + m_k \le N}$

Upper bound: enlarge the constraint to $0 < m_1 \le N$, ... , $0 < m_k \le N$

$$\mathbb{V}\mathrm{ar}[Z_N] \le \sum_{k=1}^{N} (\sigma^2)^k \sum_{\substack{0<m_1\le N \\ \cdots \\ 0<m_k \le N}} \prod_{i=1}^{k} \frac{1}{\pi\, m_i} \approx \sum_{k=1}^{N} \left(\sigma^2 \, \frac{\log N}{\pi} \right)^k$$

$$\sigma^2 = e^{\beta^2} - 1 \sim \beta^2 \sim \hat{\beta}^2 \, \frac{\pi}{\log N} \qquad \rightsquigarrow \qquad \mathbb{V}\mathrm{ar}[Z_N] \le \frac{\hat{\beta}^2}{1 - \hat{\beta}^2} \qquad \text{for } \hat{\beta} < 1$$

Lower bound: restrict the constraint to $0 < m_1 \le \frac{1}{k} N$, ... , $0 < m_k \le \frac{1}{k} N$

(exercise!)

Proof of Proposition 1: critical variance

This works for $\hat{\beta} < 1$ (subcritical regime)

Finer analysis is required for the critical regime $\hat{\beta} = 1$ i.e.

$$\beta = \frac{\sqrt{\pi}}{\sqrt{\log N}} \left(1 + \frac{\vartheta}{\log N}\right) \qquad \rightsquigarrow \qquad \mathbb{V}\mathrm{ar}\left[Z_N\right] \sim c_\vartheta \log N$$

Set $\vartheta = 0$ and (for simplicity) pretend $\sigma^2 \simeq \beta^2 = \dfrac{\pi}{\log N}$ $\quad \displaystyle\sum_{n=1}^{N} \frac{1}{n} \simeq \log N$

$$\mathbb{V}\mathrm{ar}\left[Z_N\right] \simeq \sum_{k=1}^{N} \frac{1}{(\log N)^k} \sum_{0<n_1<\ldots<n_k\leq N} \prod_{i=1}^{k} \frac{1}{n_i - n_{i-1}}$$

Proof of Proposition 1: Dickman subordinator

Introduce a triangular array of random walks on $\mathbb{N}$ (renewal processes)

$$0 = \tau_0^{(N)} < \tau_1^{(N)} < \tau_2^{(N)} < \ldots \qquad \mathrm{P}\Big(\tau_i^{(N)} - \tau_{i-1}^{(N)} = n\Big) = \frac{1}{\log N} \, \frac{\mathbb{1}_{\{1,\ldots,N\}}(n)}{n}$$

Probabilistic interpretation

$$\mathbb{V}\mathrm{ar}\big[\, Z_N \,\big] \simeq \sum_{k=1}^{N} \mathrm{P}\Big(\tau_k^{(N)} \le N\Big)$$

Theorem (Dickman subordinator) [CSZ 19]

$$\left(\frac{\tau_{s \log N}^{(N)}}{N} \right)_{s \ge 0} \xrightarrow[N\to\infty]{d} (Y_s)_{s\ge 0} \qquad \text{Lévy measure } \nu(\mathrm{d}t) = \mathbb{1}_{(0,1)}(t)\, \mathrm{d}t$$

Proof of Proposition 1: critical variance

Riemann sum approximation

$$\mathbb{V}\mathrm{ar}\big[Z_N\big] \simeq \sum_{k=1}^{N} \mathrm{P}\big(\tau_k^{(N)} \le N\big) \left(1+\tfrac{\vartheta}{\log N}\right)^k$$

$$= \log N \cdot \underbrace{\frac{1}{\log N} \sum_{s \in \frac{1}{\log N}\mathbb{N}} \mathrm{P}\left(\frac{\tau_{s\log N}^{(N)}}{N} \le 1\right) \left(1+\tfrac{\vartheta}{\log N}\right)^{s\log N}} \qquad \square$$

$$\xrightarrow[N\to\infty]{} c = \int_0^\infty \mathrm{d}s\, \mathrm{P}\big(Y_s \le 1\big)\, \mathrm{e}^{s\vartheta}$$

Proof of Proposition 2: critical covariance

$$Z_N(\mathbf{z}) = 1 + \sum_{k=1}^{N} \sum_{\substack{0<n_1<\ldots<n_k\le N \\ \mathbf{z_0}=\mathbf{z},\ z_1,\ldots,z_k\in\mathbb{Z}^2}} \prod_{i=1}^{k} q_{n_i-n_{i-1}}(z_i - z_{i-1})\, X_{n_i,z_i}$$

$$\mathbb{C}\mathrm{ov}\big[Z_N(\mathbf{z}), Z_N(\mathbf{w})\big] = \sum_{\ell=1}^{N} \sum_{y\in\mathbb{Z}^2} q_\ell(y-\mathbf{z})\, q_\ell(y-\mathbf{w})\ \sigma^2 \left\{1 + \mathbb{V}\mathrm{ar}\big[Z_{N-\ell}(y)\big]\right\}$$

$$= \underbrace{\sum_{\ell=1}^{N} q_{2\ell}(\mathbf{z}-\mathbf{w})}\ \sigma^2 \left\{1 + \mathbb{V}\mathrm{ar}\big[Z_{N-\ell}(y)\big]\right\} \quad \square$$

for $\mathbf{z} = \sqrt{N}x$, $\mathbf{w} = \sqrt{N}y$: $\quad \xrightarrow[N\to\infty]{\text{LLT}} \Phi(x-y) := \int_0^1 g_t(x-y)\,\mathrm{d}t$

Phase transition

Let us set $$Z_{N,\beta} = Z_{N,\beta}(0) = \mathrm{E}\Big[\mathrm{e}^{\beta H(S,\omega) - \frac{1}{2}\beta^2 N} \,\Big|\, S_0 = 0 \Big]$$

Key observation [Bolthausen 89]

$$Z_{N,\beta} \geq 0 \text{ is a martingale } \xrightarrow[N\to\infty]{\text{a.s.}} Z_{\infty,\beta}$$

Phase transition at $\beta_c = \beta_c^{(d)} \geq 0$ [Comets–Yoshida, Junk–Lacoin, ...]

- For $\beta \leq \beta_c$ $\quad Z_{\infty,\beta} > 0$ a.s. $\quad \mathrm{F}(\beta) := \lim_{N\to\infty} \frac{1}{N} \log Z_{N,\beta} = 0$
- For $\beta > \beta_c$ $\quad Z_{\infty,\beta} = 0$ a.s. $\quad \mathrm{F}(\beta) := \lim_{N\to\infty} \frac{1}{N} \log Z_{N,\beta} < 0$

Path properties

Polymer measure $$\mathrm{dP}_{N,\beta}(S) = \frac{\mathrm{e}^{\beta H(S,\omega) - \frac{1}{2}\beta^2 N}}{Z_{N,\beta}} \, \mathrm{dP}(S)$$

- $\beta \le \beta_c$ weak disorder [Bolthausen, Comets–Yoshida]
  - $|S_N| = O(\sqrt{N})$ diffusive under $\mathrm{P}_{N,\beta}$
- $\beta > \beta_c$ strong disorder [Carmona–Hu, Bates–Chatterjee]
  - S under $\mathrm{P}_{N,\beta}$ localizes around finitely many paths
  - super-diffusivity (mostly conj.)

Intermediate disorder

Critical point: • $\beta_c > 0$ for $d \geq 3$ • $\beta_c = 0$ for $d \leq 2$

$(d \leq 2)$ any $\beta > 0$ changes the RW behavior [disorder relevance]

$$Z_{N,\beta>0} \xrightarrow[N\to\infty]{} 0 \qquad \text{(despite } Z_{N,\beta=0} \equiv 1 \text{)}$$

Intermediate disorder $(d \leq 2)$

Can we tune $\beta = \beta_N \to 0$ so that $Z_{N,\beta_N} \xrightarrow[N\to\infty]{} \mathscr{Z}^{\xi} > 0$ random ?

$$\mathrm{P}_{N,\beta_N} \xrightarrow[N\to\infty]{} \mathscr{P}^{\xi}$$

$\rightsquigarrow$ Stochastic Heat Equation $\partial_t u = \Delta u + \beta\, \xi\, u$ ξ = white noise

Subcritical dimension $d = 1$

Consider first dimension $d = 1$

Non-trivial limit emerges under the rescaling $\quad \beta_N = \dfrac{\hat{\beta}}{N^{1/4}} \quad$ with $\hat{\beta} \in (0, \infty)$

Intermediate disorder $(d = 1)$ [Alberts–Khanin–Quastel 14]

$$Z_{N,\beta}^{\omega} \xrightarrow[N\to\infty]{d} u(1,0)$$

$$\begin{cases} \partial_t u(t,x) = \Delta_x u(t,x) + \hat{\beta}\, u(t,x)\, \xi(t,x) \\ u(0,x) \equiv 1 \end{cases}$$

Well defined in $d = 1$ for any $\hat{\beta} \geq 0$ (stochastic integral)

Critical dimension $d = 2$

Consider now dimension $d = 2$

We discovered a phase transition on the intermediate scale

$$\beta_N \sim \frac{\hat{\beta}\sqrt{\pi}}{\sqrt{\log N}} \qquad \text{with critical point} \quad \hat{\beta}_c = 1$$

Log-normality

[C.S.Z. 19]

For $\hat{\beta} < 1$ $\qquad Z_{N,\beta_N} \xrightarrow[N\to\infty]{d} e^{v\mathcal{N}(0,1)-\frac{1}{2}v^2} > 0 \qquad v^2 = \log\frac{1}{1-\hat{\beta}^2}$

For $\hat{\beta} \geq 1$ $\qquad Z_{N,\beta_N} \xrightarrow[N\to\infty]{d} 0$

Recently also "proved at criticality" [Gu–Tsai 26]