On the 2D Stochastic Heat Equation

Lecture 4

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Stochastic dynamics: foundations and applications

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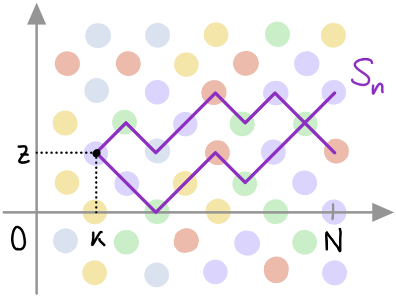
Reminders from Lecture 3

- Simple random walk on $\mathbb{Z}^2$ $\quad S = (S_n)$
- Random environment $\quad \omega(n,x) \sim \mathscr{N}(0,1)$



Partition functions (with $k=0$)

$$Z_N(z) = \mathrm{E}_z^{\mathrm{SRW}}\left[e^{\sum_{n=1}^N \{\beta\, \omega(n,S_n) - \frac{1}{2}\beta^2\}} \right]$$

Directed Polymer & Stochastic Heat Equation

Diffusive rescaled partition functions = Discretized Stochastic Heat Equation

$$Z_N(\sqrt{N}x) = u_\varepsilon(1,x)$$

Reminders from Lecture 3

We computed variance and covariances of Z_N in the critical regime

$$\beta = \frac{\sqrt{\pi}}{\sqrt{\log N}} \left(1 + \frac{\vartheta}{\log N}\right) \qquad \begin{cases} \mathbb{V}\mathrm{ar}\big[Z_N(\sqrt{N} x) \big] \sim c_\vartheta \log N \\ \mathbb{C}\mathrm{ov}\big[Z_N(\sqrt{N} x), Z_N(\sqrt{N} y) \big] \sim c_\vartheta \, \Phi(x - y) \end{cases}$$

For solution of discretized Stochastic Heat Equation

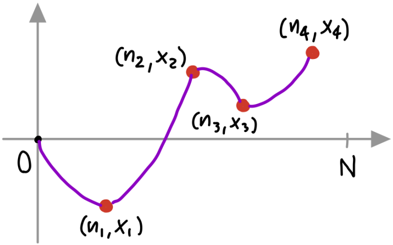
$$\mathbb{E}\big[u_\varepsilon(1, \varphi) \big] \equiv 1 \qquad \mathbb{V}\mathrm{ar}\big[u_\varepsilon(1, \varphi) \big] \xrightarrow[\varepsilon \downarrow 0]{} \underbrace{K_\vartheta(\varphi, \varphi)}_{c_\vartheta \iint \Phi(x-y)\, \varphi(x)\, \varphi(y)\, \mathrm{d}x\, \mathrm{d}y} \in (0, \infty)$$

We exploited polynomial chaos expansion of Z_N

Reminders from Lecture 3

Partition function Z_N is a **multi-linear polynomial** in the variables $X_{n,z}$

$$Z_N = 1 + \sum_{k=1}^{N} Z_N^{(k)} = 1 + \sum_{k=1}^{N} \sum_{\substack{I \subseteq \{1,\dots,N\} \times \mathbb{Z}^2 \\ |I| = k}} q(I)\, X^I$$



with $I = \{(n_1, z_1), \dots, (n_k, z_k)\}$

- $q(I) = q_{\vec{n}}(\vec{z}) = \prod_{i=1}^{k} q_{n_i - n_{i-1}}(z_i - z_{i-1})$
- $X^I = \prod_{i=1}^{k} X_{n_i, z_i}$

Main result

Theorem [CSZ 23]

Discretize the Stochastic Heat Equation on the lattice $\varepsilon\mathbb{N} \times \sqrt{\varepsilon}\,\mathbb{Z}^2$ $\qquad \varepsilon = \frac{1}{N}$

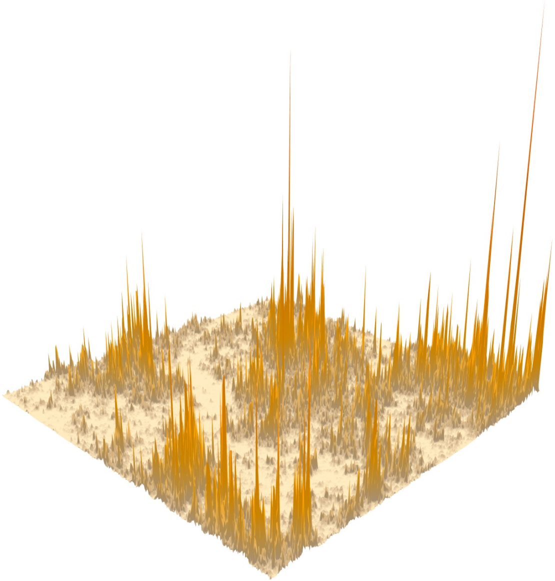
$$u_\varepsilon(t,x) \;=\; Z_N(N(1-t), \sqrt{N}x)$$

In the critical regime $\qquad \beta_\varepsilon \;=\; \dfrac{\sqrt{\pi}}{\sqrt{\log N}}\left(1 + \dfrac{\vartheta}{\log N}\right) \qquad \vartheta \in \mathbb{R}$

the solution u_ε converges to a unique and non-trivial limit SHF^ϑ

$$u_\varepsilon \;=\; \big(u_\varepsilon(t,x)\,\mathrm{d}x\big)_{t\geq 0} \xrightarrow[\varepsilon\downarrow 0]{d} \mathrm{SHF}^\vartheta \;=\; \big(\mathrm{SHF}^\vartheta(t,\mathrm{d}x)\big)_{t\geq 0}$$

The critical 2D Stochastic Heat Flow



Techniques and Ideas

Subsequential limits exist by tightness — via moments bounds

Challenge was uniqueness — lacking characterization of the limit

What we proved

$(u_\varepsilon)_{\varepsilon>0}$ is Cauchy $\qquad u_\varepsilon \approx u_{\varepsilon'} \qquad$ for small $\varepsilon, \varepsilon' > 0$

- Coarse-graining — approximate self-similarity
- Lindeberg priciple — robustness to microscopic details

In this lecture we are going to illustrate these ideas

Multi-scale description

Challenge: $u_\varepsilon \overset{d}{\approx} u_{\varepsilon'}$ for small $\varepsilon, \varepsilon' > 0$ that is

$$Z_N(\sqrt{N}x)\,\mathrm{d}x \overset{d}{\approx} Z_{N'}(\sqrt{N'}x)\,\mathrm{d}x \qquad \text{for large } N, N'$$

Concretely: $\mathscr{U}_R :=$ uniform random point in $B(0,R)$

$$Z_N(\mathscr{U}_{\sqrt{N}}) \overset{d}{\approx} Z_{N'}(\mathscr{U}_{\sqrt{N'}}) \qquad \text{for large } N, N'$$

We compare both sides to the same coarse-grained model $\mathscr{Z}_\delta$ for $\delta > 0$ small

Disorder is clustered on boxes of scales resp. δN and $\delta N'$

Multi-scale description

Graphical illustration on the blackboard

Universality

Coarse-graining approximation requires robustness to microscopic perturbations

Partition function $Z_N(\mathscr{U}_{\sqrt{N}}) = \Phi_N(X)$ is a function of microscopic variables

$$X_{n,z} = e^{\beta\omega(n,z) - \frac{1}{2}\beta^2} - 1 \qquad \mathbb{E}[X_{n,z}] = 0 \qquad \mathbb{E}[X_{n,z}^2] = e^{\beta^2} - 1$$

Lindeberg principle

Changing microscopic variables X (keeping same mean and variance)
should not change the asymptotic law of $\Phi_N(X) = Z_N(\mathscr{U}_{\sqrt{N}})$

This must be proved also for the coarse-grained model $\mathscr{Z}_\delta$ (challenging!)

Improved Lindeberg

$X = (X_1, \dots, X_n)$, $Y = (Y_1, \dots, Y_n)$ independent, centered, matching variance

Lindeberg Principle

$$\left| \mathbb{E}[h(X)] - \mathbb{E}[h(Y)] \right| \leq \sum_{k=1}^{n} \Delta_k$$

$$\Delta_k := \frac{1}{6} \max_{\xi \in \{X_k, Y_k\},\, s \in [0,1]} \left| \mathbb{E}\left[\partial_k^3 h\left(X_1, \dots, X_{k-1}, s\,\xi, Y_{k+1}, \dots, Y_n\right) \right] \right|$$

The CLT via Lindeberg

$X = (X_1, \dots, X_n)$, $Y = (Y_1, \dots, Y_n)$ i.i.d. with zero mean, unit variance & in L^3

CLT without Gaussians

$$\left| \mathbb{E}\left[\psi\left(\frac{X_1 + \ldots + X_n}{\sqrt{n}} \right) \right] - \mathbb{E}\left[\psi\left(\frac{Y_1 + \ldots + Y_n}{\sqrt{n}} \right) \right] \right| \leq \frac{C}{\sqrt{n}}$$

<u>Proof.</u> Define $h = \psi(\Phi_n)$ with $\Phi_n(x_1, \dots, x_n) := \frac{x_1 + \ldots + x_n}{\sqrt{n}}$ $\psi \in C_c^3(\mathbb{R})$

By Lindeberg $$\left| \mathbb{E}\left[\psi\left(\frac{X_1 + \ldots + X_n}{\sqrt{n}} \right) \right] - \mathbb{E}\left[\psi\left(\frac{Y_1 + \ldots + Y_n}{\sqrt{n}} \right) \right] \right| \leq \sum_{k=1}^{n} \Delta_k$$

The CLT via Lindeberg

It remains to show that $\quad \Delta_k \leq \frac{C}{n^{3/2}}$

$$\Delta_k := \frac{1}{6} \max_{\xi \in \{X_k, Y_k\},\, s \in [0,1]} \left| \mathbb{E}\left[\partial_k^3 h\left(X_1, \ldots, X_{k-1}, s\,\xi, Y_{k+1}, \ldots, Y_n\right)\right]\right|$$

By direct computation

$$\partial_k h = \psi'(\Phi_n)\,\partial_k \Phi_n = \psi'(\Phi_n)\,\frac{1}{\sqrt{n}} \qquad \partial_k^2 h = \psi''(\Phi_n)\,\frac{1}{n}$$

$$\partial_k^3 h = \psi'''(\Phi_n)\,\frac{1}{n^{3/2}} \qquad \left|\partial_k^3 h\right| \leq \frac{C}{n^{3/2}} \qquad \square$$

Application to directed polymers

See blackboard

Summarizing

Probabilistic handle on Stochastic Heat Equation via Directed Polymers

$\rightsquigarrow$ KPZ universality in $d = 1$ [Seppäläinen 12] [Alberts–Khanin–Quastel 14] [Corwin 12]

Statistical Mechanics $\leftrightsquigarrow$ Singular SPDEs

A broader picture

Extensions of Stochastic Heat Equation

- Nonlinear [Dunlap–Gu 22] [Dunlap–Graham 23]
- Lévy noise [Berger–Chong–Lacoin 23]
- Correlated Gaussian noise [Müller–Tribe 04] [Dunlap–Hairer–Li 25] [Cosco–Cottini–Donadini 25]

Related critical models

- Anisotropic KPZ equation [Cannizzaro–Erhard–Toninelli 23]
- Stochastic Burgers equation [Cannizzaro–Gubinelli–Toninelli 23] [Cannizzaro–Moulard–Toninelli 25]
- Diffusion in the curl of GFF [Cannizzaro–Haunschmidt-Sibitz–Toninelli22]

  . . . [Armstrong–Bou Rabee–Kuusi 24+] [Chatzigeorgiou–Morfe–Otto–Wang 25]

Thanks