Noise sensitivity for the 2D Stochastic Heat Equation

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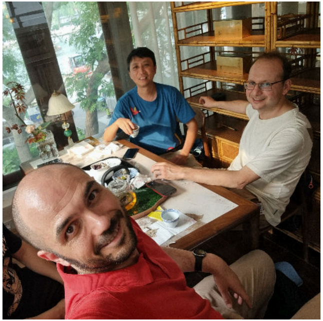


The Stochastic Heat Equation

Heat equation with multiplicative singular potential $t \geq 0,\ x \in \mathbb{R}^d$

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, u(t,x)\, \xi(t,x) \qquad \text{(SHE)}$$

$\beta \geq 0$ coupling constant

$\xi(t,x) =$ "space-time white noise"

$(d = 1)$ sub-critical: well-posed

Ito-Walsh / Robust solution theories

[Chen–Dalang 15] [Hairer–Pardoux 15]

$(d = 2)$ critical [C.S.Z. 23]

Natural candidate solution: the critical 2D Stochastic Heat Flow (SHF)

Regularisation

How we define a solution of 2D SHE?

Regularized noise $\xi_N(t,x)$ $\rightsquigarrow$ well-defined solution $u_N(t,x)$

(discretization, mollification, ...)

$$\begin{cases} \partial_t u_N(t,x) = \Delta_x u_N(t,x) + \beta\, u_N(t,x)\, \xi_N(t,x) \\ u_N(0,x) \equiv 1 \quad \text{(for simplicity)} \end{cases} \tag{reg-SHE}$$

Convergence of $u_N(t,\varphi) = \int_{\mathbb{R}^2} u_N(t,x)\,\varphi(x)\,\mathrm{d}x$ as $N \to \infty$?

Renormalisation

Convergence of the mean is easy: $\mathbb{E}\left[u_N(t,\varphi) \right] \xrightarrow[N\to\infty]{} \int_{\mathbb{R}^2} \varphi(x)\,\mathrm{d}x$

Convergence of the variance?

$$\beta \sim \frac{\hat\beta}{\sqrt{\log N}} \quad \text{for} \quad \hat\beta = \sqrt{\pi}\left(1 + \frac{\vartheta}{\log N}\right)$$

$\mathbb{V}\mathrm{ar}\left[u_N(t,\varphi) \right] \xrightarrow[N\to\infty]{} K_t^{\vartheta}(\varphi,\varphi) > 0$ [Bertini–Cancrini 98] [C.S.Z. 19]

Convergence of all higher moments [C.S.Z. 19] [Gu–Quastel–Tsai 21]

Convergence in law of $u_N(t,\varphi)$? $\iff$ of the measure $u_N(t,x)\,\mathrm{d}x$?

The critical 2D Stochastic Heat Flow

Theorem [C.S.Z. *Invent. Math.* 23]

Take $$\beta = \frac{\sqrt{\pi}}{\sqrt{\log N}}\left(1 + \frac{\vartheta}{\log N}\right)$$ for some $\vartheta \in \mathbb{R}$

Then u_N converges in law to a unique and non-trivial limit $\mathscr{U}^{\vartheta}$

$$\big(u_N(t,x)\,\mathrm{d}x\big)_{t\geq 0} \xrightarrow[N\to\infty]{d} \big(\mathscr{U}^{\vartheta}(t,\mathrm{d}x)\big)_{t\geq 0}$$

$\mathscr{U}^{\vartheta}$ = critical 2D **Stochastic Heat Flow (SHF)** = stochastic process of random measures on $\mathbb{R}^2$

SHF and Stochastic Heat Equation

The SHF is a "candidate solution" of the critical 2d Stochastic Heat Equation

$$\mathscr{U}^{\vartheta}(t, \mathrm{d}x) \qquad (\text{initial condition } 1 \text{ at time } 0)$$

We actually build a two-parameter space-time process

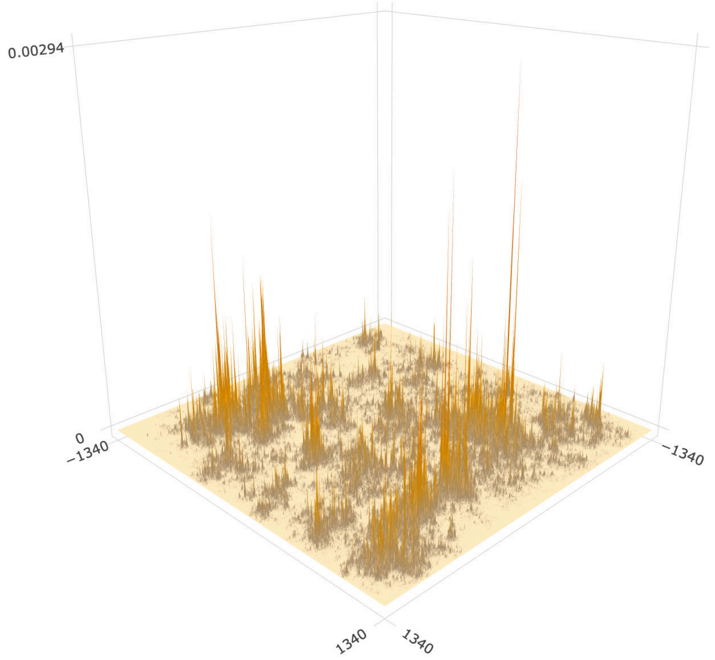
$$\left(\mathscr{U}^{\vartheta}(s, \mathrm{d}y; t, \mathrm{d}x)\right)_{0 \le s \le t < \infty} \qquad (\text{starting at time } s \text{ from } \mathrm{d}y)$$

"Flow": Chapman-Kolmogorov property for $s < t < u$ [Clark–Mian 2024+]

$$\mathscr{U}^{\vartheta}(s, \mathrm{d}y; u, \mathrm{d}z) = \int_{x \in \mathbb{R}^2} \mathscr{U}^{\vartheta}(s, \mathrm{d}y; \underbrace{t, \mathrm{d}x) \, \mathscr{U}^{\vartheta}(t, \mathrm{d}x}_{\substack{\text{non-trivial "product"} \\ \text{of measures}}}; u, \mathrm{d}z)$$

Key properties of the SHF

- a.s. $\mathscr{U}^{\vartheta}(t, \mathrm{d}x)$ is singular w.r.t. Lebesgue [C.S.Z. 2024+]
  "not a function"
- a.s. $\mathscr{U}^{\vartheta}(t, \mathrm{d}x) \in \mathscr{C}^{-\kappa}$ for any $\kappa > 0$ (in particular: non atomic)
  "*barely* not a function"
- Formulas for all moments [C.S.Z. 19] [Gu–Quastel–Tsai 21]
- Scaling covariance $a^{-1}\, \mathscr{U}^{\vartheta}(a\, t, \mathrm{d}(\sqrt{a}\, x)) \overset{d}{=} \mathscr{U}^{\vartheta + \log a}(t, \mathrm{d}x)$
- Axiomatic characterization via independence & moments [Tsai 24+]
- Universality w.r.t. approximation scheme [C.S.Z. 23] [Tsai 24+]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Noise sensitivity

Consider the following question:

Is the SHF $\mathscr{U}^{\vartheta}$ sensitive to small perturbations of the driving noise ξ ?

Problem: there is no noise ξ on the space of $\mathscr{U}^{\vartheta}$

Theorem [C.–Donadini 25+]

$$(\xi_N, u_N) \xrightarrow[N\to\infty]{d} (\xi, \mathscr{U}^{\vartheta}) \qquad \text{with } \xi \text{ and } \mathscr{U}^{\vartheta} \text{ independent}$$

Puzzling: u_N is a function of ξ_N ... but dependence is lost in the scaling limit!

Noise sensitivity

Let us rephrase the question:

Is u_N sensitive to small perturbations of the driving noise ξ_N ?

We take $\xi_N :=$ discretisation of white noise on the lattice $\frac{1}{N}\mathbb{N} \times \frac{1}{\sqrt{N}}\mathbb{Z}^2$

$$\xi_N(t,x) = N \cdot \omega(n,z) \text{ i.i.d.} \qquad \text{for } (t,x) = \left(\tfrac{n}{N}, \tfrac{z}{\sqrt{N}}\right)$$

We write $u_N(t,\varphi) = f_N(\omega)$ for a suitable function $f_N(\cdot) = f_N^{t,\varphi}(\cdot)$

$f_N(\cdot)$ is the partition function of 2D directed polymer in random environment

Noise sensitivity

Fix i.i.d. random variables $\omega = (\omega_i)_{i=1,2,\dots}$ $\qquad \mathbb{E}[\omega_i] = 0 \quad \mathbb{V}\mathrm{ar}[\omega_i] = 1$

Take a sequence of functions $f_N(\omega) \in L^2$ $\qquad \lim_{N\to\infty} \mathbb{V}\mathrm{ar}[f_N(\omega)] = \sigma^2 \in (0,\infty)$

Define "ε-perturbation" $\omega^\varepsilon = (\omega_i^\varepsilon)_{i=1,2,\dots}$ $\qquad \omega_i^\varepsilon := \begin{cases} \omega_i & \text{w. prob. } 1-\varepsilon \\ \widetilde{\omega}_i \perp\!\!\!\perp \omega_i & \text{w. prob. } \varepsilon \end{cases}$

We call $(f_N)_{N\in\mathbb{N}}$ noise sensitive if [Garban–Steif 14]

$$\lim_{N\to\infty} \mathbb{C}\mathrm{ov}\left[f_N(\omega^\varepsilon), f_N(\omega)\right] = 0 \qquad \forall \varepsilon > 0$$

Noise sensitivity

"Usual" functions are not noise sensitive, e.g. $f_N(\omega) = \dfrac{\omega_1 + \ldots + \omega_N}{\sqrt{N}}$

"Parity" is noise sensitive: $f_N(\omega) = \omega_1 \cdots \omega_N$ for symmetric $\omega_i = \pm 1$

Chaos decomposition $f_N = \mathbb{E}[f_N] + \sum_{d=1}^{\infty} f_N^{(d)}$ $\mathbb{V}\mathrm{ar}[f_N] = \sum_{d=1}^{\infty} \|f_N^{(d)}\|_2^2$

For instance $f_N^{(d)}(\omega) = \sum_{\{i_1,\ldots,i_d\}} c_N(i_1,\ldots,i_d)\, \omega_{i_1} \cdots \omega_{i_d}$ (polynomial chaos)

Spectral criterion

Noise sensitivity $\iff$ $\forall d \in \mathbb{N}$: $\|f_N^{(d)}\|_2^2 \xrightarrow[N\to\infty]{} 0$

The BKS Theorem

Boolean setting: binary functions $f(\omega)$ of binary variables ω_i

Robust condition for noise sensitivity based on influences

$$I_i(f) := \mathbb{P}\big(f(\omega^i_+) \neq f(\omega^i_-)\big) \qquad \mathscr{W}(f) := \sum_i I_i(f)^2$$

Theorem [Benjamini–Kalai–Schramm 99]

$(f_N)_{N\in\mathbb{N}}$ is noise sensitive if $\lim_{N\to\infty} \mathscr{W}(f_N) = 0$ [B.K.S. 99]

$\forall \varepsilon > 0$: $\mathrm{Cov}\left[f(\omega^\varepsilon), f(\omega)\right] \leq C\, \mathscr{W}(f)^{\alpha\,\varepsilon}$ [Keller–Kindler 13]

Influences beyond the Boolean setting

Define $\delta_i f := f - \mathbb{E}_i[f]$ with $\mathbb{E}_i[\cdot] = \mathbb{E}[\cdot|\sigma(\omega_j : j \neq i)]$ [Talagrand 94]

Two notions of influence

$$I_i^{(1)}(f) := \|\delta_i f\|_1 = \mathbb{E}\big[|\delta_i f|\big] \qquad I_i^{(2)}(f) := \|\delta_i f\|_2^2 = \mathbb{E}\big[(\delta_i f)^2\big]$$

(for Boolean f they coincide up to a factor 2)

It is the L^1 influence that is relevant for us: $\mathscr{W}(f) := \sum_i I_i^{(1)}(f)^2$

L^2 influence relevant for [Mossel–O'Donnel–Oleszkiewicz 10] [Kahn–Kalai–Linial 88]

Main result

We extend BKS in either of the following settings:

- $\mathbb{E}[|\omega_i|^q] < \infty$ for some $q > 2$ & $f(\omega)$ is a polynomial chaos
- ω_i take finitely many values & $f(\omega)$ is any function in L^2

Both ensure a suitable hypercontractivity $L^2 \to L^q$

Generalized BKS [C.–Donadini 25+]

$$\forall d \in \mathbb{N}: \qquad \left\| f^{(d)} \right\|_2^2 \leq (c_q)^d \, \mathscr{W}(f)^{1-\frac{2}{q}}$$

$$\forall \varepsilon > 0: \qquad \mathbb{C}\mathrm{ov}\left[f(\omega^\varepsilon), f(\omega) \right] \leq C \, \mathscr{W}(f)^{\alpha_q \varepsilon}$$

Back to SHE

Noise sensitivity of 2D SHE [C.–Donadini 25+]

$$\mathscr{W}(u_N(t,\varphi)) \sim \frac{c_{t,\varphi}}{\log N} \quad \Longrightarrow \quad u_N(t,\varphi) \text{ is noise sensitive}$$

Influences are stable under composition with Lipschitz functions:

$$\mathscr{W}(\phi(f)) \leq 4\,\|\phi'\|_\infty^2\,\mathscr{W}(f)$$

Enhanced noise sensitivity [C.–Donadini 25+]

$\phi(u_N(t,\varphi))$ is noise sensitive $\forall$ Lipschitz ϕ if the $\boldsymbol{\omega}_i$'s take finitely many values

$\Longrightarrow$ $u_N(t,\varphi)$ is asymptotically independent of any bounded order chaos

Conclusion

We extended the BKS Theorem beyond the Boolean setting

- Robust conditions for noise sensitivity (stable under composition)
- Quantitative bounds

Our proof generalises Keller-Kindler... (large deviations $\rightsquigarrow$ moment bounds)

... and refines it: optimal estimate for binary ω_i's

$$\mathbb{C}\mathrm{ov}\left[f(\omega^{\varepsilon}), f(\omega)\right] \leq \mathscr{W}(f)^{\frac{\varepsilon}{2-\varepsilon}+o(1)}$$

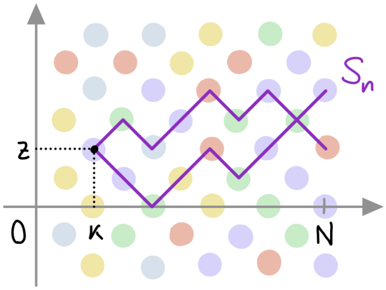
The assumption that ω_i's take finitely many values can hopefully be removed

Future direction: black noise à la Tsirelson cf. [Himwich–Parekh 24+]

Merci

Directed Polymer in Random Environment

- $S = (S_n)_{n \geq 0}$ simple random walk on $\mathbb{Z}^d$
- Independent Gaussians $\omega(n,x) \sim \mathcal{N}(0,1)$
- $H(S,\omega) := \sum_{n=k+1}^{N} \omega(n, S_n) \;\sim \mathcal{N}(0, N-k)$



Partition Functions $(k \in \mathbb{N},\ z \in \mathbb{Z}^d)$

$$Z^{\omega}_{N,\beta}(k,z) = \mathrm{E}\left[e^{\beta H(S,\omega) - \frac{1}{2}\beta^2 (N-k)} \,\middle|\, S_k = z \right]$$

Partition functions and SHE

Diff. rescaled partition functions $=$ discretized SHE solutions

$$Z^{\omega}_{N,\beta}(N(1-t),\sqrt{N}x) \;=\; u_N(t,x) \qquad \text{(time rev.)}$$

Partition functions solve a difference equation: with $\xi_N \approx \omega$

$$\begin{cases} \partial_t u_N(t,x) = \Delta_x u_N(t,x) + \underbrace{\beta\, N^{\frac{2-d}{4}}}_{\beta_{\text{SHE}}} u_N(t,x)\, \xi_N(t,x) \\ u_N(0,x) \equiv 1 \end{cases} \qquad \text{(reg-SHE)}$$

Discrete analogue of Feynman-Kac

$$u_N(t,x) \;\approx\; \mathrm{E}\left[e^{\beta_{\text{SHE}} \int_{1-t}^{1} \xi(s,B_s) - \frac{1}{2}\beta_{\text{SHE}}^2\, t} \,\middle|\, B_{1-t} = x \right]$$