On (ir)reducible Central Limit Theorems

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20 Years of the Fourth Moment Theorem

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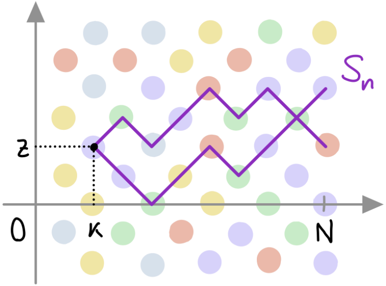
Outline

Directed Polymer in Random Environment

- $S = (S_n)_{n \geq 0}$ simple random walk on $\mathbb{Z}^d$
- Independent Gaussians $\omega(n,x) \sim \mathcal{N}(0,1)$
- $H(S,\omega) := \sum_{n=k+1}^{N} \omega(n,S_n) \sim \mathcal{N}(0, N-k)$



Partition Functions

$(k \in \mathbb{N},\ z \in \mathbb{Z}^d)$

$$Z^{\omega}_{N,\beta}(k,z) = \mathrm{E}\left[e^{\beta H(S,\omega) - \underbrace{\tfrac{1}{2}\beta^2 (N-k)}_{\text{(constant)}}} \,\middle|\, S_k = z \right]$$

$$\mathbb{E}\left[Z^{\omega}_{N,\beta}\right] = 1$$

Phase transition

Let us focus on $$Z^{\omega}_{N,\beta} = Z^{\omega}_{N,\beta}(0,0) = \mathrm{E}\left[\mathrm{e}^{\beta H(S,\omega) - \frac{1}{2}\beta^2 N} \,\middle|\, S_0 = 0 \right]$$

Key observation

[Bolthausen]

$$Z^{\omega}_{N,\beta} \geq 0 \text{ is a martingale } \xrightarrow[N\to\infty]{\text{a.s.}} Z^{\omega}_{\infty,\beta} \begin{cases} > 0 & \text{for } \beta \leq \beta_c \\ = 0 & \text{for } \beta > \beta_c \end{cases}$$

Phase transition at $\beta_c = \beta_c^{(d)} \geq 0$ [Comets–Yoshida, Junk–Lacoin, ...]

weak disorder $\beta \leq \beta_c$ (diffusivity) strong disorder $\beta > \beta_c$ (localization)

Non-trivial critical point $\beta_c > 0$ only for $d \geq 3$

Intermediate disorder

For $d \le 2$ we have $\beta_c = 0$

$$\forall \beta > 0: \qquad Z^{\omega}_{N,\beta} \xrightarrow[N\to\infty]{} 0 \qquad \left(\text{despite } Z^{\omega}_{N,\beta=0} \equiv 1\right)$$

Intermediate disorder

Can we tune $\beta = \beta_N \to 0$ so that $Z^{\omega}_{N,\beta_N} \xrightarrow[N\to\infty]{} \mathscr{Z}^{\xi} > 0$ random?

Focus on $d = 2$ (critical dimension): phase transition on intermediate scale

$$\beta_N \sim \frac{\hat{\beta}}{\sqrt{\log N}} \qquad \text{with critical point } \hat{\beta}_c = \sqrt{\pi}$$

Gaussian behavior for $d = 2$

Log-normality [C.S.Z. 19]

For $\hat{\beta} < \sqrt{\pi}$ $\qquad Z^{\omega}_{N,\beta_N} \xrightarrow[N\to\infty]{d} e^{\mathcal{N}(0,v^2) - \frac{1}{2}v^2} > 0 \qquad v^2 = \log \frac{1}{1-\frac{\hat{\beta}^2}{\pi}}$

For $\hat{\beta} \geq \sqrt{\pi}$ $\qquad Z^{\omega}_{N,\beta_N} \xrightarrow[N\to\infty]{d} 0$

Refined limit at $\hat{\beta} = \sqrt{\pi}$ $\rightsquigarrow$ critical 2D **Stochastic Heat Flow** [C.S.Z. 23]

Link with Stochastic Heat Equation

$$\partial_t u = \Delta u + \beta\, \xi\, u \qquad \xi = \text{white noise}$$

Discretized solution $u(t,x) = Z^{\omega}_{N,\beta}(N(1-t), \sqrt{N}x)$

Outline

Polynomial chaos (homogeneous sums)

Partition function is a polynomial chaos: $$Z_{N,\beta}^{\omega} = 1 + \sum_{k\geq 1} Z^{(k)}$$

$$Z^{(1)} \simeq \beta \sum_{\substack{0<n_1\leq N \\ z_1\in\mathbb{Z}^2}} \underbrace{q(n_1,z_1)}_{\mathrm{P}(S_{n_1}=z_1)} \omega(n_1,z_1)$$

$$Z^{(2)} \simeq \beta^2 \sum_{\substack{0<n_1<n_2\leq N \\ z_1,z_2\in\mathbb{Z}^2}} \underbrace{q(n_1,z_1)\, q(n_2-n_1,z_2-z_1)}_{\mathrm{P}(S_{n_1}=z_1,\ S_{n_2}=z_2)} \omega(n_1,z_1)\,\omega(n_2,z_2)$$

$$Z^{(k)} \simeq \beta^k \sum_{\substack{0<n_1<\ldots<n_k\leq N \\ z_1,\ldots,z_k\in\mathbb{Z}^2}} \underbrace{\prod_{i=1}^{k} q(n_i-n_{i-1},\, z_i-z_{i-1})}_{\mathrm{P}(S_{n_1}=z_1,\ldots,S_{n_k}=z_k)} \prod_{j=1}^{k} \omega(n_j,z_j)$$

Some CLTs

The first order chaos $Z^{(1)}$ is centred Gaussian with diverging variance

$$\mathbb{V}\mathrm{ar}[Z^{(1)}] \simeq \beta^2 \sum_{\substack{0<n_1\leq N \\ z_1\in\mathbb{Z}^2}} q(n_1,z_1)^2 \simeq \beta^2 \sum_{0<n_1\leq N} \frac{1}{\pi\, n_1} \sim \beta^2 \frac{\log N}{\pi}$$

Rescaling $\beta^2 \sim \frac{\pi \hat{\beta}^2}{\log N}$ we obtain $Z^{(1)} \xrightarrow{d} \hat{\beta}\, \mathscr{Z}_1$ with $\mathscr{Z}_1 \sim \mathscr{N}(0,1)$

The second order chaos $Z^{(2)}$ displays interesting behavior

$$Z^{(2)} \simeq \sum_{n_2-n_1>n_1} + \sum_{n_2-n_1\leq n_1} \xrightarrow{d} \hat{\beta}^2 \frac{(\mathscr{Z}_1)^2-1}{2} + \frac{\hat{\beta}^2}{\sqrt{2}} \underbrace{\mathscr{Z}_2}_{\text{independent } \mathscr{N}(0,1)}$$

A hierarchy of CLTs

Each chaos $Z^{(k)}$ yields many terms: $n_i - n_{i-1} > n_1$ VS. $n_i - n_{i-1} \leq n_1$

Building blocks: dominated sequences

$$\forall k \in \mathbb{N}: \qquad Z^{(k)}_{\text{dom}} := \sum_{\substack{0<n_1<\ldots<n_k\leq N \\ n_i - n_{i-1} \leq n_1 \ \forall i=2,\ldots,k}} (\cdots) \xrightarrow{d} \frac{\hat{\beta}^k}{\sqrt{k}} \underbrace{\mathscr{Z}_k}_{\text{independent } \mathscr{N}(0,1)}$$

It was our first meeting with the **Fourth Moment Theorem** [C.S.Z. 19]

We deduce that $Z^{(k)} \xrightarrow{d}$ explicit polynomial in $\{\mathscr{Z}_\ell : \ell \leq k\}$

Finally $\quad Z^{\omega}_{N,\beta} = 1 + \sum_{k\geq 1} Z^{(k)} \xrightarrow{d} :e^{\mathscr{Y}}: \qquad \mathscr{Y} := \sum_{k\geq 1} \frac{\hat{\beta}^k}{\sqrt{k}} \mathscr{Z}_k \sim \mathscr{N}(0, v^2)$

A different (old) approach

Computing fourth moment of $Z_{\text{dom}}^{(k)}$ is quite technical (but it is worth it!)

Simpler approach: Feller-Lindeberg CLT for triangular arrays [C. Cottini 22]

Condition $n_i - n_{i-1} \leq n_1$ can be replaced by $n_i - n_{i-1} \ll n_1 \quad \forall i = 2, \dots, k$

$\rightsquigarrow$ points $\{n_1, n_2, \dots, n_k\} \subseteq [1, N]$ are clustered around n_1

$\rightsquigarrow$ we can approximate $Z_{\text{dom}}^{(k)}$ by a sum of independent RVs

$$Z_{\text{dom}}^{(k)} \approx \sum_{j=1}^{M} \sum_{n_1, \dots, n_k \in B_j} (\dots) \qquad \text{with} \quad B_j := \left((j-1)\tfrac{N}{M}, j\tfrac{N}{M}\right] \cap \mathbb{Z}$$

Approximation $\approx$ in L^2 as $M \to \infty$ (bounds in $L^{2+\varepsilon}$ by hypercontractivity)

Reducible CLTs

A related (even simpler) approach to log-normality is [Cosco–Donadini 25]

All known CLTs for 2D directed polymers (and Stochastic Heat Equation, KPZ) can be deduced from Feller-Lindeberg: we call them reducible CLTs

> Can we characterize CLTs which are not reducible?

We investigated this question in the recent paper [C. Cottini Peccati 25]

Outline

Reducible CLTs

Reducible CLTs [C. Cottini 22]

Consider polynomial chaos of fixed order $d \in \mathbb{N}$ (for simplicity)

$$Z_N = \sum_{\{v_1,\dots,v_d\} \subseteq V_N} q_N(v_1,\dots,v_d) \prod_{i=1}^{d} \omega_{v_i}$$

Assume $\mathbb{E}[Z_N^2] \longrightarrow \sigma^2 \in (0,\infty)$ and (as $N \to \infty$)

- there are disjoint $B_1,\dots,B_M \subseteq V_N$ such that (as $N \to \infty$, $M \to \infty$)

$$\sum_{i=1}^{M} \left\{ \sum_{A \subseteq B_i} q_N(A)^2 \right\} \longrightarrow \sigma^2 \qquad \max_{i=1,\dots,M} \left\{ \sum_{A \subseteq B_i} q_N(A)^2 \right\} \longrightarrow 0$$

Then $Z_N \xrightarrow{d} \mathscr{N}(0,\sigma^2)$ (reduce to Feller-Lindeberg CLT)

Our setting

We focus on $q_N(v_1, \dots, v_d) = \mathbb{1}_{E_N}(v_1, \dots, v_d)$

$$E_N \subseteq \underbrace{V_N \times \cdots \times V_N}_{d \text{ times}} \quad \text{symmetric, finite with} \qquad |E_N| \longrightarrow \infty$$

Normalized polynomial chaos $\qquad \mathbb{E}[Z_N^2] = 1$

$$Z_N = \frac{1}{\sqrt{d!\,|E_N|}} \sum_{v_1 \neq \dots \neq v_d \in V_N} \mathbb{1}_{E_N}(v_1, \dots, v_d) \prod_{i=1}^{d} \omega_{v_i}$$

Reducible if there are disjoint $B_1^{(N)}, \dots, B_M^{(N)} \subseteq V_N$ s.t. $\qquad$ (as $N \to \infty$, $M \to \infty$)

$$\sum_{i=1}^{M} |E_N \cap (B_i \times \ldots \times B_i)| \sim |E_N| \qquad \max_{i=1,\dots,M} |E_N \cap (B_i \times \ldots \times B_i)| = o(|E_N|)$$

Irreducible CLT

We call the sequence Z_N irreducible if it is not reducible

Polynomial chaos Z_N $\iff$ Symmetric sets $E_N \subseteq (V_N)^d$

We provide sufficient conditions for irreducibility in terms of

- a notion of combinatorial dimension [Blei 79]
- spectral properties of the (hyper)graph V_N with edges E_N

Recall: Z_N satisfies a CLT $\iff$ $\mathbb{E}[Z_N^4] \to 3$ [Nourdin, Peccati, Reinert 10]

Combining these conditions we obtain irreducible CLTs

Combinatorial dimension

Definition [Blei 79]

A sequence of sets $E_N \subseteq (V_N)^d$ has combinatorial dimension $(d \geq 2)$

$$\alpha \in [1, d] \qquad \text{(possibly non-integer)}$$

if there are $0 < c < C < \infty$ such that $\forall N \in \mathbb{N}$

- $|E_N| \geq c\,|V_N|^\alpha$
- $|E_N \cap (A_1 \times \cdots \times A_d)| \leq C \max_{i=1,\ldots,d} |A_i|^\alpha$ $\quad \forall A_1, \ldots, A_d \subseteq V_N$

Irreducibility via combinatorial dimension

Theorem [C. Cottini Peccati 25]

For $d \geq 2$, let the sets $E_N \subseteq (V_N)^d$ have combinatorial dimension $\alpha > 1$

Then the corresponding polynomial chaos Z_N is irreducible

Moreover, for any $d \geq 3$ and $b \in \{2, \ldots, d-1\}$, we can build sets E_N with combinatorial dimension

$$\alpha = \frac{d}{b} \in (1, d)$$

such that $Z_N \xrightarrow{d} \mathcal{N}(0,1)$ ($\rightsquigarrow$ irreducible CLT)

Some ideas from the proof

- Assume $\sum_{i=1}^{M} |E_N \cap (B_i \times \ldots \times B_i)| \sim |E_N|$ and comb. dim. E_N is $\alpha > 1$

  Then $\max_{i=1,\ldots,M} |E_N \cap (B_i \times \ldots \times B_i)| \geq c\,|E_N|$ $\rightsquigarrow$ irreducibility

- Construction of E_N with $\alpha = \frac{d}{b}$ is obtained with $V_N = \{1, \ldots, N\}^b$

  Explicit $E_N \subseteq V_N$: "fractional cartesian product"

- Random construction available for $d = 2$ [Blei, Körner 84]

Irridubility via spectral graph properties

Fix $d = 2$ and symmetric sets $E_N \subseteq V_N \times V_N$ $\rightsquigarrow$ undirected graphs $(V_N, \mathscr{E}_N)$

Normalized Laplacian $\quad L_N := I - D_N^{-1/2} \underbrace{\mathbb{1}_{E_N}}_{\text{adjacency matrix}} D_N^{-1/2} \quad D_N =$ diag. matrix with degrees

Eigenvalues $\quad 0 = \mu_1^{(N)} \leq \mu_2^{(N)} \leq \cdots \leq \mu_{|V_N|}^{(N)} \leq 2$

Theorem [C. Cottini Peccati 25]

Assume that for some $k \geq 2$ $\qquad \liminf_{N\to\infty} \mu_k^{(N)} > 0$

Then the corresponding polynomial chaos Z_N is irreducible

Some ideas from the proof

Edge expansion of a subset $S \subseteq V_N$

$$\varphi(S) := \frac{E(S, S^c)}{\mathrm{Vol}(S)} = \frac{\#\text{ edges connecting } S \text{ and } S^c}{\#\text{ edges starting from } S}$$

Connected to reducibility via Cheeger's inequalities

- Easy direction: for any disjoint $B_1, \ldots, B_k \subseteq V_N$

$$\mu_k^{(N)} \leq \max_{j=1,\ldots,k} \varphi(B_j)$$

- Difficult direction: $\exists$ partition $B_1, \ldots, B_k \subseteq V_N$ [Lee, Gharan, Trevisan 14]

$$\max_{j=1,\ldots,k} \varphi(B_j) \leq C\, k^4 \sqrt{\mu_k^{(N)}}$$

Outline

Conclusion

We presented some CLTs for polynomial chaos from 2D directed polymers

Originally proved via Fourth Moment Theorem

Then shown to be reducible to Feller-Lindeberg CLT

We recently investigated irreducibile CLTs [C. Cottini Peccati 25]

Sufficient conditions via combinatorial dimension and spectral graph properties

Many interesting examples, but necessary conditions still open

Happy birthday, 4th Moment Theorem!