Renewal Theory, Disordered Systems, and Stochastic PDEs

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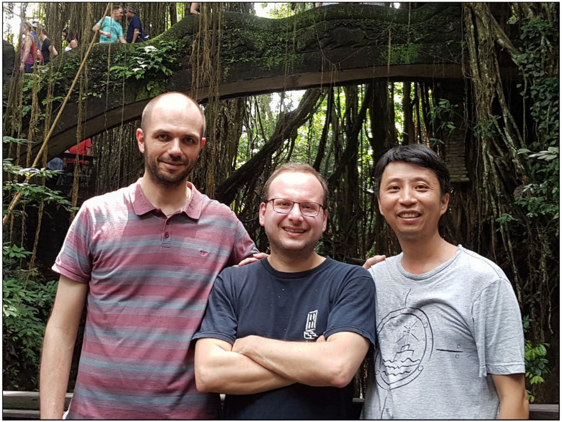
Lévy processes and random walks

A workshop in celebration on Ron Doney's 80th birthday

Based on joint works with



Nikos Zygouras (Warwick) and Rongfeng Sun (NUS)

Overview

1. Renewal Theorem for ultra-heavy tailed renewal processes in the domain of attraction of the Dickman subordinator

- [CSZ19] *The Dickman subordinator, renewal theorems, and disordered systems*, EJP (2019)

2. Link with Disordered Systems and Singular Stochastic PDEs
    - Model of Directed Polymer in Random Environment
    - $2d$ Stochastic Heat Equation with space-time white noise

- [CSZ21] *The Critical 2d Stochastic Heat Flow*, arXiv (2021)

To start with

I first met Ron through his papers

Back in 2003, I was a PhD student in Paris working on polymer models

This is how I got interested in random walks conditioned to stay positive

I discovered fluctuation theory from Feller's magnificent book

Looking for finer results, I stumbled into the beautiful paper

- L. Alili, R.A. Doney
  Wiener-hopf factorization revisited and some applications
  Stoch. Stoch. Rep. (1999)

No electronic version, genuine photocopy (I still have it!)

A few months later

I got the opportunity to meet Ron in person on November 6, 2003

It was on the occasion of Loïc Chaumont's *Habilitation*, held in the same place where I was doing my PhD (Chevaleret)

I didn't know I would have worked with Ron and Loïc in coming years!

In 2014 I wrote my very first paper, on RWs conditioned to stay positive

- J. Bertoin, R.A. Doney
  On conditioning a random walk to stay nonnegative, AOP (1994)

About the same time, I discovered the fundamental paper

- R.A. Doney
  One-sided local large deviation and renewal theorems in the case of infinite mean, PTRF (1999)

Ron's works have been a constant presence in my research

Later on

I met again Ron at St. Flour Summer School in 2005

We didn't have close contacts in the following years, but we did keep working on related topics in fluctuation and renewal theory...

... and in 2015 we happened to be working on the very same problem, the Strong Renewal Theorm, for which we decided to join our efforts

This eventually led to our paper

- F. Caravenna, R.A. Doney
  One-sided local large deviation and renewal theorems in the case of infinite mean, EJP (2019)

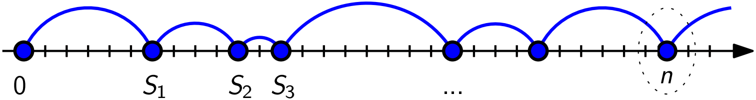
which is one of the most difficult problems I have worked on

The influence of Ron is also present in the work that I present today

The classical renewal theorem

Random walk $S_k := X_1 + X_2 + \dots X_k$ with positive increments

(X_i) i.i.d. $\quad X_i \in \mathbb{N} = \{1, 2, \dots\}$ $\quad$ aperiodic



Renewal function $\quad u_n := \mathrm{P}(S \text{ visits } n) = \sum_{k \geq 0} \mathrm{P}(S_k = n)$

Renewal Theorem (Erdos, Feller, Pollard 1949)

$$\lim_{n\to\infty} u_n = \frac{1}{\mathrm{E}[X]}$$ also when $\mathrm{E}[X] = \infty$

The case of heavy tails

When $\mathrm{E}[X] = \infty$ we have $u_n \to 0$. At which rate?

Tail Assumption

$$\mathrm{P}(X > n) \underset{n\to\infty}{\sim} \frac{\ell_n}{n^\alpha} \qquad \alpha \in (0,1) \qquad \ell.\ \text{slowly varying}$$

Theorem

(Doney 1997) (Garsia, Lamperti 1962)

$$u_n \underset{n\to\infty}{\sim} \frac{c}{\mathrm{E}[X \wedge n]} = \frac{c_\alpha}{\ell_n\, n^{1-\alpha}} \qquad \text{with} \qquad c_\alpha := \frac{\sin \pi\alpha}{\pi}$$

if local assumption holds: $\mathrm{P}(X = n) \le C\, \dfrac{\ell_n}{n^{1+\alpha}}$

In [CarDon19] we proved necessary and sufficient conditions

A case of ultra-heavy tails

Let us now focus on the extreme case $\alpha = 0$

$$\mathrm{P}(X = n) = p_n \sim \frac{1}{n}$$

This makes sense with a truncation at scale N

$$\mathrm{P}(X^{(N)} = n) = \frac{p_n \, \mathbb{1}_{\{1 \le n \le N\}}}{p_1 + \ldots + p_N} \sim \frac{1}{n} \frac{\mathbb{1}_{\{1 \le n \le N\}}}{\log N}$$

Triangular array of renewal processes

$$S_k^{(N)} = X_1^{(N)} + \ldots + X_k^{(N)}$$

Renewal function (exponentially weighted) for $\vartheta \in \mathbb{R}$

$$u_n^{(N)} := \sum_{k \ge 0} \left(1 + \tfrac{\vartheta}{\log N}\right)^k \mathrm{P}(S_k^{(N)} = n)$$

Main result

Since $\mathrm{E}[X^{(N)}] \sim \dfrac{N}{\log N}$ we expect $u_n^{(N)} \approx \dfrac{\log N}{N}$ as $n \approx N \to \infty$

Theorem (Strong Renewal Theorem [CSZ19])

For any $\delta > 0$

$$u_n^{(N)} \sim \frac{\log N}{N} \, G_\vartheta\left(\tfrac{n}{N}\right) \qquad \textit{uniformly for } \delta N \le n \le N$$

where

$$G_\vartheta(t) := \int_0^\infty \frac{\mathrm{e}^{(\vartheta-\gamma)s} \, s \, t^{s-1}}{\Gamma(s+1)} \, \mathrm{d}s$$

Where does the function $G_\vartheta(t)$ come from?

The Dickman subordinator

Our renewal process $S^{(N)}$ is attracted to a pure jump Lévy process Y

$$\left(\frac{S^{(N)}_{\lfloor s \log N \rfloor}}{N}\right)_{s \ge 0} \xrightarrow[N\to\infty]{d} Y = (Y_s)_{s\ge 0}$$

called the Dickman subordinator

- Lévy measure $$\nu^Y(\mathrm{d}t) := \frac{1}{t}\, \mathbb{1}_{(0,1)}(t)\, \mathrm{d}t$$
- Explicit density $$\frac{\mathrm{P}(Y_s \in \mathrm{d}t)}{\mathrm{d}t} = \frac{\mathrm{e}^{-\gamma s}\, s\, t^{s-1}}{\Gamma(s+1)} \quad \text{for } t \in (0,1)$$

$G_\vartheta(t)$ is the (exponentially weighted) renewal function of Y

$$G_\vartheta(t) = \int_0^\infty \mathrm{e}^{\vartheta s}\, \frac{\mathrm{P}(Y_s \in \mathrm{d}t)}{\mathrm{d}t}\, \mathrm{d}s$$

Key tools

- Sharp local bound

$$\mathrm{P}\big(S_k^{(N)} = n\big) \le C\, k\, \mathrm{P}\big(X^{(N)} = n\big)\, \mathrm{P}\big(X^{(N)} \le n\big)^{k-1} \exp\Big\{-c\, \tfrac{k}{\log n} \log^+ \tfrac{k}{\log n}\Big\}$$

  - Intuitively "a single jump brings $S^{(N)}$ close to n"
  - Super-exponential decay in k

- Recursive formulas

$$u_n^{(N)} = \sum_{0 \le a < \frac{n}{2} \le b \le n} u_a^{(N)}\, \mathrm{P}\big(X^{(N)} = b - a\big)\, u_{n-b}^{(N)} .$$

to obtain local renewal estimates (l.h.s.) from integrated ones (r.h.s.)

Other cases $\alpha = 0$ in [Nagaev 08] [Nagaev, Wachtel 12] [Alexander, Berger 16]

Directed Polymer in Random Environment

Example of disordered model in statistical mechanics

"random walk interacting with a random medium" (in a Gibbsian way)

Introduced in the 1980s to describe interfaces in Ising model

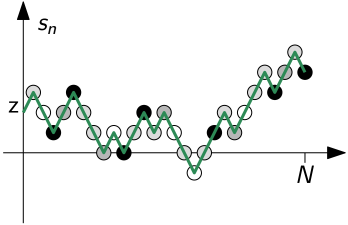
[Imbrie, Spencer JSP 88] [Bolthausen CMP 89]

A stream of mathematical research in the last 25 years

- Localization phenomena
- Super-diffusivity
- KPZ universality class

St. Flour 2016 Lecture notes by Francis Comets

Directed Polymer Partition Function



- $s = (s_n)_{n \geq 0}$ simple random walk on $\mathbb{Z}^d$
- $\omega(n, z)$ independent $N(0, 1)$ (disorder)
- $H_N(s, \omega) := \sum_{n=1}^{N} \omega(n, s_n)$

Directed Polymer Partition Functions ($N \in \mathbb{N}$, $z \in \mathbb{Z}^d$)

$$Z(N, z) := \mathrm{E}\left[e^{\beta H_N(s,\omega) - \frac{1}{2}\beta^2 N}\right] = \frac{1}{(2d)^N} \sum_{\substack{s=(s_0,\ldots,s_N) \\ \text{s.r.w. path} \\ \text{with } s_0 = z}} e^{\beta H_N(s,\omega) - \frac{1}{2}\beta^2 N}$$

Diffusive rescaling

We focus on $d = 2$. Let us average $Z(N, z)$ on the diffusive scale

$$Z(N, \varphi) := \frac{1}{N} \sum_{z \in \mathbb{Z}^2} Z(N, z)\, \varphi(\tfrac{z}{\sqrt{N}}) \qquad \varphi \in C_c(\mathbb{R}^2)$$

- Mean easily converges: (since $\mathbb{E}[Z(N, z)] = 1$)

$$\mathbb{E}[Z(N, \varphi)] \xrightarrow[N \to \infty]{} \int_{\mathbb{R}^2} \varphi(x)\, \mathrm{d}x$$

- Variance converges if we critically rescale $\beta = \beta_N \sim \sqrt{\frac{\pi}{\log N}}$

$$\mathbb{V}\mathrm{ar}[Z(N, \varphi)] \xrightarrow[N \to \infty]{} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \varphi(x)\, K(x, y)\, \varphi(y)\, \mathrm{d}x\, \mathrm{d}y$$

with $K(x, y) \sim C \log \frac{1}{|x - y|}$ [Bertini, Cancrini 98]

Critical scaling limit

Theorem (Critical 2d Stochastic Heat Flow [CSZ21])

With the critical rescaling

$$\beta = \beta_N = \sqrt{\frac{\pi}{\log N}}\left(1 + \frac{\vartheta}{\log N}\right) \qquad \text{for } \vartheta \in \mathbb{R}$$

we have the joint convergence in distribution over $t \geq 0$, $\varphi \in C_c(\mathbb{R}^2)$

$$Z(Nt, \varphi) \xrightarrow[N\to\infty]{d} \mathscr{Z}(t, \varphi) = \int_{\mathbb{R}^2} \varphi(x)\, \mathscr{Z}(t, \mathrm{d}x)$$

The limiting process $\mathscr{Z}(t, \mathrm{d}x)$ *is called Critical 2d Stochastic Heat Flow*

"Explicit" formulas for all moments of $\mathscr{Z}(t, \varphi)$ [Gu, Quastel, Tsai 21]

Link with the Dickman subordinator

Scaling limit of the variance

$$\mathbb{V}\mathrm{ar}[Z(N,\varphi)] \approx \int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\, K_N(x,y)\, \varphi(y)\, \mathrm{d}x\, \mathrm{d}y$$

Explicit kernel

$$K_N(x,y) = \beta^2 \sum_{1\le m<n\le N} \mathrm{P}(s_m = \sqrt{N}(x-y)) \cdot u^{(N)}_{n-m}$$

where $u^{(N)}$ = renewal function of ultra-heavy tailed renewal process

$$\lim_{N\to\infty} K_N(x,y) = \pi \iint_{0<s<t<1} \underbrace{g_s(x-y)}_{\text{heat kernel on } \mathbb{R}^2} \cdot \underbrace{G_\vartheta(t-s)}_{\substack{\text{renewal function of the}\\ \text{Dickman subordinator}}} \, \mathrm{d}s\, \mathrm{d}t$$

Polynomial chaos

- Simple random walk kernel on $\mathbb{Z}^2$ $\qquad q(n,z) = \mathrm{P}(s_n = z)$
- New i.i.d. centred random variables $\qquad \tilde{\omega}(n,z) := \dfrac{e^{\beta\omega(n,x) - \frac{1}{2}\beta^2} - 1}{\beta}$

Polynomial chaos

Equivalent rewriting of the partition function

$$Z(N,z) = 1 + \beta \sum_{\substack{1 \le \ell \le N \\ x \in \mathbb{Z}^2}} q(\ell,x)\,\tilde{\omega}(\ell,x)$$
$$+\beta^2 \sum_{\substack{1 \le \ell < m \le N \\ x,y \in \mathbb{Z}^2}} q(\ell,x)\,q(m-\ell,y-x)\,\tilde{\omega}(\ell,x)\,\tilde{\omega}(m,y) + \ldots$$

Enters renewal theory

It follows that

$$\mathbb{V}\mathrm{ar}[Z(N,z)] = \beta^2 \sum_{\substack{1\le\ell\le N\\ x\in\mathbb{Z}^2}} q(\ell,x)^2 \; + \; \beta^4 \sum_{\substack{1\le\ell<m\le N\\ x,y\in\mathbb{Z}^2}} q(\ell,x)^2\, q(m-\ell,y-x)^2 + \ldots$$

$$= \beta^2 \sum_{1\le\ell\le N} q(2\ell,0) \; + \; \beta^4 \sum_{1\le\ell<m\le N} q(2\ell,0)\, q(2(m-\ell),0) + \ldots$$

$$\sim \beta^2 \sum_{1\le\ell\le N} \frac{1}{\pi\ell} \; + \; \beta^4 \sum_{1\le\ell<m\le N} \frac{1}{\pi\ell}\,\frac{1}{\pi(m-\ell)} + \ldots$$

With the critical rescaling of $\beta = \beta_N \sim \sqrt{\frac{\pi}{\log N}}$ we obtain

$$\mathbb{V}\mathrm{ar}[Z(N,z)] \sim \mathrm{P}(S_1^{(N)} \le N) + \mathrm{P}(S_2^{(N)} \le N) + \ldots \; = \sum_{n=1}^{N} u_n^{(N)}$$

The Stochastic Heat Equation

Very interesting, yet ill-defined stochastic PDE on $\mathbb{R}^d$

$$\partial_t u(t,x) = \Delta u(t,x) + \beta\, u(t,x)\, \xi(t,x) \qquad \text{(SHE)}$$

with initial condition $u(0,x) \equiv 1$ (say)

$\xi(t,x) =$ white noise on $\mathbb{R}^{1+d}$ (space-time white noise)

ξ is very irregular $\rightsquigarrow$ product $u\,\xi$ is classically ill-defined

For $d = 1$ SHE makes sense via Ito-Walsh stochastic integration (1980s)

Also "pathwise" approach via Regularity Structures [Hairer Pardoux 15]

For $d \geq 2$ the situation is open

The Critical 2d Stochastic Heat Equation

We now fix $d = 2$. To make sense of SHE, we can regularize the noise

- mollification in space: $\xi_\delta = \xi * g_\delta \quad \delta > 0$
- discretization in space-time: $\xi(t,x) \rightsquigarrow \omega(\frac{n}{N}, \frac{z}{\sqrt{N}})$ i.i.d. $N(0,1)$

$\rightsquigarrow$ SHE becomes a difference equation on the rescaled lattice $\frac{\mathbb{N}}{N} \times \frac{\mathbb{Z}}{\sqrt{N}}$

The solution $u(n,z)$ of the discretized SHE is...

... the Directed Polymer partition function $Z(n,z)$

(dicrete Feynman-Kac formula)

The Critical 2d Stochastic Heat Flow $\mathscr{Z}(t, \mathrm{d}x)$ is a natural candidate for the ill-defined solution of the 2d Stochastic Heat Equation (with critically rescaled noise strength!)

Conclusions

Renewal Theory is a beautiful research area, classical and recent

It is also an incredibly useful tool for many different models

These include random polymer models (in highly non-obvious ways!). . .

. . . and it may also be useful for some Stochastic PDEs

Thanks... and Happy Birthday, Ron!

(Picture taken from MFO website)