2D directed polymers and Stochastic Heat Flow

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Large Scale Stochastic Dynamics

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Collaborators

Based on joint works with

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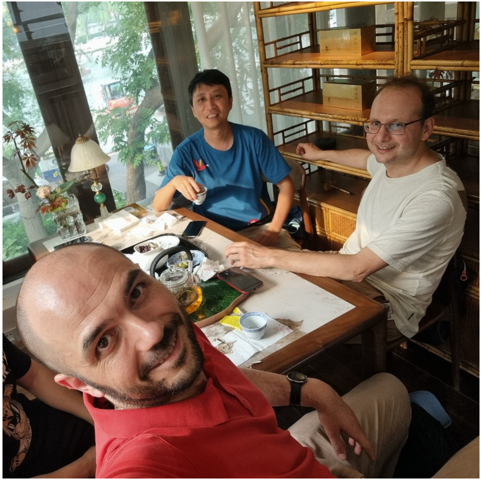

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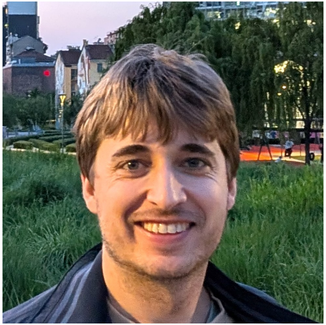

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Outline

1. Erwin
2. Directed Polymers
3. Stochastic Heat Equation
4. Stochastic Heat Flow

Erwin

I started as a postdoc of Erwin's 20 years ago

There was a celebration in Zurich for his 60th birthday

Erwin was a hero of my PhD

- His very first paper [AOP 1976, 5 pages]
- His copolymer paper with Frank den Hollander
- Our first meeting in Giambattista Giacomin's office

I was both excited and humbled to be his postdoc

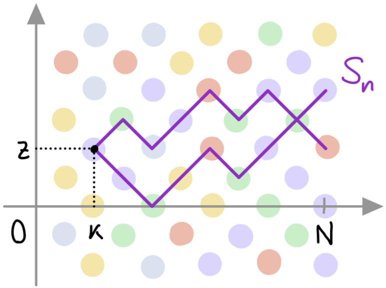
It was a great experience, which shaped the direction of my future research

> *"I don't have any special ideas — I just need to think about the problem long enough until I see the solution"*

Outline

Directed Polymer in Random Environment

- $S = (S_n)_{n \geq 0}$ simple random walk on $\mathbb{Z}^d$
- Independent Gaussians $\omega(n,x) \sim \mathcal{N}(0,1)$
- $H(S,\omega) := \sum_{n=k+1}^{N} \omega(n, S_n) \sim \mathcal{N}(0, N-k)$



Partition Functions $(k \in \mathbb{N},\ z \in \mathbb{Z}^d)$

$$Z_{N,\beta}^{\omega}(k,z) = \mathrm{E}\left[e^{\beta H(S,\omega) - \underbrace{\tfrac{1}{2}\beta^2 (N-k)}_{\text{(constant)}}} \,\middle|\, S_k = z \right] \qquad \mathbb{E}\left[Z_{N,\beta}^{\omega}\right] = 1$$

Phase transition

Let us focus on $$Z^{\omega}_{N,\beta} = Z^{\omega}_{N,\beta}(0,0) = \mathrm{E}\left[e^{\beta H(S,\omega) - \frac{1}{2}\beta^2 N} \,\middle|\, S_0 = 0 \right]$$

Key observation

[Bolthausen 89]

$$Z^{\omega}_{N,\beta} \geq 0 \text{ is a martingale } \xrightarrow[N\to\infty]{\text{a.s.}} Z^{\omega}_{\infty,\beta}$$

Phase transition at $\beta_c = \beta_c^{(d)} \geq 0$ [Comets–Yoshida, Junk–Lacoin, ...]

- For $\beta \leq \beta_c$ $\quad Z^{\omega}_{\infty,\beta} > 0$ a.s. $\quad \mathrm{F}(\beta) := \lim_{N\to\infty} \frac{1}{N} \log Z^{\omega}_{N,\beta} = 0$
- For $\beta > \beta_c$ $\quad Z^{\omega}_{\infty,\beta} = 0$ a.s. $\quad \mathrm{F}(\beta) := \lim_{N\to\infty} \frac{1}{N} \log Z^{\omega}_{N,\beta} < 0$

Path properties

Polymer measure $$\mathrm{dP}^{\omega}_{N,\beta}(S) = \frac{e^{\beta H(S,\omega) - \frac{1}{2}\beta^2 N}}{Z^{\omega}_{N,\beta}} \, \mathrm{dP}(S)$$

- $\beta \leq \beta_c$ weak disorder [Bolthausen, Comets–Yoshida]
  - $|S_N| = O(\sqrt{N})$ diffusive under $\mathrm{P}^{\omega}_{N,\beta}$
- $\beta > \beta_c$ strong disorder [Carmona–Hu, Bates–Chatterjee]
  - S under $\mathrm{P}^{\omega}_{N,\beta}$ localizes around finitely many paths
  - super-diffusivity (mostly conj.)

Intermediate disorder

Critical point: • $\beta_c > 0$ for $d \geq 3$ • $\beta_c = 0$ for $d \leq 2$

$(d \leq 2)$ any $\beta > 0$ changes the RW behavior [disorder relevance]

$$Z^{\omega}_{N,\beta>0} \xrightarrow[N\to\infty]{} 0 \qquad (\text{despite } \; Z^{\omega}_{N,\beta=0} \equiv 1)$$

Intermediate disorder $(d \leq 2)$

Can we tune $\beta = \beta_N \to 0$ so that $Z^{\omega}_{N,\beta_N} \xrightarrow[N\to\infty]{} \mathscr{Z}^{\xi} > 0$ random ?

$$\mathrm{P}^{\omega}_{N,\beta_N} \xrightarrow[N\to\infty]{} \mathscr{P}^{\xi}$$

$\rightsquigarrow$ Stochastic Heat Equation $\partial_t u = \Delta u + \beta\,\xi\,u$ $\quad\xi$ = white noise

Non-trivial phase transition for $d = 2$

We mainly focus on $d = 2$ (critical dimension)

Phase transition on the intermediate scale

$$\beta_N \sim \frac{\hat{\beta}}{\sqrt{\log N}} \qquad \text{with critical point} \quad \hat{\beta}_c = \sqrt{\pi}$$

Log-normality [C.S.Z. 19]

For $\hat{\beta} < \sqrt{\pi}$ $\quad Z^{\omega}_{N,\beta_N} \xrightarrow[N\to\infty]{d} e^{v\mathcal{N}(0,1)-\frac{1}{2}v^2} > 0 \qquad v^2 = \log \frac{1}{1-\pi/\hat{\beta}^2}$

For $\hat{\beta} \geq \sqrt{\pi}$ $\quad Z^{\omega}_{N,\beta_N} \xrightarrow[N\to\infty]{d} 0$

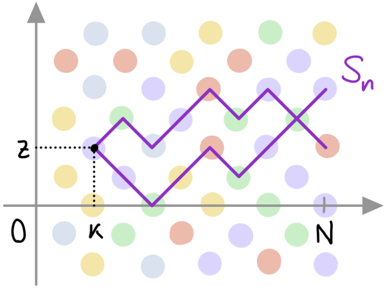
What happens at $\hat{\beta} = \sqrt{\pi}$?

Outline

Partition functions and SHE

Partition functions $Z^{\omega}_{N,\beta}(k,z)$

satisfy a discretised Stochastic Heat Equation



Diffusive rescaling (time rev.)

$$u_N(t,x) := Z^{\omega}_{N,\beta}\left(N(1-t), \sqrt{N}x\right)$$

$$\begin{cases} \partial_t u_N(t,x) = \Delta_x u_N(t,x) + \underbrace{\beta\, N^{\frac{2-d}{4}}}_{\beta_{\text{SHE}}} \; \underbrace{\xi_N(t,x)}_{\to \text{ white noise}} \; u_N(t,x) \\ u_N(0,x) \equiv 1 \end{cases} \qquad \text{(disc-SHE)}$$

Sub-critical dimension $d = 1$

Removing discretization $N \to \infty$ yields (formally) Stochastic Heat Equation

$$\begin{cases} \partial_t u(t,x) = \Delta_x u(t,x) + \beta_{\mathrm{SHE}}\, u(t,x)\, \xi(t,x) \\ u(0,x) \equiv 1 \end{cases} \qquad \text{(SHE)}$$

Well defined in $d = 1$ (sub-critical) for any $\beta_{\mathrm{SHE}} \geq 0$

Intermediate disorder $(d = 1)$ [Alberts–Khanin–Quastel 14]

Rescaling $\beta_N = \dfrac{\beta_{\mathrm{SHE}}}{N^{1/4}}$ $\qquad Z_{N,\beta}^{\omega}\big(N(1-t), \sqrt{N}x\big) \xrightarrow[N\to\infty]{d} u(t,x)$

Critical dimension $d = 2$

SHE is ill-defined in dimension $d = 2$ (and higher) ξ white noise

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, \xi(t,x)\, u(t,x) \qquad \text{(SHE)}$$

No solution $\rightsquigarrow$ no candidate limit of $\boxed{u_N(t,x) = Z^{\omega}_{N,\beta}\big(N(1-t), \sqrt{N}x\big)}$

Main question: non-trivial limit?

$$u_N(t,\varphi) := \int_{\mathbb{R}^2} \varphi(x)\, \underbrace{u_N(t,x)\,\mathrm{d}x}_{\text{random measure}} \xrightarrow[N\to\infty]{d} \mathscr{U}(t,\varphi) = \int_{\mathbb{R}^2} \varphi(x)\, \underbrace{\mathscr{U}(t,\mathrm{d}x)}_{\text{random measure}} \quad ?$$

The need for renormalization

We take $\beta = \beta_N \sim \frac{\sqrt{\pi}}{\sqrt{\log N}} \left(1 + \frac{\vartheta}{\log N}\right)$

Randomness of $u_N(t,x)\,\mathrm{d}x$ persists as $N \to \infty$ despite $\beta_N \to 0$

- $\mathbb{V}\mathrm{ar}\left[u_N(t,\varphi)\right] \xrightarrow[N\to\infty]{} K_t^{\vartheta}(\varphi,\varphi) > 0$ [Bertini–Cancrini 98]
- Convergence of higher moments [C.S.Z. 19] [Gu–Quastel–Tsai 21]

Trivial convergence for fixed $\beta > 0$ [Berger C. Turchi 25]

$$u_N(t,x)\,\mathrm{d}x \xrightarrow[N\to\infty]{\mathbb{P}} 0 \quad \text{vaguely}$$

(on compact sets)

Outline

The critical 2D Stochastic Heat Flow

Theorem [C.S.Z. 23]

Take $\beta = \frac{\sqrt{\pi}}{\sqrt{\log N}}\left(1 + \frac{\vartheta}{\log N}\right)$ for some $\vartheta \in \mathbb{R}$

Then $u_N(t,\varphi)$ converges in law to a unique and non-trivial limit $\mathscr{U}^{\vartheta}(t,\varphi)$

$$\big(u_N(t,x)\,\mathrm{d}x\big)_{t\geq 0} \xrightarrow[N\to\infty]{d} \big(\mathscr{U}^{\vartheta}(t,\mathrm{d}x)\big)_{t\geq 0}$$

$\mathscr{U}^{\vartheta}$ = critical 2D **Stochastic Heat Flow (SHF)** = stochastic process of random measures on $\mathbb{R}^2$

SHF and Stochastic Heat Equation

The SHF is a "candidate solution" of the critical 2d Stochastic Heat Equation

$$\mathscr{U}^{\vartheta}(t, \mathrm{d}x) \qquad (\text{starting at time } 0 \text{ from } \mathscr{U}^{\vartheta}(t, \mathrm{d}x) \equiv 1)$$

We actually build a two-parameter space-time process

$$\left(\mathscr{U}^{\vartheta}(s, \mathrm{d}y; t, \mathrm{d}x)\right)_{0 \le s \le t < \infty} \qquad (\text{starting at time } s \text{ from } \mathrm{d}y)$$

"Flow": Chapman-Kolmogorov property for $s < t < u$ [Clark–Mian 2024+]

$$\mathscr{U}^{\vartheta}(s, \mathrm{d}y; u, \mathrm{d}z) = \int_{x \in \mathbb{R}^2} \mathscr{U}^{\vartheta}(s, \mathrm{d}y; \underbrace{t, \mathrm{d}x)\, \mathscr{U}^{\vartheta}(t, \mathrm{d}x}_{\substack{\text{non-trivial "product"} \\ \text{of measures}}}; u, \mathrm{d}z)$$

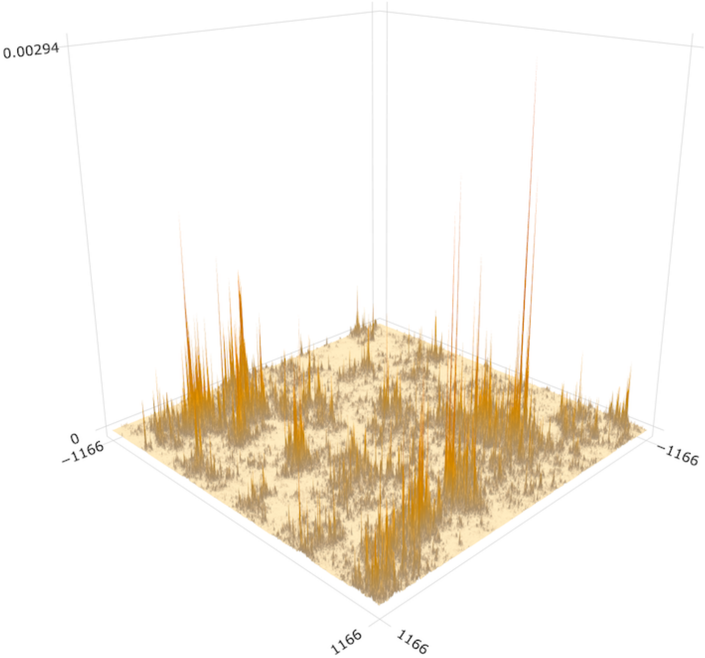
Key properties of SHF

- Scaling covariance $\quad a^{-1}\,\mathscr{U}^{\vartheta}(\mathrm{a}\,t, \mathrm{d}(\sqrt{\mathrm{a}}\,x)) \stackrel{d}{=} \mathscr{U}^{\vartheta+\log a}(t, \mathrm{d}x)$
- $\mathbb{E}[\mathscr{U}^{\vartheta}(t,\mathrm{d}x)] = \mathrm{d}x \qquad \mathbb{E}[\mathscr{U}^{\vartheta}(t,\mathrm{d}x)\,\mathscr{U}^{\vartheta}(t,\mathrm{d}y)] = \underbrace{K^{\vartheta}(t, x-y)}_{\approx \log|x-y|^{-1}}\,\mathrm{d}x\,\mathrm{d}y$
- Formulas for all moments [C.S.Z. 19] [Gu–Quastel–Tsai 21]
- Axiomatic characterization via independence & moments [Tsai 24+]
- Universality w.r.t. approximation scheme [C.S.Z. 23] [Tsai 24+]
- Moment growth $\quad \mathbb{E}[\mathscr{U}^{\vartheta}(t,\varphi)^h] \geq \exp(\exp(c\,h))$ [Ganguly–Nam 25+]

Singularity and regularity of SHF

- a.s. $\mathscr{U}^{\vartheta}(t, \mathrm{d}x)$ is singular w.r.t. Lebesgue [C.S.Z. 25+]
  "not a function"
- a.s. $\mathscr{U}^{\vartheta}(t, \mathrm{d}x) \in \mathscr{C}^{-\kappa}$ for any $\kappa > 0$ (in particular: non atomic)
  "*barely* not a function"
- Not a GMC on $\mathbb{R}^2$: $\mathscr{U}^{\vartheta}(t, \mathrm{d}x) \neq \mathrm{e}^{X(x)}\,\mathrm{d}x$ for Gaussian X [C.S.Z. 23]
  with $\mathbb{C}\mathrm{ov}[X(x), X(y)] \sim \log\log\frac{1}{|x-y|}$
- Conditional GMC on path space $C([0,\infty), \mathscr{M}(\mathbb{R}^2))$ [Clark–Tsai 25+]

Let us show a simulation of SHF where we zoom in (equiv. we send $\vartheta \to -\infty$)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0
−1166
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Strong disorder for SHF

The mass of SHF $\mathscr{U}^{\vartheta}(t, \mathrm{d}x)$ escapes to infinity as $\vartheta \to \infty$ and/or $t \to \infty$

Vanishing mass in large balls

[Berger–C.–Turchi 25+]

$$\mathscr{U}^{\vartheta}(t, B_{\vartheta,t}) \xrightarrow[t \vee \vartheta \to \infty]{\mathbb{P}} 0 \qquad \text{with} \quad B_{\vartheta,t} = B\Big(0, \mathrm{e}^{c\, t\, \mathrm{e}^{\vartheta}} \sqrt{t}\Big)$$

Free energy bound

[Berger–C.–Turchi 25+]

$$\mathrm{F}(\beta) \leq -c\, \mathrm{e}^{-\frac{\pi}{\sigma^2(\beta)}} \qquad \text{with} \quad \sigma^2(\beta) = \mathrm{e}^{\beta^2} - 1$$

SHF and white noise

Does SHF $\mathscr{U}^{\vartheta}(t, \mathrm{d}x)$ satisfy a SPDE driven by white noise $\xi(t,x)$? **NO!**

- SHF $\mathscr{U}^{\vartheta}(t, \mathrm{d}x)$ is the limit of discretized SHE solutions $u_N(t,x)\,\mathrm{d}x$
- White noise $\xi(t,x)$ is the limit of regularised noise $\xi_N(t,x)$

Theorem [C.–Donadini 25+]

$$(u_N, \xi_N) \xrightarrow[N\to\infty]{d} (\mathscr{U}^{\vartheta}, \xi) \qquad \mathscr{U}^{\vartheta} \text{ and } \xi \textbf{ independent}$$

Deduced from noise sensitivity of u_N

Independently: $\mathscr{U}^{\vartheta}$ is a "black noise" [Gu–Tsai 25+]

Conclusions

The critical 2D SHF $\mathscr{U}^{\vartheta}$ is a universal object

It arises from directed polymers partition functions (or regularisation of SHE)

Many interesting properties, as well as many open problems

Connection to directed polymers: super-diffusivity?

Happy birthday, Erwin!