The 2D Stochastic Heat Equation and related critical models

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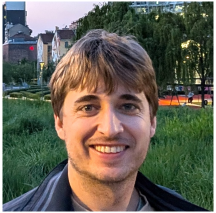

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Survey

From disordered systems to the Critical 2D Stochastic Heat Flow

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Abstract. We review our joint work on the scaling limits of disordered systems, linking the notion of disorder relevance/irrelevance to that of sub/super-criticality of singular SPDEs. This line of research culminated in the construction of the Critical 2D Stochastic Heat Flow (SHF), a universal process which provides a non-trivial solution to the Stochastic Heat Equation in dimension 2, a critical singular SPDE that lies beyond the reach of existing solution theories. The SHF also offers a rare example of a non-Gaussian scaling limit for a disordered system at its phase transition point in the critical dimension.

https://arxiv.org/abs/2511.08479

In a nutshell

Stochastic Heat Equation (SHE)

$$\partial_t u(t,x) = \underbrace{\Delta_x u(t,x)}_{\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i{}^2} u(t,x)} + \beta\, \xi(t,x)\, u(t,x) \qquad t \geq 0,\ x \in \mathbb{R}^d$$

Singular random potential $\xi(t,x)$ "space-time white noise"

Main result

We construct a natural candidate solution of SHE in space dimension $d = 2$

called the critical 2D Stochastic Heat Flow (SHF)

Why is it interesting?

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, \xi(t,x)\, u(t,x) \qquad \text{(SHE)}$$

- Fundamental PDE + universal random potential $\xi(t,x)$

  white noise = "continuum" i.i.d. random variables

- KPZ equation [Kardar–Parisi–Zhang *PRL* 86]

$$\partial_t h(t,x) = \Delta_x h(t,x) + |\nabla_x h(t,x)|^2 + \beta\, \xi(t,x) \qquad \text{(KPZ)}$$

Cole–Hopf transformation $h(t,x) = \log u(t,x)$

Why is it difficult?

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, \xi(t,x)\, u(t,x) \qquad \text{(SHE)}$$

$\xi(t,x)$ is a distribution $\rightsquigarrow$ u(t,x) expected $\begin{cases} \text{non-smooth function} & d = 1 \\ \text{genuine distribution} & d \geq 2 \end{cases}$

Product $\xi(t,x)\, u(t,x)$ unclear: no classical space to solve SHE (as a PDE)

Stochastic integral for $d = 1$ [Ito/Walsh, Da Prato–Zabczyk]

SHE solution $u(t,x) > 0$ starting from $u(0,\cdot) \geq 0$ $\rightsquigarrow$ "KPZ solution" $h(t,x) = \log u(t,x)$

The role of dimension

Revolution in 2010s: **robust solution theories** for **sub-critical SPDEs**

[Hairer *Invent. Math.* 14] [Gubinelli–Imkeller–Perkowski *Forum Math Pi* 15] [. . .]

These theories apply to SHE and KPZ only for $d = 1$

Space-time blow-up $\tilde{u}(t,x) := u(\varepsilon^2 t, \varepsilon x)$

$$\partial_t \tilde{u}(t,x) = \Delta_x \tilde{u}(t,x) + \varepsilon^{\frac{2-d}{2}} \beta \, \tilde{\xi}(t,x) \, \tilde{u}(t,x)$$

$$\text{as } \varepsilon \downarrow 0 \text{ the noise strength } \begin{cases} \text{vanishes} & d < 2 \quad \text{sub-critical} \\ \text{unchanged} & d = 2 \quad \textbf{critical} \\ \text{diverges} & d > 2 \quad \text{super-critical} \end{cases}$$

What can we do?

We focus on the **critical dimension** $d = 2$

Regularized noise $\xi_N(t,x)$ by discretization on scale $\frac{1}{N}$ (or mollification, ...)

$\rightsquigarrow$ Well-defined solution $u_N(t,x)$

$$\begin{cases} \partial_t u_N(t,x) = \Delta_x u_N(t,x) + \beta\, \xi_N(t,x)\, u_N(t,x) \\ u_N(t,0) \equiv 1 \quad \text{(for simplicity)} \end{cases} \qquad \text{(reg-SHE)}$$

Fix $\xi_N(t,x) \xrightarrow[N\to\infty]{} \xi(t,x)$

Does $u_N(t,x)$ converge to an interesting limit?

Which notion of convergence?

Do not expect pointwise convergence $\qquad \varphi : \mathbb{R}^2 \to \mathbb{R}$

Space-average $\qquad u_N(t,\varphi) := \int_{\mathbb{R}^2} \varphi(x)\, u_N(t,x)\, \mathrm{d}x \xrightarrow[N\to\infty]{d} \mathscr{U}(t,\varphi) \qquad (?)$

$\iff$ random measure on $\mathbb{R}^2$ $\qquad \underbrace{u_N(t,x)}_{\geq 0 \ \ \mathbb{E}[\cdot]=1}\, \mathrm{d}x \xrightarrow[N\to\infty]{d} \mathscr{U}(t,\mathrm{d}x) \qquad (?)$

Convergence? *No interesting limit $\forall$ fixed β* [Berger–C.–Turchi 25+]

$\forall$ density $\varphi \in C_c(\mathbb{R}^2)$ $\qquad u_N(t,\varphi) \xrightarrow[N\to\infty]{} \begin{cases} 1 & \text{if } \beta = 0 \\ 0 & \text{if } \beta > 0 \end{cases}$

The need for renormalization

We take $\beta = \beta_N \sim \frac{\sqrt{\pi}}{\sqrt{\log N}} \left(1 + \frac{\vartheta}{\log N}\right)$

$$\partial_t u_N(t,x) = \Delta_x u_N(t,x) + \beta_N \, \xi_N(t,x) \, u_N(t,x) \qquad \text{(reg-SHE)}$$

Formally $\beta_N \, \xi_N(t,x) \, u_N(t,x) \longrightarrow 0 \ \ldots$ *but actually not!*

- $\mathbb{V}\mathrm{ar}\big[u_N(t,\varphi) \big] \xrightarrow[N\to\infty]{} K_t^{\vartheta}(\varphi,\varphi) > 0$ [Bertini–Cancrini *J Phys A* 98]
- Higher moments converge [C.S.Z. *CMP* 19] [Gu–Quastel–Tsai *PMP* 21]

Does $u_N(t,\varphi)$ converge to an interesting limit ? $\big(\Longleftrightarrow \ u_N(t,x)\,\mathrm{d}x \big)$

Main result

Theorem

[C.S.Z. *Invent. Math.* 23]

Take
$$\beta_N = \frac{\sqrt{\pi}}{\sqrt{\log N}}\left(1 + \frac{\vartheta}{\log N}\right) \quad \text{for some } \vartheta \in \mathbb{R}$$

Then u_N converges in law to a unique and non-trivial limit $\mathscr{U}^\vartheta$

$$u_N = \big(u_N(t,x)\,\mathrm{d}x\big)_{t\geq 0} \xrightarrow[N\to\infty]{d} \mathscr{U}^\vartheta = \big(\mathscr{U}^\vartheta(t,\mathrm{d}x)\big)_{t\geq 0}$$

$\mathscr{U}^\vartheta$ = critical 2D **Stochastic Heat Flow (SHF)** = stochastic process of measures on $\mathbb{R}^2$

SHF and Stochastic Heat Equation

The SHF is a "candidate solution" of the critical 2d Stochastic Heat Equation

$$\mathscr{U}^{\vartheta}(t, \mathrm{d}x) \qquad \text{(initial condition 1 at time 0)}$$

We actually build the SHF as a two-parameter space-time process

$$\left(\mathscr{U}^{\vartheta}(s, \mathrm{d}y;\, t, \mathrm{d}x)\right)_{0 \le s \le t < \infty} \qquad \text{(starting at time } s \text{ from } \mathrm{d}y\text{)}$$

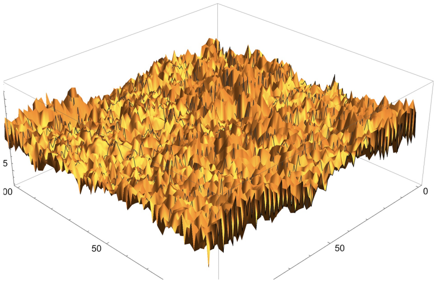
Why "flow"? Chapman-Kolmogorov for $s < t < u$ [Clark–Mian 24+]

$$\mathscr{U}^{\vartheta}(s, \mathrm{d}y;\, u, \mathrm{d}z) = \int_{x \in \mathbb{R}^2} \mathscr{U}^{\vartheta}(s, \mathrm{d}y;\, \underbrace{t, \mathrm{d}x)\, \mathscr{U}^{\vartheta}(t, \mathrm{d}x}_{\substack{\text{non-trivial "product"} \\ \text{of measures}}};\, u, \mathrm{d}z)$$

How does the SHF look?

We can efficiently simulate the SHF via $u_N(t,x)$ time $O(N^2)$

Some not-so-randomly picked realizations [M. Mucciconi, N. Zygouras]



KPZ $\approx \log u_N(t,x)$

Alps, Italy/France (vecteezy.com)

Properties of the SHF

- a.s. $\mathscr{U}^{\vartheta}(t, \mathrm{d}x)$ is singular w.r.t. Lebesgue [C.S.Z. 25+]

  "not a function"

- a.s. $\mathscr{U}^{\vartheta}(t, \mathrm{d}x) \in \mathscr{C}^{-\kappa}$ for any $\kappa > 0$ (in particular: non atomic)

  "*barely* not a function"

- $\mathbb{E}[\mathscr{U}^{\vartheta}(t, \mathrm{d}x)] = \mathrm{d}x \qquad \mathbb{E}[\mathscr{U}^{\vartheta}(t, \mathrm{d}x)\,\mathscr{U}^{\vartheta}(t, \mathrm{d}y)] = \underbrace{K^{\vartheta}(t, x-y)}_{\approx \log|x-y|^{-1}}\, \mathrm{d}x\, \mathrm{d}y$

- Formulas/estimates for moments (integer and fractional) [C.S.Z. *CMP* 19]

  [Gu–Quastel–Tsai *PMP* 21] [Ganguly–Nam 25+] [Berger–C.–Turchi 25+]

- Scaling covariance $a^{-1}\,\mathscr{U}^{\vartheta}(\mathrm{a}\,t, \mathrm{d}(\sqrt{\mathrm{a}}\,x)) \overset{d}{=} \mathscr{U}^{\vartheta + \log a}(t, \mathrm{d}x)$

SHF and white noise

Does SHF $\mathscr{U}^{\vartheta}(t, \mathrm{d}x)$ satisfy a SPDE driven by white noise $\xi(t,x)$?

- SHF $\mathscr{U}^{\vartheta}(t, \mathrm{d}x)$ is the limit of regularized SHE solutions $u_N(t,x)\,\mathrm{d}x$

$$\begin{cases} \partial_t u_N(t,x) = \Delta_x u_N(t,x) + \beta\, u_N(t,x)\, \xi_N(t,x) \\ u_N(0,x) \equiv 1 \quad \text{(for simplicity)} \end{cases} \qquad \text{(reg-SHE)}$$

- White noise $\xi(t,x)$ is the limit of regularized noise $\xi_N(t,x)$

$$\langle \xi, \psi \rangle = \lim_{N\to\infty} \int \xi_N(t,x)\, \psi(t,x)\, \mathrm{d}x \qquad \text{in distribution} \qquad \forall \psi \in C_c^\infty$$

No equation for the SHF

Theorem [C.–Donadini 25+]

$$\big(\xi_N, u_N\big) \xrightarrow[N\to\infty]{d} \big(\xi, \mathscr{U}^{\vartheta}\big) \qquad \text{with } \xi \text{ and } \mathscr{U}^{\vartheta} \textbf{ independent}$$

Puzzling: u_N is a **function** of ξ_N ... yet dependence is lost in the limit!

$\mathscr{U}^{\vartheta}$ cannot solve a SPDE driven by ξ

Same result for white noise mollification [Gu–Tsai 25+]

Proof: $\mathscr{U}^{\vartheta}$ is a "black noise" (à la Tsirelson)

Axiomatic characterization

Theorem [Tsai 24+]

Let $\mathscr{Z} = \big(\mathscr{Z}_{s,t}(\mathrm{d}x, \mathrm{d}y)\big)_{s \le t}$ be a stochastic process on $\mathscr{M}_+(\mathbb{R}^2 \times \mathbb{R}^2)$ satisfying

- continuity of $(s,t) \mapsto \mathscr{Z}_{s,t}$
- independence of $\mathscr{Z}_{s,t}$ and $\mathscr{Z}_{t,u}$ $\quad \forall s < t < u$
- convolution $\mathscr{Z}_{s,u} = \mathscr{Z}_{s,t} * \mathscr{Z}_{t,u}$ (Chapman-Kolmogorov) $\quad \forall s < t < u$
- moments $\mathbb{E}\big[\prod_{i=1}^n \mathscr{Z}_{s,t}(\varphi_i, \psi_i)\big]$ for $n = 1, 2, 3, 4$ coincide with those of $\mathscr{U}^\vartheta$ for some $\vartheta \in \mathbb{R}$

Then $\mathscr{Z}$ has the same distribution as the SHF $\mathscr{U}^\vartheta$

Strong disorder

The SHF emerges in the critical regime $\beta = \frac{\sqrt{\pi}}{\sqrt{\log N}} \left(1 - \frac{\vartheta}{\log N}\right)^{-1}$

What about the strong disorder regime $\vartheta = \vartheta_N \to \infty$? $\frac{\sqrt{\pi}}{\sqrt{\log N}} < \beta \leq O(1)$

Theorem [Berger–C.–Turchi 25+]

Local extinction $\displaystyle\int_{B(0,R)} u_N(t,x)\,\mathrm{d}x \xrightarrow[\vartheta \to +\infty]{d} 0$ for $R = \mathrm{e}^{c\, f_\beta\, N}$, $f_\beta = \mathrm{e}^{-\frac{\pi}{\beta^2}}$

Corollary: $\mathscr{U}_1^{\vartheta}(B(0,R)) \xrightarrow[\vartheta \to +\infty]{d} 0$ for $R = \mathrm{e}^{c\,\mathrm{e}^{\vartheta}}$

Weak disorder

What about the weak disorder regime $\vartheta = \vartheta_N \to -\infty$? $\qquad 0 < \beta < \frac{\sqrt{\pi}}{\sqrt{\log N}}$

Regularized (multiplicative) SHE "converge" to additive SHE $v(t,x)$

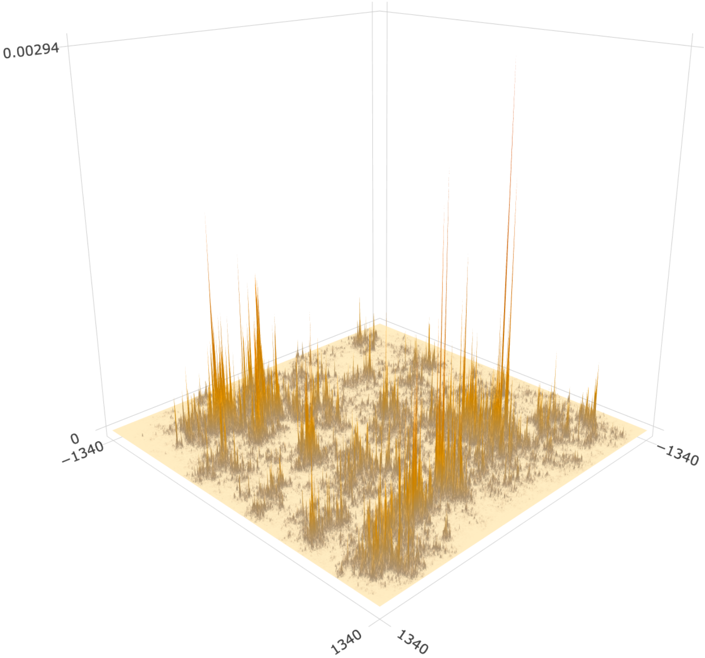
$$\partial_t v(t,x) = \Delta_x v(t,x) + \xi(t,x) \qquad \text{(Edwards-Wilkinson)}$$

Theorem [C.S.Z. *AAP* 17] [C.–Cottini–Rossi *AAP* 25]

$$\sqrt{|\vartheta|}\,\left\{ u_N(t,x)\,\mathrm{d}x - \mathrm{d}x \right\} \xrightarrow[\vartheta\to-\infty]{d} v(t,x)\,\mathrm{d}x$$

$$\sqrt{|\vartheta|}\,\left\{ \mathscr{U}^{\vartheta}(t,\mathrm{d}x) - \mathrm{d}x \right\} \xrightarrow[\vartheta\to-\infty]{d} v(t,x)\,\mathrm{d}x$$

Results on moments and maxima [Cosco–Nakajima–Zeitouni] [Liu–Zygouras]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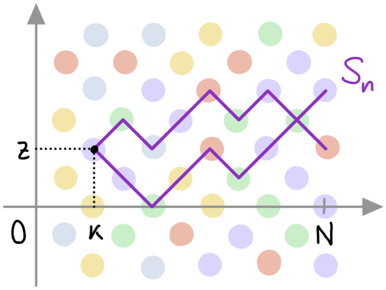
Directed Polymer in Random Environment

- $S = (S_n)_{n \geq 0}$ simple random walk on $\mathbb{Z}^d$
- Independent Gaussians $\omega(n,x) \sim \mathcal{N}(0,1)$
- $H(S,\omega) := \sum_{n=k+1}^{N} \omega(n, S_n) \sim \mathcal{N}(0, N-k)$



Partition Functions

$(k \in \mathbb{N},\ z \in \mathbb{Z}^d)$

$$Z^{\omega}_{N,\beta}(k,z) = \mathrm{E}\left[e^{\beta H(S,\omega) - \frac{1}{2}\beta^2 (N-k)} \,\middle|\, S_k = z \right]$$

Partition functions and SHE

Diff. rescaled partition functions = discretized SHE solutions

$$Z^{\omega}_{N,\beta}\big(N(1-t),\sqrt{N}x\big) \;=\; u_N(t,x) \qquad \text{(time rev.)}$$

Partition functions solve a difference equation: with $\xi_N \approx \omega$

$$\begin{cases} \partial_t u_N(t,x) = \Delta_x u_N(t,x) + \underbrace{\beta\, N^{\frac{2-d}{4}}}_{\beta_{\mathrm{SHE}}}\; \xi_N(t,x)\, u_N(t,x) \\ u_N(0,x) \;\equiv\; 1 \end{cases} \qquad \text{(reg-SHE)}$$

Discrete analogue of Feynman-Kac

$$u_N(t,x) \;\approx\; \mathrm{E}\left[\mathrm{e}^{\beta_{\mathrm{SHE}} \int_{1-t}^{1} \xi(s,B_s) - \frac{1}{2}\beta_{\mathrm{SHE}}^2\, t} \,\middle|\, B_{1-t} = x \right]$$

Techniques

Existence of subsequential limits $u_{N_k}(t,x)\,\mathrm{d}x \to \mathscr{U}^{\vartheta}(t,\mathrm{d}x)$ is easy. Uniqueness?

Cauchy sequence

$$u_N(t,x)\,\mathrm{d}x \approx u_M(t,x)\,\mathrm{d}x \qquad \text{for large } N, M$$

- Coarse-graining (tool: sharp second moment computations)
- Lindeberg priciple (tool: higher moment bounds)

Probabilistic handle on the SHE solution $u_N(t,x)$ via directed polymers

Correlation of $u_N(t,x)$ $\rightsquigarrow$ overlap of random walks

Summarizing

Singular Stochastic PDEs such as SHE and KPZ

closely linked to Directed Polymers

(discretized solutions $\leftrightsquigarrow$ partition functions)

Role of dimension d and disorder strength β

Critical dimension $d = 2$ + critical disorder strength $\beta = \beta_N$

Scaling limit of partition functions $\rightsquigarrow$ Stochastic Heat Flow $\mathscr{U}^{\vartheta}(t, \mathrm{d}x)$

Many features are understood, but several questions are still open

Future challenges

- Finer regularity properties of the SHF
- Universality
- SHF as a Markov process
- Martingale problem [Nakashima 25+] [Chen 25+]
- Critical 2d KPZ? "How to take log of $\mathscr{U}^{\vartheta}$?"

Related models

Stochastic Heat Equation with Lévy noise [Berger–Chong–Lacoin *CMP* 23]

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, \xi(t,x)\, u(t,x) \qquad \mathbb{P}(\xi > t) \approx t^{-\alpha}$$

Anisotropic KPZ equation [Cannizzaro–Erhard–Toninelli *CPAM* 23, *Duke* 23]

$$\partial_t h(t,x) = \Delta_x h(t,x) + \beta \left\{ (\partial_{x_1} h(t,x))^2 - (\partial_{x_2} h(t,x))^2 \right\} + \xi(t,x)$$

Stochastic Burgers equation [Cannizzaro–Gubinelli–Toninelli *CMP* 23]
[Cannizzaro–Moulard–Toninelli 25+]

$$\partial_t \eta(t,x) = \Delta_x \eta(t,x) + w \cdot |\nabla \eta(t,x)|^2 + \nabla \cdot \xi(t,x)$$

Merci