The 2D Stochastic Heat Equation in the strong disorder limit

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Renormalization from Quantum Field Theory to Statistical Mechanics and Complex Systems

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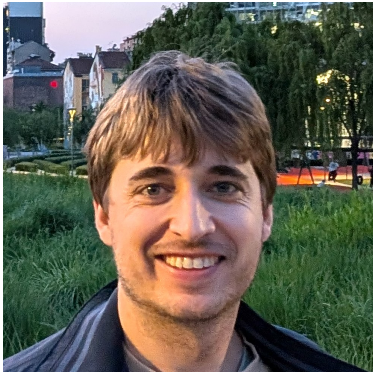

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Outline

Stochastic Heat Equation

Heat equation with multiplicative singular potential $t \geq 0,\ x \in \mathbb{R}^d$

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, \xi(t,x)\, u(t,x) \qquad \text{(SHE)}$$

$\beta \geq 0$ coupling constant

$\xi(t,x) =$ "space-time white noise"

Gaussian with cov. $\delta(t-t')\,\delta(x-x')$

$(d=1)$ sub-critical: well-posed (Ito-Walsh / Robust solution theories)

$(d=2)$ critical: How to define a solution?

Regularisation

Regularized noise $\xi_1(t,x)$ $\quad\rightsquigarrow\quad$ well-defined solution $u_1(t,x)$

(discretized on scale 1)

$$\partial_t u_1(t,x) = \Delta_x u_1(t,x) + \beta\, \xi_1(t,x)\, u_1(t,x)$$

Diffusive rescaling $\quad u_N(t,x) := u_1\big(N\,t, \sqrt{N}\,x\big)$ $\quad$ (large-scale properties of u_1)

$$\partial_t u_N(t,x) = \Delta_x u_N(t,x) + \beta\, \tilde{\xi}_N(t,x)\, u_N(t,x)$$

with $\tilde{\xi}_N(t,x) := N\,\xi_1\big(N\,t, \sqrt{N}\,x\big) \xrightarrow[N\to\infty]{d}$ white noise $\xi(t,x)$

(discretized on scale $1/\sqrt{N}$)

Convergence?

Two reasons for studying $u_N(t,x)$ as $N \to \infty$

- Large-scale (diffusive) behavior of the regularized SHE solution $u_1(t,x)$
- Approximation of the ill-defined original SHE with space-time white noise

Convergence of $u_N(t,\varphi) = \int_{\mathbb{R}^2} u_N(t,x)\,\varphi(x)\,\mathrm{d}x$? $\quad [\varphi \in C_c \quad \varphi \geq 0 \quad \int \varphi = 1]$

NO! Theorem [Berger C. Turchi 25]

For every $\beta > 0$ we have $\qquad u_N(t,\varphi) \xrightarrow[N\to\infty]{d} 0$

$$\partial_t u_N = \Delta_x u_N + \beta\, \xi_N\, u_N$$

Finer results later!

Outline

1. Stochastic Heat Equation
2. The critical regime
3. Strong disorder
4. Directed polymers
5. Sketch of the proof
6. Conclusions

Renormalisation or "taming infinities"

Why local extinction $u_N(t,\varphi) \xrightarrow[N\to\infty]{} 0$? Set $u_N(0,x) = 1$ (say)

- Mean is constant $\mathbb{E}\left[u_N(t,\varphi)\right] \equiv 1$
- 2nd moment explodes $\mathbb{E}\left[u_N(t,\varphi)^2\right] \longrightarrow \infty$ $\forall$ fixed $\beta > 0$

We can tame the moment divergence by tuning $\beta = \beta_N \to 0$ as $N \to \infty$

- $\mathbb{E}\left[u_N(t,\varphi)^2\right] \longrightarrow K_t(\varphi,\varphi) > 0$ [Bertini Cancrini 98] [C.S.Z. 19]
- Higher moments converge too [C.S.Z. 19] [Gu Quastel Tsai 21]

Convergence $u_N \longrightarrow \mathscr{U}$? $\partial_t u_N = \Delta_x u_N + \beta\, \xi_N\, u_N \partial_t \mathscr{U} = \Delta_x \mathscr{U} + 0$???

The critical 2D Stochastic Heat Flow

Theorem [C.S.Z. 23]

Critical window $\beta^2 = \frac{\pi}{\log N}\left(1 + \frac{\vartheta}{\log N}\right)$ for some $\vartheta \in \mathbb{R}$

u_N converges in law to a unique and non-trivial limit $\mathscr{U}^\vartheta$

$$\left(u_N(t,\varphi) \quad u_N(t,x)\,\mathrm{d}x \right)_{t\geq 0} \xrightarrow[N\to\infty]{d} \left(\mathscr{U}^\vartheta(t,\varphi)\ \mathscr{U}^\vartheta(t,\mathrm{d}x) \right)_{t\geq 0}$$

$\mathscr{U}^\vartheta$ = critical 2D **Stochastic Heat Flow (SHF)** = stochastic process of random measures on $\mathbb{R}^2$

Key properties of the SHF

- a.s. $\mathscr{U}^{\vartheta}(t, \mathrm{d}x)$ is singular w.r.t. Lebesgue [C.S.Z. arXiv 25]

  "not a function"

- a.s. $\mathscr{U}^{\vartheta}(t, \mathrm{d}x) \in \mathscr{C}^{-\kappa}$ for any $\kappa > 0$ (in particular: non atomic)

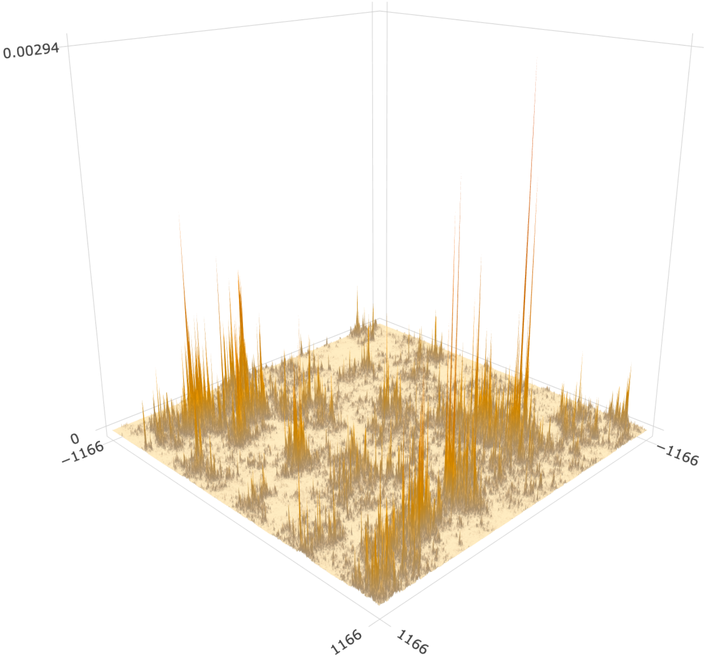
  "*barely* not a function"

- Formulas for all moments [C.S.Z. 19] [Gu–Quastel–Tsai 21]

- Scaling covariance $a^{-1}\, \mathscr{U}^{\vartheta}(\mathrm{a}\, t, \mathrm{d}(\sqrt{\mathrm{a}}\, x)) \overset{d}{=} \mathscr{U}^{\vartheta + \log a}(t, \mathrm{d}x)$

- Axiomatic characterization via independence & moments [Tsai 24+]

- Universality w.r.t. approximation scheme [C.S.Z. 23] [Tsai 24+]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Outline

SHF and SHE

The SHF $\mathscr{U}^{\vartheta}(t, \mathrm{d}x)$ provides a "solution" to the ill-defined 2D SHE

$$\partial_t u(t,x) = \Delta u(t,x) + \beta\, \xi(t,x)\, u(t,x) \qquad u(0,\cdot) \equiv 1$$

Noise regularization $\xi = \tilde{\xi}_N$ + Critical scaling $\beta = \beta_N(\vartheta) \to 0$

$$\beta^2 = \frac{\pi}{\log N}\left(1 - \frac{\vartheta}{\log N}\right)^{-1} = \frac{\pi}{\log N - \vartheta} \qquad \Longleftrightarrow \qquad \boxed{\vartheta = \log N - \frac{\pi}{\beta^2}}$$

Strong disorder

Any regime of $\vartheta = \vartheta(N, \beta) \to \infty$ $\qquad 1 \ll \vartheta \le \underbrace{\log N - O(1)}_{\text{fixed } \beta > 0}$

Local extinction

We show that u_N locally vanishes for strong disorder

Theorem [Berger C. Turchi 25]

$$u_N(t,\varphi) \xrightarrow[N\to\infty]{d} 0 \qquad \text{as soon as } \vartheta \to \infty$$

(including fixed $\beta > 0$)

Speed of convergence? We obtain quantitative bounds

Fractional moments

$$\mathbb{E}\big[u_N(t,\varphi)^\gamma\big] \quad \text{with } \gamma \in (0,1)$$

Truncated mean

$$\mathbb{E}\big[u_N(t,\varphi) \wedge 1\big] = \mathbb{P}\big(u_N(t,\varphi) > U(0,1)\big)$$

We set for simplicity $t = 1$ (no loss of generality)

Quantitative bounds

Theorem [Berger C. Turchi 25]

There are c, C such that

$$c \exp(-C e^{\vartheta}) \leq \sup_{\varphi \in \mathscr{M}_1\left(e^{c e^{\vartheta}}\right)} \mathbb{E}\left[u_N(1,\varphi) \wedge 1\right] \leq C \exp(-c e^{\vartheta})$$

$\forall \varepsilon \in (0,1)$ there are $c_\varepsilon, C_\varepsilon$ such that

$$c_\varepsilon \exp(-C e^{\vartheta}) \leq \sup_{\varphi \in \mathscr{M}_1\left(e^{c e^{\vartheta}}\right)} \mathbb{P}\left(u_N(1,\varphi) \geq \varepsilon\right) \leq C_\varepsilon \exp(-c e^{\vartheta})$$

LB: 2^{nd} moment method UB: coarse-graining + change of measure

Superdiffusivity

Rewrite the rate $\mathrm{e}^{c\,\mathrm{e}^{\vartheta}} = \mathrm{e}^{c\,f_\beta\,N}$ with $f_\beta = \mathrm{e}^{-\frac{\pi}{\beta^2}}$ $\left(\vartheta = \log N - \frac{\pi}{\beta^2}\right)$

Super-rescaled solution $u_N^c(t,x) := u_N\left(t\,,\mathrm{e}^{c\,\mathrm{e}^{\vartheta}}\,x\right) = u_1\left(N\,t\,,\mathrm{e}^{c\,f_\beta\,N}\sqrt{N}\,x\right)$

Theorem [Berger C. Turchi 25]

There are $0 < a < b < \infty$ such that

$$\forall \text{ density } \varphi \in C_c(\mathbb{R}^2) \qquad \int_{\mathbb{R}^2} \varphi(x)\, u_N^c(t,x)\,\mathrm{d}x \xrightarrow[N\to\infty]{d} \begin{cases} 0 & \text{if } c < a \\ 1 & \text{if } c > b \end{cases}$$

Conjecture: non trivial limit $u_N^{\hat{c}}(t,x)\,\mathrm{d}x \xrightarrow[\vartheta\to\infty]{d} \mathscr{M}(\mathrm{d}x)$ for some $\hat{c} \in (a,b)$

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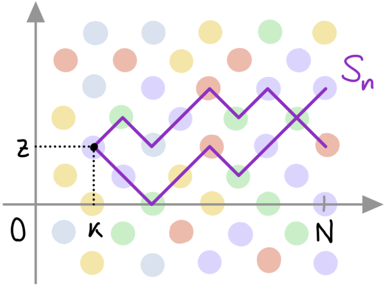
Directed polymers partition functions

- $S = (S_n)_{n \geq 0}$ simple random walk on $\mathbb{Z}^d$
- Independent Gaussians $\omega(n, x) \sim \mathcal{N}(0, 1)$
- $H(S, \omega) := \sum_{n=k+1}^{N} \omega(n, S_n)$



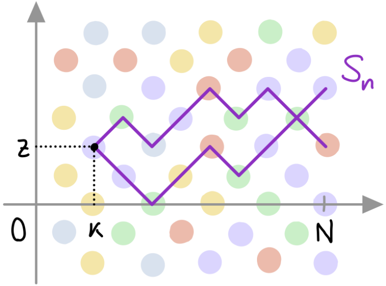
Partition Functions $(k \in \mathbb{N},\ z \in \mathbb{Z}^d)$

$$Z^{\omega}_{N,\beta}(k, z) = \mathrm{E}\left[\mathrm{e}^{\beta H(S,\omega) - \frac{1}{2}\beta^2 (N-k)} \,\middle|\, S_k = z \right] \qquad \mathbb{E}\left[Z^{\omega}_{N,\beta}\right] = 1$$

Partition functions and SHE

Partition functions $Z^{\omega}_{N,\beta}(k,z)$ are solutions of

discretised Stochastic Heat Equation



Diffusive rescaling (+ time reversal)

$$u_N(t,x) := Z^{\omega}_{N,\beta}(N(1-t), \sqrt{N}x)$$

$$\begin{cases} \partial_t u_N(t,x) = \Delta_x u_N(t,x) + \beta \underbrace{\xi_N(t,x)}_{\text{white noise discr. on scale } 1/\sqrt{N}} u_N(t,x) \\ u_N(0,x) \equiv 1 \end{cases} \qquad \text{(disc-SHE)}$$

Local extinction and free energy

Quantitative bounds for $Z^{\omega}_{N,\beta}(f) \to 0$ uniformly over $N \in \mathbb{N}$, $\beta \in (0, \beta_0)$

Theorem [Berger C. Turchi 25]

$$c \exp(-C e^{\vartheta}) \;\le\; \sup_{f \in \mathscr{M}_1^{\mathrm{disc}}\left(e^{c e^{\vartheta}} \sqrt{N}\right)} \mathbb{E}\left[Z^{\omega}_{N,\beta}(f) \wedge 1\right] \;\le\; C \exp(-c e^{\vartheta})$$

Free energy: $Z^{\omega}_{N,\beta}(0) = e^{\mathrm{F}(\beta) N + o(N)} \xrightarrow[N \to \infty]{} 0$ [Lacoin 10, Berger Lacoin 17]

Theorem [Berger C. Turchi 25]

$$-\frac{c'}{\beta^8} \exp\left(-\frac{\pi}{\beta^2}\right) \le \mathrm{F}(\beta) \le -c \exp\left(-\frac{\pi}{\beta^2}\right)$$

Outline

Proof of the UB: coarse-graining

$$\sup_{f \in \mathscr{M}_1^{\mathrm{disc}}\left(\mathrm{e}^{c\,\mathrm{e}^{\vartheta}}\sqrt{N}\right)} \mathbb{E}\left[Z_{N,\beta}^{\omega}(f) \wedge 1\right] \;\leq\; C\,\exp\left(-c\,\mathrm{e}^{\vartheta}\right) \qquad \text{for any } \vartheta = \vartheta_N \to \infty$$

- Change of scale argument: reduce $\mathscr{M}_1^{\mathrm{disc}}\left(\mathrm{e}^{c\,\mathrm{e}^{\vartheta}}\sqrt{N}\right)$ to $\mathscr{M}_1^{\mathrm{disc}}(\sqrt{N})$
- Coarse-graining argument: $\begin{cases} \text{reduce } \vartheta = \vartheta_N \to \infty \text{ to fixed } \vartheta \in \mathbb{R} \\ \text{replace } \exp\left(-c\,\mathrm{e}^{\vartheta}\right) \text{ by any } f(\vartheta) \to 0 \end{cases}$

Key bound

$$\sup_{f \in \mathscr{M}_1^{\mathrm{disc}}(\sqrt{N})} \mathbb{E}\left[Z_{N,\beta}^{\omega}(f) \wedge 1\right] \;\leq\; \frac{C}{\vartheta} \qquad \text{for fixed } \vartheta \in \mathbb{R}$$

Proof of the UB: change of measure

Change of scale: $$\sup_{f \in \mathcal{M}_1^{\mathrm{disc}}(\sqrt{N})} \mathbb{E}\left[Z_{N,\beta}^{\omega}(f) \wedge 1\right] \le \frac{2}{\varepsilon} \sup_{f \in \mathcal{M}_1^{\mathrm{disc}}(\sqrt{\varepsilon N})} \mathbb{E}\left[Z_{N,\beta}^{\omega}(f) \wedge 1\right]$$

Idea: for f on scale $\sqrt{\varepsilon N}$, partition function $Z_{N,\beta}^{\omega}(f)$ is almost point-to-plane

Size-biased law $$\tilde{\mathbb{P}}(\mathrm{d}\omega) := Z(\omega)\, \mathbb{P}(\mathrm{d}\omega) \qquad \text{for } Z(\omega) = Z_{N,\beta}^{\omega}(f)$$

Change of measure

$$\mathbb{E}[Z \wedge 1] \le \mathbb{P}(A) + \tilde{\mathbb{P}}(A^c) \qquad \text{for any event } A$$

Optimal with $A = \{Z > 1\}$ (but we don't know Z ...)

Proof of the UB: choice of a proxy

Take X with $\mathbb{E}[X]=0$ and set $A=\left\{X>\frac{1}{2}\tilde{\mathbb{E}}[X]\right\}$

By Chebychev $\qquad \mathbb{P}(A)\leq 4\,\dfrac{\mathbb{V}\mathrm{ar}[X]}{\tilde{\mathbb{E}}[X]^2} \qquad \tilde{\mathbb{P}}(A^c)\leq 4\,\dfrac{\tilde{\mathbb{V}}\mathrm{ar}[X]}{\tilde{\mathbb{E}}[X]^2}$

We take X as a manageable proxy of Z: restrict the chaos expansion of Z to

$$I=\{(n_1,x_1),\ldots,(n_k,x_k)\} \qquad \text{with} \quad \begin{cases}\mathrm{width}(I)=n_k-n_1\leq \varepsilon N\,,\\ |I|=k\leq \log(\varepsilon N)\end{cases}$$

We finally estimate $\mathbb{V}\mathrm{ar}[X]$, $\tilde{\mathbb{E}}[X]$ (2^{nd} moment) and $\tilde{\mathbb{V}}\mathrm{ar}[X]$ (3^{rd} moment)

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Conclusions

2D SHE with discretized noise $\rightsquigarrow$ directed polymer partition function

Strong disorder regime: local extinction with quantitative bounds

$$\vartheta = \log N - \frac{\pi}{\beta^2} \longrightarrow \infty \qquad \text{(including fixed } \beta > 0\text{)}$$

Mass escapes to infinity at spatial scale $\exp\left(c\,\mathrm{e}^{\vartheta}\right) = \exp\left(c\,\mathrm{e}^{-\pi/\beta^2} N\right)$

We expect analogous result for the mollified SHE (in progress)

Robust proof: coarse-graining + change of scale + change of measure

Q. Berger, F.C., N. Turchi. arXiv: 2508.02478

Thanks