# The 2d Stochastic Heat Equation and related critical models

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# Grazie

Thanks to the **committee** and to all the **people involved in the organization** of this amazing week in Rome

Thank you for staying here till the end

It is a honor, and an emotional moment, to be here today

# Outline

1. Continuum
2. Discrete
3. Conclusion

# Antefatto (backstory)

I first met probability 23 years ago in Pisa

"Introduction to SDEs" by Da Prato

Exciting environment around Caracciolo and Flandoli

Gubinelli, F Toninelli, Zambotti (+Faggionato, Montanari, Sportiello, Gambassi...)

I took a (random) path in Paris, Zurich, Padova and then Milano

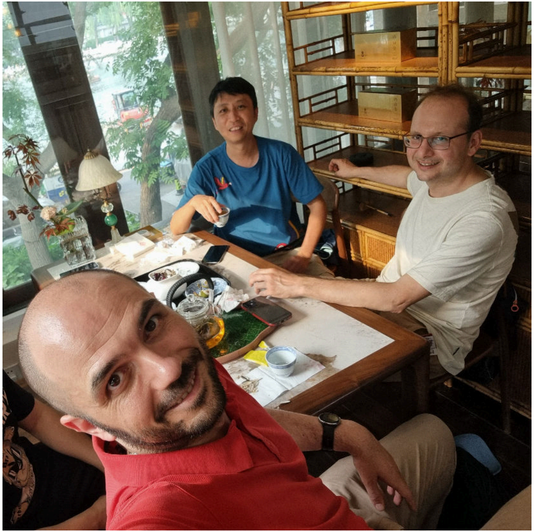
I witnessed the impressive expansion of the Italian probability community

Collaboration has been the most fundamental aspect

# Collaborators

This talk stems from works with

Rongfeng Sun and Nikos Zygouras



# In a nuthshell

Stochastic Heat Equation (SHE)

$$\partial_t u(t,x) = \underbrace{\Delta_x u(t,x)}_{\sum_{i=1}^{d} \frac{\partial^2}{\partial {x_i}^2} u(t,x)} + \beta\, u(t,x)\, \xi(t,x) \qquad t \geq 0,\ x \in \mathbb{R}^d$$

Singular random potential $\xi(t,x)$ "space-time white noise"

**Main result** [C.S.Z. *Invent. Math.* 2023]

We construct a natural candidate solution of SHE in space dimension $d = 2$

the Critical 2d Stochastic Heat Flow

# Why is it interesting?

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, u(t,x)\, \xi(t,x) \qquad \text{(SHE)}$$

- Fundamental PDE + universal random potential $\xi(t,x)$

  white noise = "continuum" i.i.d. random variables

- KPZ equation [Kardar–Parisi–Zhang *PRL* 86]

$$\partial_t h(t,x) = \Delta_x h(t,x) + |\nabla_x h(t,x)|^2 + \beta\, \xi(t,x) \qquad \text{(KPZ)}$$

Cole-Hopf transformation $h(t,x) = \log u(t,x)$

# Why is it difficult?

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, u(t,x)\, \xi(t,x) \qquad \text{(SHE)}$$

$\xi(t,x)$ is a distribution $\rightsquigarrow$ u(t,x) expected: $\begin{cases} \text{non-smooth function} & d=1 \\ \text{genuine distribution} & d \geq 2 \end{cases}$

Product $u(t,x)\,\xi(t,x)$ unclear: no classical space to solve SHE (as a PDE)

Stochastic integral? **Yes** for $d = 1$ [Ito–Walsh, Da Prato–Zabczyk]

SHE solution $u(t,x) > 0$

starting from $u(0,\cdot) > 0$

$\rightsquigarrow$

"KPZ solution" $h(t,x) = \log u(t,x)$

[Bertini–Giacomin *CMP* 97]

# Why is it difficult?

Revolution in 2010s: **robust solution theories** for **sub-critical SPDEs**

[Hairer *Invent. Math.* 14] [Gubinelli–Imkeller–Perkowski *Forum Math Pi* 15] [. . . ]

SHE and KPZ: only apply for $d = 1$

Role of dimension: space-time blow-up $\tilde{u}(t,x) := u(\varepsilon^2 t, \varepsilon x)$

$$\partial_t \tilde{u}(t,x) = \Delta_x \tilde{u}(t,x) + \varepsilon^{\frac{2-d}{2}} \beta \, \tilde{u}(t,x) \, \tilde{\xi}(t,x)$$

$$\text{as } \varepsilon \downarrow 0 \text{ the noise formally } \begin{cases} \text{vanishes} & d < 2 \\ \text{stays constant} & d = 2 \quad \rightsquigarrow \text{ critical dimension} \\ \text{diverges} & d > 2 \end{cases}$$

# What can we do?

Henceforth we fix $d = 2$

Regularized noise $\xi_N(t,x)$ $\rightsquigarrow$ well-defined solution $u_N(t,x)$

(discretization, mollification, Fourier cutoff, ...)

$$\partial_t u_N(t,x) = \Delta_x u_N(t,x) + \beta\, u_N(t,x)\, \xi_N(t,x) \qquad \text{(reg-SHE)}$$

Fix $\xi_N(t,x) \xrightarrow[N\to\infty]{} \xi(t,x)$

Does $u_N(t,x)$ converge to some interesting limit?

No! Unless some kind of renormalization is performed

# Which notion of convergence?

Do not expect pointwise convergence

Space-average $\varphi : \mathbb{R}^2 \to \mathbb{R}$

$$u_N(t,\varphi) := \int_{\mathbb{R}^2} \varphi(x)\, u_N(t,x)\, \mathrm{d}x \xrightarrow[N\to\infty]{d} \mathscr{U}(t,\varphi) \qquad (?)$$

i.e. convergence as random measure on $\mathbb{R}^2$ $u_N(t,x) \geq 0$

$$u_N(t,x)\, \mathrm{d}x \xrightarrow[N\to\infty]{d} \mathscr{U}(t,\mathrm{d}x) \qquad (?)$$

Still no interesting limit without renormalization

# What is renormalization?

We take $\beta = \beta_N \sim \frac{\hat{\beta}}{\sqrt{\log N}} \xrightarrow[N\to\infty]{} 0$ for specific $\hat{\beta} \in (0,\infty)$

$$\begin{cases} \partial_t u_N(t,x) = \Delta_x u_N(t,x) + \beta_N\, u_N(t,x)\, \xi_N(t,x) \\ u_N(0,x) \equiv 1 \quad \text{(say)} \end{cases} \qquad \text{(reg-SHE)}$$

*Formally* $\beta_N\, u_N\, \xi_N \longrightarrow 0$ *but actually not!* (singular product)

## Proposition

[Bertini–Cancrini *J. Phys. A* 98] [C.S.Z. *EJP* 19]

for $\hat{\beta} = \sqrt{\pi}$ $\qquad \mathbb{V}\mathrm{ar}\big[u_N(t,\varphi)\big] \xrightarrow[N\to\infty]{} K_t(\varphi,\varphi) > 0$

# Main result

## Theorem

Take $\beta_N \sim \frac{\sqrt{\pi}}{\sqrt{\log N}}$ more precisely for some $\vartheta \in \mathbb{R}$

$$\beta_N = \frac{\sqrt{\pi}}{\sqrt{\log N}} \left(1 + \frac{\vartheta}{\log N}\right)$$

Then $u_N$ converges in law to a unique and non-trivial limit $\mathscr{U}^{\vartheta}$

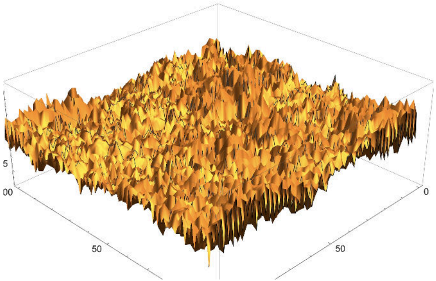
$$\big(u_N(t,x)\,\mathrm{d}x\big)_{t\ge 0} \xrightarrow[N\to\infty]{d} \big(\mathscr{U}^{\vartheta}(t,\mathrm{d}x)\big)_{t\ge 0}$$

$\mathscr{U}^{\vartheta}$ = critical 2d Stochastic Heat Flow = stochastic process of random measures on $\mathbb{R}^2$

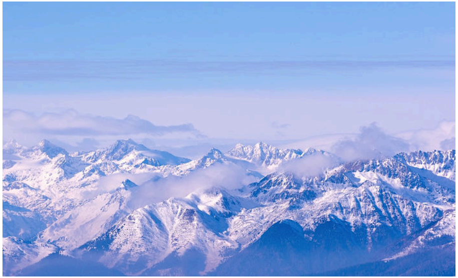
# Intermezzo 1. How does the SHF look like?

We can efficiently simulate the SHF via $u_N(t,x)$ time $O(N^2)$

Some not-so-randomly picked realizations [M. Mucciconi, N. Zygouras]




KPZ $\approx \log u_N(t,x)$



Alps, Italy (vecteezy.com)

# Back to business

The SHF is a "candidate solution" of the critical 2d Stochastic Heat Equation

$$\mathscr{U}^{\vartheta}(t, \mathrm{d}x) \qquad \text{“initial condition 1 at time 0”}$$

We actually build the SHF as a two-parameter space-time process

$$\big(\mathscr{U}^{\vartheta}(s, \mathrm{d}y; t, \mathrm{d}x)\big)_{0 \le s \le t < \infty} \qquad \text{“starting at time } s \text{ from } \mathrm{d}y\text{”}$$

Why "flow"? Chapman-Kolmogorov property for $s < u < t$ [Clark 2024+]

$$\mathscr{U}^{\vartheta}(s, \mathrm{d}y; t, \mathrm{d}x) = \int_{z \in \mathbb{R}^2} \mathscr{U}^{\vartheta}(s, \mathrm{d}y; \underbrace{u, \mathrm{d}z)\, \mathscr{U}^{\vartheta}(u, \mathrm{d}z}_{\substack{\text{non-trivial “product”} \\ \text{of measures}}}; t, \mathrm{d}x)$$

# Key properties of the SHF

- a.s. $\mathscr{U}^{\vartheta}(t, \mathrm{d}x)$ is singular w.r.t. Lebesgue [C.S.Z. *in preparation*]

  "not a function"

- a.s. $\mathscr{U}^{\vartheta}(t, \mathrm{d}x) \in \mathscr{C}^{-\kappa}$ for any $\kappa > 0$ (in particular: non atomic)

  "*barely* not a function"

- Diffusive rescaling $a^{-1}\, \mathscr{U}^{\vartheta}(a\, t, \mathrm{d}(\sqrt{a}\, x)) \overset{d}{=} \mathscr{U}^{\vartheta + \log a}(t, \mathrm{d}x)$

- $\mathbb{E}[\mathscr{U}^{\vartheta}(t, \mathrm{d}x)] = \mathrm{d}x \qquad \mathbb{E}[\mathscr{U}^{\vartheta}(t, \mathrm{d}x)\, \mathscr{U}^{\vartheta}(t, \mathrm{d}y)] = \underbrace{K_t^{\vartheta}(x-y)}_{\approx \log|x-y|^{-1}}\, \mathrm{d}x\, \mathrm{d}y$

- Formulas for higher moments [C.S.Z. *CMP* 19] [Gu–Quastel–Tsai *PMP* 21]

# Gaussian Multiplicative Chaos?

Much studied class of random measures: Gaussian Multiplicative Chaos (GMC)

$$\mathscr{M}(\mathrm{d}x) = \text{“}\, \mathrm{e}^{X(x) - \frac{1}{2}\mathbb{V}\mathrm{ar}[X(x)]}\, \mathrm{d}x \,\text{”} \qquad X(\cdot) \text{ generalized Gaussian field}$$

**Theorem** [C.S.Z. *AoP* 23]

The critical 2d Stochastic Heat Flow $\mathscr{U}^{\vartheta}(t, \mathrm{d}x)$ is not a GMC

Not the usual GMC: $\mathscr{U}^{\vartheta}$ log-correlated $\rightsquigarrow$ $X$ log-log-correlated

**Conjecture** $h_N(t,x) = \log u_N(t,x)$

The critical 2d KPZ solution (yet to be found) may be non Gaussian ?

# Further properties

Long time behavior (follows from singularity)

$$\forall R: \qquad \mathscr{U}_t^{\vartheta}(B(0,R)) \xrightarrow[t\to\infty]{d} 0$$

"mass escapes to infinity"

## Conjectures

$$\frac{\mathscr{U}_t^{\vartheta}(B(0,\sqrt{t}))}{t} \xrightarrow[t\to\infty]{d} 0$$

"superdiffusivity"

$$\iff \qquad \mathscr{U}_1^{\vartheta}(B(0,1)) \xrightarrow[\vartheta\to+\infty]{d} 0$$

"strong disorder"

# Related models

Stochastic Heat Equation with Lévy noise [Berger–Chong–Lacoin *CMP* 23]

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, u(t,x)\, \xi(t,x) \qquad \mathbb{P}(\xi > t) \approx t^{-\alpha}$$

Well-posedness (+ intermittency) under optimal assumptions

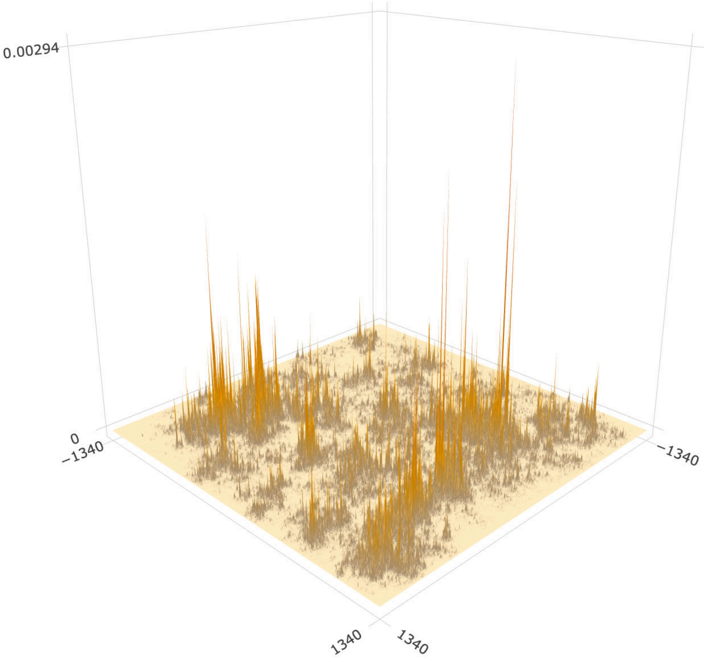
Anisotropic KPZ equation [Cannizzaro–Erhard–Toninelli *CPAM* 23, *Duke* 23]

$$\partial_t h(t,x) = \Delta_x h(t,x) + \beta \left\{ (\partial_{x_1} h(t,x))^2 - (\partial_{x_2} h(t,x))^2 \right\} + \xi(t,x)$$

- Suitable regularization + $\beta_N \approx (\log N)^{-1/2}$ $\rightsquigarrow$ non trivial Gaussian limit
- Fixed $\beta > 0$ $\rightsquigarrow$ logarithmic superdiffusivity

# Intermezzo 2. Let's play again with the SHF

Simulation of the SHF $\mathscr{U}^{\vartheta}(1, \mathrm{d}x) \approx u_N(1, x)$ for $N = 50\,000$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0
−1340
1340
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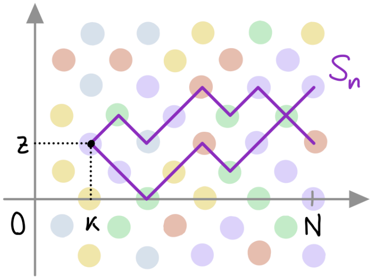
# Directed Polymer in Random Environment

- $S = (S_n)_{n \geq 0}$ simple random walk on $\mathbb{Z}^2$
- Independent standard Gaussians $\omega(n, x)$
- $H(S, \omega) := \sum_{n=k+1}^{N} \omega(n, S_n) \sim \mathcal{N}(0, N-k)$




## Partition Functions $(k \in \mathbb{N},\ z \in \mathbb{Z}^2)$

$$Z_N^{\omega}(k, z) = \mathrm{E}\left[ e^{\beta H(S,\omega) - \frac{1}{2}\beta^2 (N-k)} \,\middle|\, S_k = z \right]$$

# Feynman-Kac

Partition functions = discretized SHE solutions (up to time rev.)

$$Z_N^{\omega}(Nt, \sqrt{N}x) \;=\; u_N(1-t, x)$$

They solve a difference equation with $\xi_N \approx \omega$

$$\begin{cases} \partial_t u_N(t,x) = \Delta_x u_N(t,x) + \beta_N \, u_N(t,x) \, \xi_N(t,x) \\ u_N(0,x) \equiv 1 \end{cases} \qquad \text{(reg-SHE)}$$

Discrete analogue of Feynman-Kac

$$u(1-t,x) \;=\; \mathrm{E}\left[ e^{\int_t^1 \beta \, \xi(s, B_s) - \frac{1}{2}\beta^2 (1-t)} \,\middle|\, B_t = x \right]$$

# Phase transition

We look at $u_N(1,x) = Z_N^{\omega}(0,\sqrt{N}x)$ for $\beta \sim \frac{\hat\beta}{\sqrt{\log N}}$

## Theorem (log-normality) [C.S.Z. *AAP* 17]

- for $\hat\beta < \sqrt{\pi}$ $\quad u_N(1,x) \xrightarrow[N\to\infty]{d} e^{\sigma \mathcal{N}(0,1) - \frac{1}{2}\sigma^2}$

  with $\sigma^2 = \log \frac{1}{1-(\hat\beta^2/\pi)}$

- for $\hat\beta \geq \sqrt{\pi}$ $\quad u_N(1,x) \xrightarrow[N\to\infty]{d} 0$

Alternative proofs of log-normality [C.–Cottini *EJP* 22] [Cosco–Donadini 24+]

# Random walk insight

$$\mathbb{E}\Big[u_N(1,x)\,u_N(1,y)\Big] = \mathrm{E}\Big[e^{\beta^2 \sum_{i=1}^{N} \mathbb{1}_{S_i=S_i'}} \,\Big|\, S_0 = x\sqrt{N},\, S_0' = y\sqrt{N}\Big]$$

For $x = y$ $$\frac{\pi}{\log N}\left\{\sum_{i=1}^{N} \mathbb{1}_{S_i=S_i'}\right\} \xrightarrow[N\to\infty]{d} \mathrm{Exp}(1)$$ [Erdös–Taylor 60]

Recent generalizations [Lygkonis–Zygouras 24] [Cosco–Zeitouni 21-24]

Explains scaling $\beta \sim \frac{\hat{\beta}}{\sqrt{\log N}}$ with critical value $\hat{\beta} = \sqrt{\pi}$

# Sub-critical regime $\hat\beta < \sqrt{\pi}$

Averaged solution $u_N(t,\varphi) = \int_{\mathbb{R}^2} \varphi(x)\, u_N(t,x)\,\mathrm{d}x$ has vanishing variance $\approx \frac{1}{\log N}$

## Edwards-Wilkinson fluctuations for SHE [C.S.Z. *AAP* 17]

$$\left(\hat\beta < \sqrt{\pi}\right) \qquad \frac{\sqrt{\log N}}{\hat\beta}\left\{u_N(t,\varphi) - \mathbb{E}\big[u_N(t,\varphi)\big]\right\} \xrightarrow[N\to\infty]{d} v(\varphi)$$

solution of additive Stochastic Heat Equation with additional independent noise

$$\partial_t v(t,x) = \Delta_x v(t,x) + \xi(t,x) + \frac{\hat\beta}{\sqrt{\pi - \hat\beta^2}}\,\xi'(t,x) \qquad \text{(EW)}$$

# Sub-critical regime $\hat{\beta} < \sqrt{\pi}$

Same result for averaged KPZ solution $h_N(t,\varphi) = \int_{\mathbb{R}^2} \varphi(x)\, h_N(t,x)\, \mathrm{d}x$

## Edwards-Wilkinson fluctuations for KPZ [C.S.Z. *AoP* 20]

$$(\hat{\beta} < \sqrt{\pi}) \qquad \frac{\sqrt{\log N}}{\hat{\beta}} \left\{ h_N(t,\varphi) - \mathbb{E}\big[h_N(t,\varphi)\big] \right\} \xrightarrow[N\to\infty]{d} v(\varphi)$$

Intense research [Chatterjee–Dunlop *AoP* 20] [Gu *SPDE* 20] [Dunlap–Gu *AoP* 20] [Tao *SPDE* 22] [Nakajima–Nakashima *EJP* 23] [Tao 23+] [Dunlap–Graham 23+]

Higher dimensions & related [Comets–Cosco–Mukerjee, Lygkonis–Zygouras, Cosco–Zeitouni, Junk, Junk–Lacoin, . . . ]

# Quasi-critical regime

Interpolate between sub-critical regime $\hat{\beta} < \sqrt{\pi}$ and critical regime $\hat{\beta} = \sqrt{\pi}$

$$\beta_N = \frac{\sqrt{\pi}}{\sqrt{\log N}} \left( 1 - \frac{\vartheta_N}{\log N} \right) \qquad \text{for} \quad 1 \ll \vartheta_N \ll \log N$$

Quasi-critical regime $\leftrightsquigarrow$ behavior of the SHF $\mathscr{U}^{\vartheta}(t, \mathrm{d}x)$ as $\vartheta \to -\infty$

- Edwards-Wilkinson fluctuations hold for $u_N(t, \varphi)$ [C.–Cottini–Rossi 2023+]
- Currently investigating log-normality of $u_N(t, x)$ [Berger–C.–Turchi 2024+]

# Techniques

Directed polymers give us a probabilistic handle on the SHE solution $u_N(t,x)$

Correlation of $u_N(t,x)$ $\rightsquigarrow$ overlap of random walks $\rightsquigarrow$ renewal theory

Key tools: polynomial chaos, hypercontractivity, concentration inequalities

(some fail in the quasi-critical and critical regimes)

For the SHF $\mathscr{U}^{\vartheta}(t,\mathrm{d}x)$ we exploit coarse-graining $+$ Lindeberg priciple

# To conclude

We introduced the critical 2d Stochastic Heat Flow $\mathscr{U}^{\vartheta}(t, \mathrm{d}x)$

Scaling limit of regularized solutions of 2d Stochastic Heat Equation

$\Longleftrightarrow$ directed polymer partition functions

Universal process of random measures on $\mathbb{R}^2$ with many explicit features,

yet several open questions

# Future challenges

- Finer regularity properties
- Intrinsic characterization of the SHF — to please Massimiliano :-)
- SHF as a Markov process, e.g. martingale problem — [M. Nakashima]
- Sensitivity and black noise features — [Himwich–Parekh 24+]
- Universality for directed polymers — cf. [C.–Toninelli–Torri *AoP* 17]
- Critical 2d KPZ? How to take log of $\mathscr{U}^{\vartheta}$?
- ...

Grazie