The Critical 2D Stochastic Heat Flow

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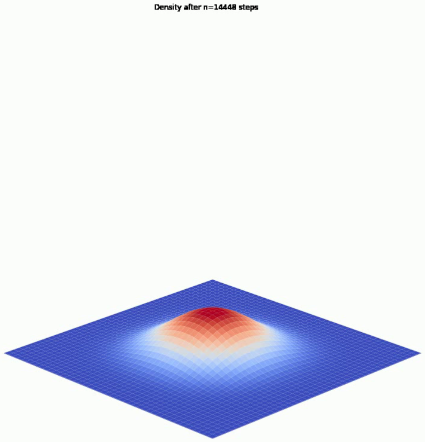
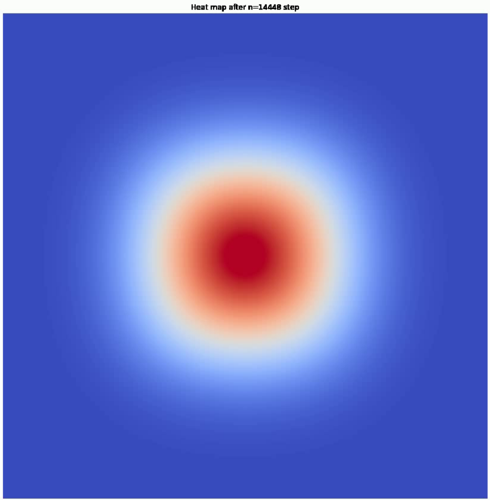
Heat Equation $\partial_t u(t,x) = \Delta_x u(t,x)$ (video by Q. Berger)





In a nutshell

Stochastic Heat Equation ill-defined!

$$\partial_t u(t,x) = \underbrace{\Delta_x u(t,x)}_{\sum_{i=1}^{d} \frac{\partial^2}{\partial {x_i}^2} u(t,x)} + \beta \underbrace{\xi(t,x)}_{\substack{\text{“space-time white noise”} \\ \text{cov. } \delta(t-t')\,\delta(x-x')}} \cdot u(t,x)$$

We construct a “non-trivial solution” in space dimension $d = 2$

Critical 2D Stochastic Heat Flow (SHF)

Singular SPDEs $\leftrightsquigarrow$ Statistical Mechanics

Outline

1. 2D Stochastic Heat Equation

2. Construction of SHF

3. Properties of SHF

4. SHF and Directed Polymers

Why is it interesting?

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, \xi(t,x) \cdot u(t,x)$$

- Fundamental PDE + universal random potential

- KPZ equation [Kardar–Parisi–Zhang 86]

$$\partial_t h(t,x) = \Delta_x h(t,x) + |\nabla_x h(t,x)|^2 + \beta\, \xi(t,x)$$

Cole–Hopf: $h(t,x) = \log u(t,x)$

Why is it difficult?

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, \xi(t,x) \cdot u(t,x)$$

$\xi(t,x)$ is a distribution $\rightsquigarrow$ u(t,x) expected $\begin{cases} \text{non-smooth funct.} & d=1 \\ \text{distribution} & d \geq 2 \end{cases}$

Ill-defined product $\xi(t,x) \cdot u(t,x)$ $\rightsquigarrow$ Singular SPDEs

How to make sense of the equation? $u(0,y) = \psi_0(y)$

Stochastic integral

Duhamel's principle: integral formulation $\qquad g_t(x) = \text{heat kernel}$

$$u(t,x) = \int_{\mathbb{R}^d} g_t(x-y)\, \underbrace{\psi_0(y)}_{\text{initial datum}}\, \mathrm{d}z + \iint_{(0,t)\times\mathbb{R}^d} g_{t-s}(x-y)\, u(s,y)\, \beta\, \xi(s,y)\, \mathrm{d}s\, \mathrm{d}y$$

Works for $d = 1$

- stochastic integral [Ito/Walsh] [Da Prato–Zabczyk]
- well-posedness, positivity [Chen–Dalang 15] [Müller 91]

Breaks down for $d \geq 2$ $\qquad g_t(x) \notin L^2(\mathrm{d}t, \mathrm{d}x)$

Pathwise stochastic analysis

Revolution in 2010s: robust solution theories for **subcritical singular SPDEs**

- Regularity Structures [Hairer]
- Paracontrolled Distributions [Gubinelli–Imkeller–Perkowski]
- Energy Solutions [Gonçalves–Jara]
- Renormalization Group Approach [Kupiainen, Duch]

Stochastic Heat Equation and KPZ: subcritical only for $d = 1$

These approaches do not apply for $d \geq 2$

The role of dimension

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, \xi(t,x)\, u(t,x)$$

Space-time blow-up $\tilde{u}(t,x) := u(\delta^2 t, \delta x)$

$$\partial_t \tilde{u}(t,x) = \Delta_x \tilde{u}(t,x) + \delta^{\frac{2-d}{2}}\, \beta\, \tilde{\xi}(t,x)\, \tilde{u}(t,x)$$

As $\delta \downarrow 0$ the noise strength $\begin{cases} \text{vanishes} & d < 2 & \text{sub-critical} \\ \text{unchanged} & d = 2 & \textbf{critical} \\ \text{diverges} & d > 2 & \text{super-critical} \end{cases}$

We will henceforth focus on the **critical dimension** $d = 2$

What can we do?

Regularized noise $\xi_\varepsilon(t,x)$ on scale $\varepsilon > 0$ (discretization, mollification, ...)

Well-defined equation $\begin{cases} \partial_t u_\varepsilon(t,x) = \Delta_x u_\varepsilon(t,x) + \beta\, \xi_\varepsilon(t,x)\, u_\varepsilon(t,x) \\ u_\varepsilon(0,x) = \psi_0(x) \geq 0 \in C_b \qquad (\text{e.g. } \psi_0 \equiv 1) \end{cases}$

Fix $\xi_\varepsilon(t,x) \xrightarrow[\varepsilon \downarrow 0]{} \xi(t,x)$ Does the solution u_ε converge?

The answer depends subtly on the noise strength β

Which notion of convergence?

Space-average $\quad u_\varepsilon(t,\varphi) := \int_{\mathbb{R}^2} \varphi(x)\, u_\varepsilon(t,x)\, \mathrm{d}x \xrightarrow[\varepsilon\downarrow 0]{d} \mathscr{U}(t,\varphi) \qquad \varphi \in C_c$

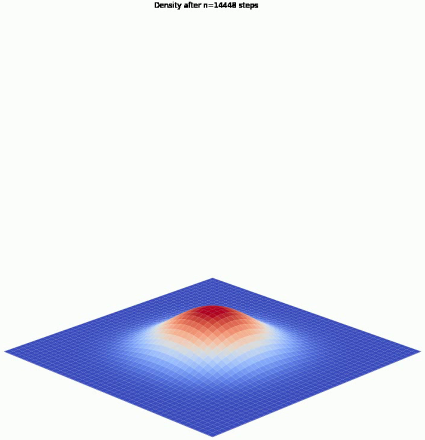
$\iff \quad u_\varepsilon(t,x)\, \mathrm{d}x \xrightarrow[\varepsilon\downarrow 0]{d} \mathscr{U}(t,\mathrm{d}x)$ vague convergence of random measures

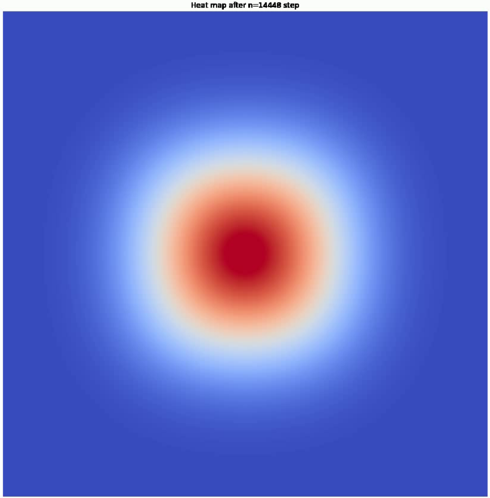
Theorem [Berger–C.–Turchi 25]

Fixed noise strength β: trivial limit $\mathscr{U} = \textit{non random}$

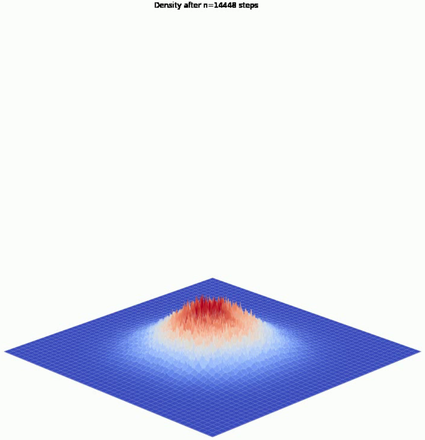
$$u_\varepsilon(t,\cdot) \xrightarrow[\varepsilon\downarrow 0]{} \begin{cases} \text{heat equation sol.} & \text{if } \beta = 0 \\ 0 & \text{if } \beta > 0 \end{cases}$$

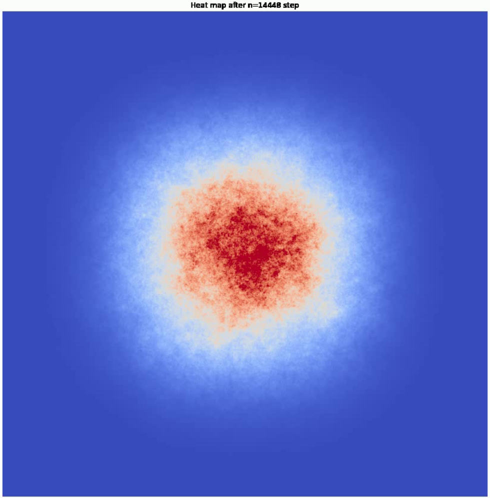
Discretization: $\beta = 0$ (by Q. Berger)



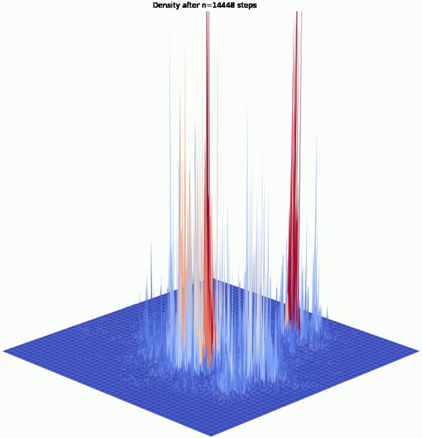


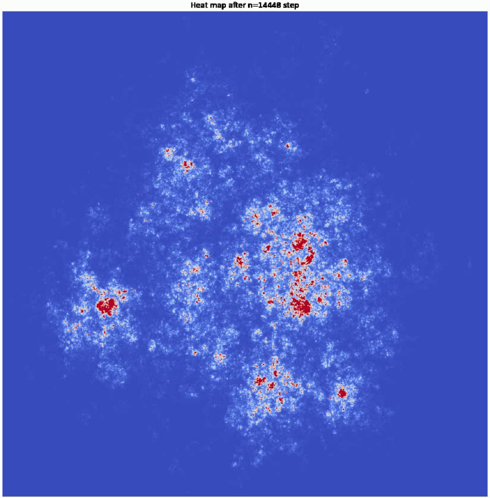
Discretization: β very small (by Q. Berger)



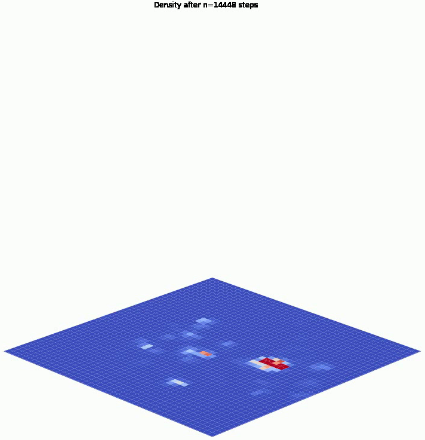


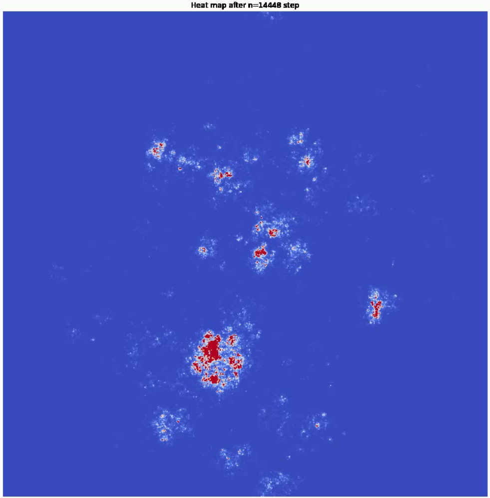
Discretization: β "critical" (by Q. Berger)





Discretization: β "supercritical" (by Q. Berger)





The need for renormalization

Critical scaling

($\rightsquigarrow$ phase transition)

$$\beta = \beta_\varepsilon = \frac{\hat{\beta}_{\mathrm{crit}}}{\sqrt{|\log \varepsilon|}} \left(1 + \frac{\vartheta}{|\log \varepsilon|} \right)$$

our discretization

$$\hat{\beta}_{\mathrm{crit}} = \sqrt{\pi}$$

Noise strength vanishes as $\varepsilon \downarrow 0$, yet **the effect of noise persists**

- $\mathbb{V}\mathrm{ar}\big[u_\varepsilon(t, \varphi) \big] \xrightarrow[\varepsilon \downarrow 0]{} K_t^{\vartheta}(\varphi, \varphi) > 0$ [Bertini–Cancrini 98] [C.S.Z. 19]

- All moments converge [C.S.Z. 19] [Gu–Quastel–Tsai 21]

  $\mathbb{E}\big[u_\varepsilon(t, \varphi)^q \big] \geq \exp(\exp(c\, q))$ [Ganguly–Nam 25]

Main result

Theorem [C.S.Z. 23]

Discretize the Stochastic Heat Equation on the lattice $\varepsilon\mathbb{N} \times \sqrt{\varepsilon}\,\mathbb{Z}^2$

Rescale $$\beta_\varepsilon = \frac{\sqrt{\pi}}{\sqrt{|\log \varepsilon|}}\left(1 + \frac{\vartheta}{|\log \varepsilon|}\right) \qquad \vartheta \in \mathbb{R}$$

The solution u_ε converges to a unique and non-trivial limit SHF^ϑ

$$u_\varepsilon = \big(u_\varepsilon(t,x)\,\mathrm{d}x\big)_{t\geq 0} \xrightarrow[\varepsilon\downarrow 0]{d} \mathrm{SHF}^\vartheta = \big(\mathrm{SHF}^\vartheta(t,\mathrm{d}x)\big)_{t\geq 0}$$

Measure-valued process on $\mathbb{R}^2$ Critical 2D **Stochastic Heat Flow**

Techniques and Ideas

Subsequential limits exist by tightness via moments bounds

Challenge is uniqueness lacking characterization of the limit

What we proved

$(u_\varepsilon)_{\varepsilon>0}$ is Cauchy $u_\varepsilon \approx u_{\varepsilon'}$ for small $\varepsilon, \varepsilon' > 0$

- Coarse-graining approximate self-similarity
- Lindeberg priciple robustness to microscopic details

These ideas played an important role in subsequent developments

SHF and Stochastic Heat Equation

SHF $=$ limit of solutions of renormalized 2D Stochastic Heat Equation

$$\mathrm{SHF}^{\vartheta}(t, \mathrm{d}x) \qquad \mathrm{SHF}^{\vartheta}(0, \psi_0\,;\, t, \mathrm{d}x) = \int_{\mathbb{R}^2} \psi_0(y)\, \mathrm{SHF}^{\vartheta}(0, \mathrm{d}y\,;\, t, \mathrm{d}x)$$

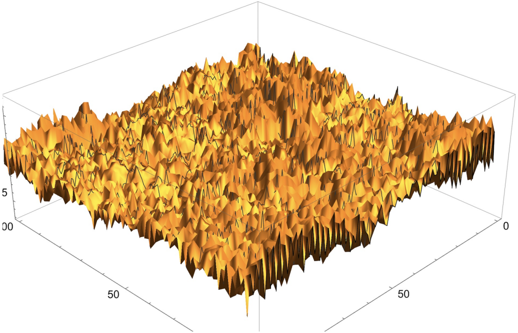
$\rightsquigarrow$ 2 space-time parameters $\big(\mathrm{SHF}^{\vartheta}(s, \mathrm{d}y\,;\, t, \mathrm{d}x)\big)_{0 \le s \le t < \infty}$

Flow property/Chapman–Kolmogorov: for $s < t < u$ [Clark–Mian 24]

$$\mathrm{SHF}^{\vartheta}(s, \mathrm{d}y\,;\, u, \mathrm{d}z) = \int_{x \in \mathbb{R}^2} \mathrm{SHF}^{\vartheta}(s, \mathrm{d}y\,;\, \underbrace{t, \mathrm{d}x)\, \mathrm{SHF}^{\vartheta}(t, \mathrm{d}x}_{\text{non-trivial ``product'' of measures}}\,;\, u, \mathrm{d}z)$$

How does the SHF look like?

Some not-so-randomly picked realizations [M. Mucciconi, N. Zygouras]



KPZ $\approx \log u_\varepsilon(t,x)$

Alps, Italy (by Mg-k)

Universality of SHF

We constructed SHF$^\vartheta$ via discretization of the Stochastic Heat Equation

Universality: other regularizations should also lead to SHF$^\vartheta$

Theorem [Tsai 24]

Stochastic Heat Equation with mollified noise $\xi_\varepsilon = \xi * \rho_\varepsilon$

Rescale $$\beta_\varepsilon = \frac{\sqrt{2\pi}}{\sqrt{|\log \varepsilon|}} \left(1 + \frac{\vartheta + c}{|\log \varepsilon|}\right) \qquad \vartheta \in \mathbb{R}$$

The solution u_ε converges to SHF$^\vartheta$ as $\varepsilon \downarrow 0$

$\rightsquigarrow$ Far reaching universality [Drillick, Hou, Parekh 26+]

Characterization of SHF

Theorem [Tsai 24]

Let $\mathscr{Z} = \big(\mathscr{Z}_{s,t}(\mathrm{d}x, \mathrm{d}y)\big)_{s \le t}$ be a stochastic process on $\mathscr{M}_+(\mathbb{R}^2 \times \mathbb{R}^2)$ satisfying

- continuity of $(s,t) \mapsto \mathscr{Z}_{s,t}$
- independence of $\mathscr{Z}_{s_1,t_1}\,,\ \mathscr{Z}_{s_2,t_2}\,,\ \ldots\,,\ \mathscr{Z}_{s_k,t_k}$ $\qquad \forall s_1 < t_1 < \ldots < s_k < t_k$
- flow property (Chapman-Kolmogorov) $\mathscr{Z}_{s,u} = \mathscr{Z}_{s,t} \bullet \mathscr{Z}_{t,u}$ $\qquad \forall s < t < u$
- same moments $\mathbb{E}\big[\prod_{i=1}^n \mathscr{Z}_{s,t}(\varphi_i, \psi_i)\big]$ as SHF^{ϑ} $\qquad$ up to $n = 4$

Then

$$\mathscr{Z} \overset{d}{=} \mathrm{SHF}^{\vartheta}$$

SHF and noise

Stochastic Heat Equation with regularized noise (discretized/mollified)

$$\partial_t u_\varepsilon(t,x) = \Delta_x u_\varepsilon(t,x) + \beta_\varepsilon\, u_\varepsilon(t,x)\, \xi_\varepsilon(t,x)$$

As $\varepsilon \downarrow 0$ $\quad u_\varepsilon \longrightarrow \mathrm{SHF}^{\vartheta} \qquad \xi_\varepsilon \longrightarrow \xi \qquad \beta_\varepsilon = \beta_\varepsilon(\vartheta) \longrightarrow 0$

Theorem [C.–Donadini 25] [Gu–Tsai 25]

$$\big(u_\varepsilon\,,\,\xi_\varepsilon\big) \xrightarrow[\varepsilon\downarrow 0]{d} \big(\mathrm{SHF}^{\vartheta}\,,\,\xi\big)$$ with SHF^{ϑ} and ξ **independent**

$\rightsquigarrow$ **No SPDE** driven by ξ for SHF^{ϑ}

- Noise sensitivity [C.–Donadini 25]
- Black noise [Gu–Tsai 25]

Basic Properties of SHF

- $\mathrm{SHF}^{\vartheta}(t, \mathrm{d}x) \neq$ Gaussian (positive random measure)

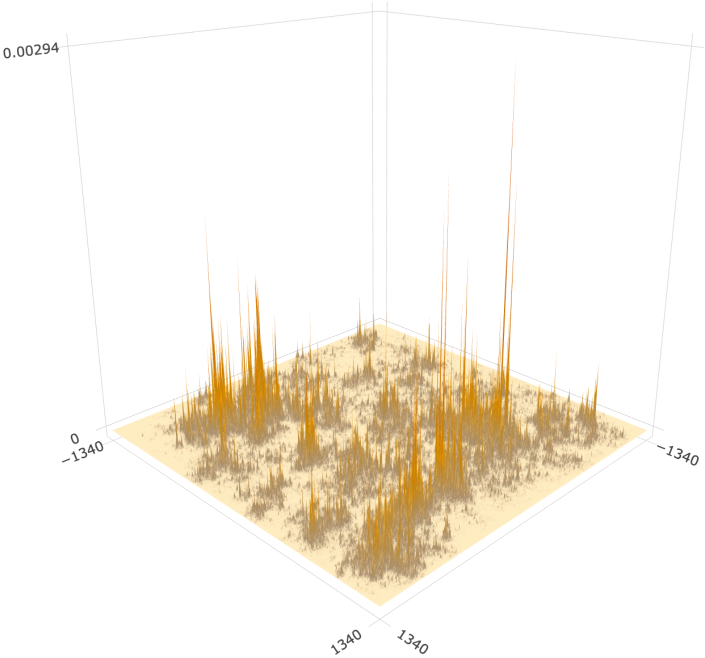
- $\mathrm{SHF}^{\vartheta}(t, \mathrm{d}x) \neq$ GMC $\simeq \exp(\text{Gaussian})$ [C.S.Z. 23]

  Conditional GMC structure on path space [Clark–Tsai 25]

- Moment formulas [C.S.Z. 19] [Gu–Quastel–Tsai 21] [Ganguly–Nam 25]

  $\rightsquigarrow$ [Cosco, Nakajima, Zeitouni] [Lygkonis, Zygouras] [Liu, Zygouras] ...

- Scaling covariance $$\frac{\mathrm{SHF}^{\vartheta}(\alpha\, t\,,\, \mathrm{d}(\sqrt{\alpha}\, x))}{\alpha} \stackrel{d}{=} \mathrm{SHF}^{\vartheta + \log \alpha}(t\,,\, \mathrm{d}x)$$



0.00294
0
−1340
−1340
1340
1340

Singularity and Regularity

- $\mathrm{SHF}^{\vartheta}(t, \mathrm{d}x)$ singular w.r.t. Lebesgue [C.S.Z. 25+]
  "not a function"

- $\mathrm{SHF}^{\vartheta}(t, \mathrm{d}x) \in \mathscr{C}^{-\kappa}$ for any $\kappa > 0$ $\rightsquigarrow$ non atomic
  "*barely* not a function"

- Small ball $\mathrm{SHF}^{\vartheta}(1, g_{\varepsilon^2}) = |\log \varepsilon|^{-1/2+o(1)}$ [Gu–Tsai 26]
  $+$ Gaussian fluctuations

- Strict positivity $\mathrm{SHF}^{\vartheta}(t, B) > 0$ $\forall$ ball B [Clark–Tsai 25] [Nakashima 25]

More

- Strong disorder $\quad \mathrm{SHF}^{\vartheta}(t,B) \xrightarrow[\vartheta\to\infty]{d} 0 \quad \forall$ ball B [Clark–Tsai 25]

  $+$ sharp rates [Berger.–C.–Turchi 25]

- Weak disorder $\quad \mathrm{SHF}^{\vartheta}(t,\mathrm{d}x) \xrightarrow[\vartheta\to-\infty]{d} \mathrm{d}x \quad$ for $\psi_0 \equiv 1$

  $+$ Gaussian fluctuations (Edwards-Wilkinson) [C.–Cottini–Rossi 25]

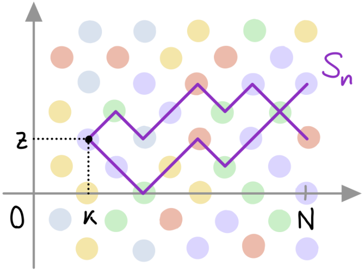
- Martingale problem [Nakashima 25] [Chen 25]

- Many questions still open!

  Fractal properties $\quad$ Superdiffusivity $\quad$ KPZ $=$ "log SHF" $\quad \ldots$

Directed Polymer in Random Environment

- Simple random walk on $\mathbb{Z}^2$ $\quad S = (S_n)_{n \geq 0}$
- Independent Gaussians $\quad \omega(n,x) \sim \mathcal{N}(0,1)$
- Interaction $\quad H_N(S,\omega) := \sum_{n \leq N} \omega(n, S_n)$



Polymer measure $\quad P_N(S) = \dfrac{e^{\beta H_N(S,\omega)}}{Z_N} P^{\mathrm{SRW}}(S)$

Partition Functions $\quad Z_N(k,z) = \mathrm{E}_{(k,z)}\left[e^{\beta H_N(S,\omega)}\right] \,/\, \mathbb{E}[\ldots]$

Stochastic Heat Equation and Feynman–Kac

Partition functions solve discretized Stochastic Heat Equation (Duhamel)

$$-\partial_k Z_N(k,z) = \Delta_z Z_N(k,z) + \left(\mathrm{e}^{\beta\omega(k,z)} - 1\right) Z_N(k,z)$$

Discretized Feynman–Kac $\qquad Z_N = \mathrm{E}\left[\mathrm{e}^{\beta H_N(S,\omega)}\right] \approx \mathrm{E}\left[\mathrm{e}^{\beta \int \xi(s,B_s)\,\mathrm{d}s}\right]$

Diffusive rescaling by factor N $\rightsquigarrow$ Discretization on scale $\varepsilon = \frac{1}{\sqrt{N}}$

$$Z\left(N(1-t), \sqrt{N}x\right) = u_\varepsilon(t,x)$$

Conclusion

Probabilistic handle on Stochastic Heat Equation via Directed Polymers

$\rightsquigarrow$ KPZ universality in $d = 1$ [Seppäläinen 12] [Alberts–Khanin–Quastel 14]

Statistical Mechanics	$\leftrightsquigarrow$	Singular SPDEs
Disorder (ir)relevance	$\leftrightsquigarrow$	super/subcriticality

Beyond the SHF

A broader picture

Extensions of Stochastic Heat Equation

- Nonlinear [Dunlap–Gu 22] [Dunlap–Graham 23]
- Lévy noise [Berger–Chong–Lacoin 23]
- Correlated Gaussian noise [Müller–Tribe 04] [Dunlap–Hairer–Li 25] [Cosco–Cottini–Donadini 25]

Related critical SPDEs

- Anisotropic KPZ equation [Cannizzaro–Erhard–Toninelli 23]
- Stochastic Burgers equation [Cannizzaro–Gubinelli–Toninelli 23] [Cannizzaro–Moulard–Toninelli 25]

...

Thanks