Noise Sensitivity and Critical 2D Directed Polymers

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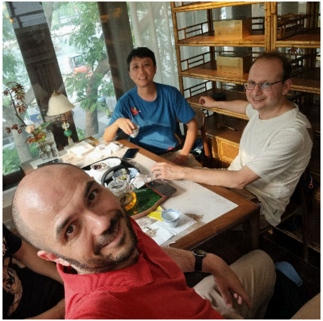


The Stochastic Heat Equation

Heat equation with multiplicative singular potential $t \geq 0,\ x \in \mathbb{R}^d$

$$\partial_t u(t,x) = \underbrace{\Delta_x u(t,x)}_{\sum_{i=1}^{d} \frac{\partial^2}{\partial {x_i}^2} u(t,x)} + \beta\, \xi(t,x)\, u(t,x) \qquad \text{(SHE)}$$

Coupling constant $\beta \geq 0$

Singular random potential $\xi(t,x)$ "space-time white noise"

Critical dimension $d = 2$ [C.S.Z. 23]

Natural candidate solution: the critical 2D Stochastic Heat Flow (SHF)

Why is it interesting?

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, \xi(t,x)\, u(t,x) \qquad \text{(SHE)}$$

- Fundamental PDE $+$ universal random potential $\xi(t,x)$

  white noise $=$ "continuum" i.i.d. random variables

- KPZ equation [Kardar–Parisi–Zhang *PRL* 86]

$$\partial_t h(t,x) = \Delta_x h(t,x) + |\nabla_x h(t,x)|^2 + \beta\, \xi(t,x) \qquad \text{(KPZ)}$$

Cole–Hopf transformation $h(t,x) = \log u(t,x)$

Why is it difficult?

$$\partial_t u(t,x) = \Delta_x u(t,x) + \beta\, \xi(t,x)\, u(t,x) \qquad \text{(SHE)}$$

$\xi(t,x)$ is a distribution $\rightsquigarrow$ u(t,x) expected $\begin{cases} \text{non-smooth function} & d=1 \\ \text{genuine distribution} & d \geq 2 \end{cases}$

Product $\xi(t,x)\, u(t,x)$ unclear: no classical space to solve SHE (as a PDE)

Stochastic integral for $d = 1$ [Ito/Walsh, Da Prato–Zabczyk]

SHE solution $u(t,x) > 0$ starting from $u(0,\cdot) \geq 0$ $\rightsquigarrow$ "KPZ solution" $h(t,x) = \log u(t,x)$

Regularisation

Henceforth $d = 2$: How to define a solution of 2D SHE?

Regularized noise $\xi_N(t,x)$ $\rightsquigarrow$ well-defined solution $u_N(t,x)$

(discretization, mollification, ...)

$$\begin{cases} \partial_t u_N(t,x) = \Delta_x u_N(t,x) + \beta\, u_N(t,x)\, \xi_N(t,x) \\ u_N(0,x) \equiv 1 \quad \text{(for simplicity)} \end{cases} \qquad \text{(reg-SHE)}$$

Convergence of $u_N(t,\varphi) := \int_{\mathbb{R}^2} \varphi(x)\, u_N(t,x)\, \mathrm{d}x$ as $N \to \infty$?

Renormalisation

Convergence of the mean: $\mathbb{E}\big[u_N(t,\varphi)\big] \xrightarrow[N\to\infty]{} \int_{\mathbb{R}^2} \varphi(x)\,\mathrm{d}x$

Convergence of the variance?

$$\beta \sim \frac{\hat\beta}{\sqrt{\log N}} \quad \text{for} \quad \hat\beta = \sqrt{\pi}\left(1 + \frac{\vartheta}{\log N}\right)$$

$\mathbb{V}\mathrm{ar}\big[u_N(t,\varphi)\big] \xrightarrow[N\to\infty]{} K_t^{\vartheta}(\varphi,\varphi) > 0$ [Bertini–Cancrini 98] [C.S.Z. 19]

Convergence of all higher moments [C.S.Z. 19] [Gu–Quastel–Tsai 21]

Convergence in law of $u_N(t,\varphi)$? $\iff$ of the random measure $u_N(t,x)\,\mathrm{d}x$?

The critical 2D Stochastic Heat Flow

Theorem [C.S.Z. *Invent. Math.* 23]

Take $\beta = \frac{\sqrt{\pi}}{\sqrt{\log N}} \left(1 + \frac{\vartheta}{\log N} \right)$ for some $\vartheta \in \mathbb{R}$

Then u_N converges in law to a unique and non-trivial limit $\mathscr{U}^{\vartheta}$

$$\big(u_N(t,x)\, \mathrm{d}x \big)_{t \geq 0} \xrightarrow[N\to\infty]{d} \big(\mathscr{U}^{\vartheta}(t, \mathrm{d}x) \big)_{t \geq 0}$$

$\mathscr{U}^{\vartheta}$ = critical 2D **Stochastic Heat Flow (SHF)** = stochastic process of measures on $\mathbb{R}^2$

SHF and Stochastic Heat Equation

The SHF is a "candidate solution" of the critical 2d Stochastic Heat Equation

$$\mathscr{U}^{\vartheta}(t, \mathrm{d}x) \qquad (\text{initial condition } 1 \text{ at time } 0)$$

We actually build a two-parameter space-time process

$$\left(\mathscr{U}^{\vartheta}(s, \mathrm{d}y; t, \mathrm{d}x)\right)_{0 \le s \le t < \infty} \qquad (\text{starting at time } s \text{ from } \mathrm{d}y)$$

"Flow": Chapman-Kolmogorov property for $s < t < u$ [Clark–Mian 2024+]

$$\mathscr{U}^{\vartheta}(s, \mathrm{d}y; u, \mathrm{d}z) = \int_{x \in \mathbb{R}^2} \mathscr{U}^{\vartheta}(s, \mathrm{d}y; \underbrace{t, \mathrm{d}x)\, \mathscr{U}^{\vartheta}(t, \mathrm{d}x}_{\substack{\text{non-trivial "product"} \\ \text{of measures}}}; u, \mathrm{d}z)$$

Key properties of the SHF

- a.s. $\mathscr{U}^{\vartheta}(t, dx)$ is singular w.r.t. Lebesgue [C.S.Z. 2024+]

  "not a function"

- a.s. $\mathscr{U}^{\vartheta}(t, dx) \in \mathscr{C}^{-\kappa}$ for any $\kappa > 0$ (in particular: non atomic)

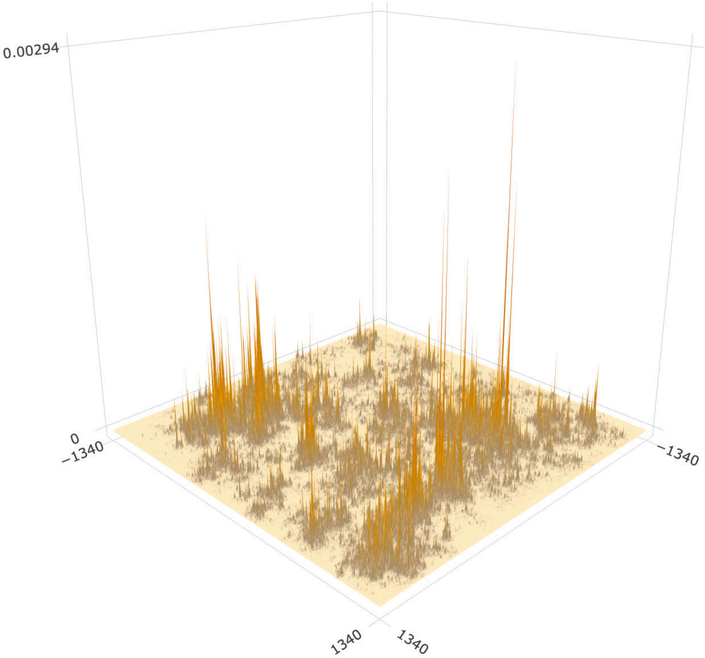
  "*barely* not a function"

- Formulas for all moments [C.S.Z. 19] [Gu–Quastel–Tsai 21]

- Scaling covariance $a^{-1}\,\mathscr{U}^{\vartheta}(a\,t, d(\sqrt{a}\,x)) \overset{d}{=} \mathscr{U}^{\vartheta+\log a}(t, dx)$

- Axiomatic characterization via independence & moments [Tsai 24+]

- Universality w.r.t. approximation scheme [C.S.Z. 23] [Tsai 24+]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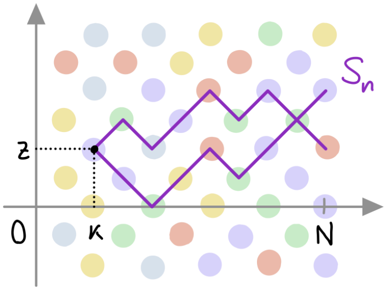
Directed polymer in random environment

- $S = (S_n)_{n\geq 0}$ simple random walk on $\mathbb{Z}^d$
- Independent Gaussians $\omega(n,x) \sim \mathcal{N}(0,1)$
- $H(S,\omega) := \sum_{n=k+1}^{N} \omega(n, S_n) \sim \mathcal{N}(0, N-k)$



Partition Function $(k \in \mathbb{N},\ z \in \mathbb{Z}^d)$

$$Z_{N,\beta}^{\omega}(k,z) = \mathrm{E}\left[e^{\beta H(S,\omega) - \frac{1}{2}\beta^2 (N-k)} \,\middle|\, S_k = z \right]$$

Partition function and SHE

Diff. rescaled partition function = discretized SHE solution

$$Z^{\omega}_{N,\beta}\big(N(1-t),\sqrt{N}x\big) \;=\; u_N(t,x) \qquad \text{(time rev.)}$$

Partition functions solve a difference equation: with $\xi_N \approx \omega$

$$\begin{cases} \partial_t u_N(t,x) = \Delta_x u_N(t,x) + \underbrace{\beta\, N^{\frac{2-d}{4}}}_{\beta_{\mathrm{SHE}}}\, \xi_N(t,x)\, u_N(t,x) \\ u_N(0,x) \equiv 1 \end{cases} \qquad \text{(reg-SHE)}$$

Discrete analogue of Feynman-Kac

$$u_N(t,x) \;\approx\; \mathrm{E}\Big[\mathrm{e}^{\beta_{\mathrm{SHE}} \int_{1-t}^{1} \xi(s,B_s) - \frac{1}{2}\beta_{\mathrm{SHE}}^2\, t} \,\Big|\, B_{1-t} = x \Big]$$

Critical 2D directed polymers

We can write $$u_N(1,\varphi) = \mathscr{Z}_N^{\omega}(\varphi) := \sum_{z\in\mathbb{Z}^2} \frac{1}{N}\,\varphi\Big(\frac{z}{\sqrt{N}}\Big)\, Z_{N,\beta}^{\omega}(0,z) \qquad (t=1)$$

e.g. $\varphi(x) = \mathbb{1}_{\{|x|\le 1\}}$ $\rightsquigarrow$ Directed Polymer started uniformly in $B(0,\sqrt{N})$

Take $$\beta = \beta_N = \frac{\sqrt{\pi}}{\sqrt{\log N}}\Big(1 + \frac{\vartheta}{\log N}\Big)$$ for some $\vartheta \in \mathbb{R}$

Critical 2D Directed Polymers and SHF

$$\mathscr{Z}_N^{\omega}(\varphi) \xrightarrow[N\to\infty]{d} \mathscr{U}^{\vartheta}(1,\varphi) := \int_{\mathbb{R}^2} \varphi(x)\,\mathscr{U}^{\vartheta}(1,\mathrm{d}x)$$

Noise sensitivity for 2D directed polymers?

Fix a probability density $\varphi(\cdot)$ on $\mathbb{R}^2$ (we focus on $t = 1$)

$$\mathscr{Z}_N^{\omega}(\varphi) \xrightarrow[N\to\infty]{d} \mathscr{U}^{\vartheta}(1, \varphi)$$

$\mathscr{Z}_N^{\omega}(\varphi)$ is a function $f_N(\omega)$ of i.i.d. random variables $\omega = (\omega(n,z))_{n\in\mathbb{N}, z\in\mathbb{Z}^2}$

Is f_N sensitive to small perturbations of the random variables ω?

$\iff$ is u_N sensitive to small perturbations of the driving noise ξ_N?

Noise sensitivity

Fix i.i.d. random variables $\omega = (\omega_i)_{i=1,2,\ldots}$ $\qquad \mathbb{E}[\omega_i] = 0 \quad \mathbb{V}\text{ar}[\omega_i] = 1$

Take a sequence of functions $f_N(\omega) \in L^2$ $\qquad \lim_{N\to\infty} \mathbb{V}\text{ar}[f_N(\omega)] = \sigma^2 \in (0,\infty)$

Define "ε-perturbation" $\omega^\varepsilon = (\omega_i^\varepsilon)_{i=1,2,\ldots}$ $\qquad \omega_i^\varepsilon := \begin{cases} \omega_i & \text{w. prob. } 1-\varepsilon \\ \widetilde{\omega}_i \perp\!\!\!\perp \omega_i & \text{w. prob. } \varepsilon \end{cases}$

We call $(f_N)_{N\in\mathbb{N}}$ noise sensitive if

$$\lim_{N\to\infty} \mathbb{C}\text{ov}\left[f_N(\omega^\varepsilon), f_N(\omega)\right] = 0 \qquad \forall \varepsilon > 0$$

Spectral criterion

Chaos decomposition $$f_N = \mathbb{E}[f_N] + \sum_{d=1}^{\infty} f_N^{(d)} \qquad \mathbb{V}\mathrm{ar}[f_N] = \sum_{d=1}^{\infty} \| f_N^{(d)} \|_2^2$$

For instance $$f_N^{(d)}(\omega) = \sum_{\{i_1,\dots,i_d\}} c_N(i_1,\dots,i_d)\, \omega_{i_1} \cdots \omega_{i_d}$$ (polynomial chaos)

Suitable decomposition for any $f_N(\omega) \in L^2$ [Efron-Stein]

Spectral criterion

Noise sensitivity $\Longleftrightarrow$ $\forall d \in \mathbb{N}$: $\| f_N^{(d)} \|_2^2 \xrightarrow[N\to\infty]{} 0$

Influences

Define $\delta_i f := f - \mathbb{E}_i[f]$ with $\mathbb{E}_i[\cdot] = \mathbb{E}[\cdot | \sigma(\omega_j : j \neq i)]$ [Talagrand 94]

Two notions of influence

$$I_i^{(1)}(f) := \|\delta_i f\|_1 = \mathbb{E}\big[|\delta_i f|\big] \qquad I_i^{(2)}(f) := \|\delta_i f\|_2^2 = \mathbb{E}\big[(\delta_i f)^2\big]$$

(for Boolean f they coincide up to a factor 2)

Sum of squared L^1 influences: $$\mathscr{W}(f) := \sum_i I_i^{(1)}(f)^2$$

General BKS

Consider either of the following settings:

- $\mathbb{E}[|\omega_i|^q] < \infty$ for some $q > 2$ & $f(\omega)$ is a polynomial chaos
- ω_i take finitely many values & $f(\omega)$ is any function in L^2

We extend [Benjamini–Kalai–Schramm 99] [Keller–Kindler 13]

Generalized BKS [C.–Donadini 25+]

$$\forall \varepsilon > 0: \qquad \mathbb{C}\mathrm{ov}\left[f(\omega^\varepsilon), f(\omega)\right] \leq C\,\mathscr{W}(f)^{\alpha_q \varepsilon}$$

$$\forall d \in \mathbb{N}: \qquad \left\|f^{(d)}\right\|_2^2 \leq (c_q)^d\,\mathscr{W}(f)^{1-\frac{2}{q}}$$

2D directed polymers

Noise sensitivity of 2D Directed Polymers [C.–Donadini 25+]

$$\mathscr{W}\left(\mathscr{Z}_N^{\omega}(\varphi)\right) \sim \frac{c_\varphi}{\log N} \qquad \Longrightarrow \qquad \mathscr{Z}_N^{\omega}(\varphi) \ \text{ is noise sensitive}$$

Stability of influences: $\mathscr{W}\left(\psi(f)\right) \leq 4\,\|\psi'\|_\infty^2\, \mathscr{W}(f)$ for Lipschitz ψ

Enhanced noise sensitivity [C.–Donadini 25+]

$\psi\left(\mathscr{Z}_N^{\omega}(\varphi)\right)$ noise sensitive $\forall$ Lipschitz ψ (if ω_i's take finitely many values)

$\Longrightarrow$ $\mathscr{Z}_N^{\omega}(\varphi)$ asymptotically independent of any bounded order chaos

Back to 2D Stochastic Heat Equation

Interesting implication for 2D SHE $\qquad \beta_N \sim \frac{\sqrt{\pi}}{\sqrt{\log N}}\left(1 + \frac{\vartheta}{\log N}\right)$

$$\partial_t u_N(t,x) = \Delta_x u_N(t,x) + \beta_N\, u_N(t,x)\, \xi_N(t,x) \qquad \text{(reg-SHE)}$$

$u_N \longrightarrow \mathscr{U}^{\vartheta}$ critical 2D SHF $\qquad \xi_N \longrightarrow \xi$ space-time white noise

Theorem [C.–Donadini 25+]

$$(\xi_N, u_N) \xrightarrow[N\to\infty]{d} (\xi, \mathscr{U}^{\vartheta})$$ with ξ and $\mathscr{U}^{\vartheta}$ independent

Puzzling: u_N is a <u>function</u> of ξ_N (!)

No SPDE for $\mathscr{U}^{\vartheta}$ driven by ξ

Conclusion

We extend the BKS Theorem beyond the Boolean setting

- Quantitative bounds — extending [Keller–Kindler]
- Robust conditions for noise sensitivity — (stable under composition)

We show noise sensitivity for critical 2D Directed Polymers

Motivations/implications for the critical 2D Stochastic Heat Equation

Future direction: black noise à la Tsirelson — cf. [Himwich–Parekh 24+]

Bedankt

Axiomatic characterization

Recently Li-Cheng Tsai provided an axiomatic characterization of the SHF

Theorem [Tsai 24+]

Let $\mathscr{Z} = \big(\mathscr{Z}_{s,t}(\mathrm{d}x,\mathrm{d}y)\big)_{s\le t}$ be a stochastic process on $\mathscr{M}_+(\mathbb{R}^2 \times \mathbb{R}^2)$ satisfying

- continuity of $(s,t) \mapsto \mathscr{Z}_{s,t}$
- independence of $\mathscr{Z}_{s,t}$ and $\mathscr{Z}_{t,u}$ $\forall s < t < u$
- convolution $\mathscr{Z}_{s,u} = \mathscr{Z}_{s,t} * \mathscr{Z}_{t,u}$ (Chapman-Kolmogorov) $\forall s < t < u$
- moments $\mathbb{E}\big[\prod_{i=1}^n \mathscr{Z}_{s,t}(\varphi_i, \psi_i)\big]$ for $n = 1, 2, 3, 4$ coincide with those of $\mathscr{U}^{\vartheta}$

Then $\mathscr{Z}$ has the same distribution as the SHF $\mathscr{U}^{\vartheta}$