The critical 2d Stochastic Heat Flow

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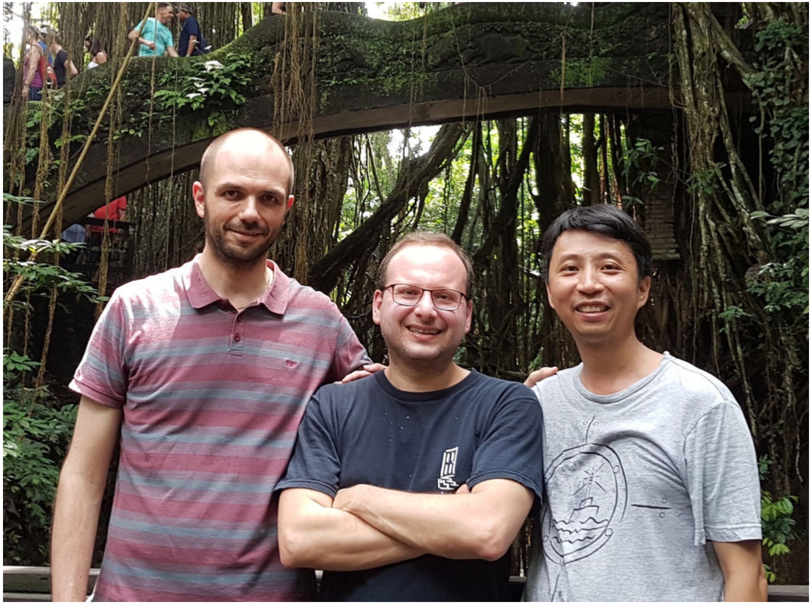
Based on joint works with



Nikos Zygouras and Rongfeng Sun

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  THE CRITICAL 2D STOCHASTIC HEAT FLOW
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- [CSZ 19b] Commun. Math. Phys. 372 (2019)
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The Stochastic Heat Equation

For $t > 0$, $x \in \mathbb{R}^d$:

$$(\text{SHE}) \qquad \begin{cases} \partial_t u(t,x) = \Delta u(t,x) + \beta\, \xi(t,x)\, u(t,x) \\ u(0,x) \equiv 1 \end{cases}$$

(for simplicity)

- $\beta > 0$ coupling constant
- $\xi(t,x)$ "space-time white noise" (very irregular)

GOAL: Construct the natural candidate solution $u(t,x)$ for $d = 2$

$\downarrow$

" STOCHASTIC HEAT FLOW "

Overview

I. Presentation: why it is interesting, why it is difficult

II. Main results

III. Ideas and Techniques

IV. Conclusions & Perspectives

I. Presentation

The Stochastic Heat Equation

$$\partial_t u(t,x) = \underbrace{\overset{\partial^2_{x_1} + \ldots + \partial^2_{x_d}}{\Delta} u(t,x)}_{\text{HEAT EQUATION}} + \beta \underbrace{\xi(t,x)}_{\text{POTENTIAL / NOISE}} u(t,x) \qquad \text{(SHE)}$$

- $u(t,x)$ = density of diffusing particles at time t, space x
- $\xi(t,x)$ = rate at which particles are generated ($\xi > 0$) / killed ($\xi < 0$)

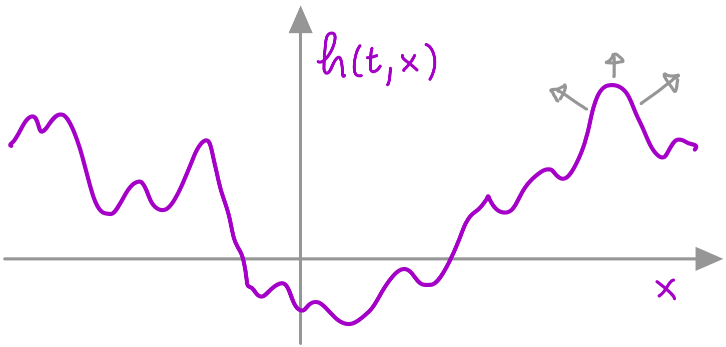
SPACE-TIME WHITE NOISE: "canonical" yet challenging!

THE KARDAR-PARISI-ZHANG EQUATION

[PRL 1986]

If $\xi(t,x)$ is regular, then $h(t,x) := \log u(t,x)$ solves

$$\partial_t h(t,x) = \underbrace{\Delta h(t,x)}_{\text{SMOOTHING}} + \underbrace{|\nabla h(t,x)|^2}_{\perp \text{ GROWTH}} + \beta \underbrace{\xi(t,x)}_{\text{NOISE}} \qquad \text{(KPZ)}$$



If $\xi(t,x)$ is NOT regular?

SHE can help us make sense of KPZ

WHITE NOISE

$\xi(t,x)$ = WHITE NOISE on $\mathbb{R} \times \mathbb{R}^d$ (space-time)

It is a random element of $\mathcal{D}'(\mathbb{R}^{1+d}) = \{$ distributions on $\mathbb{R}^{1+d}\}$

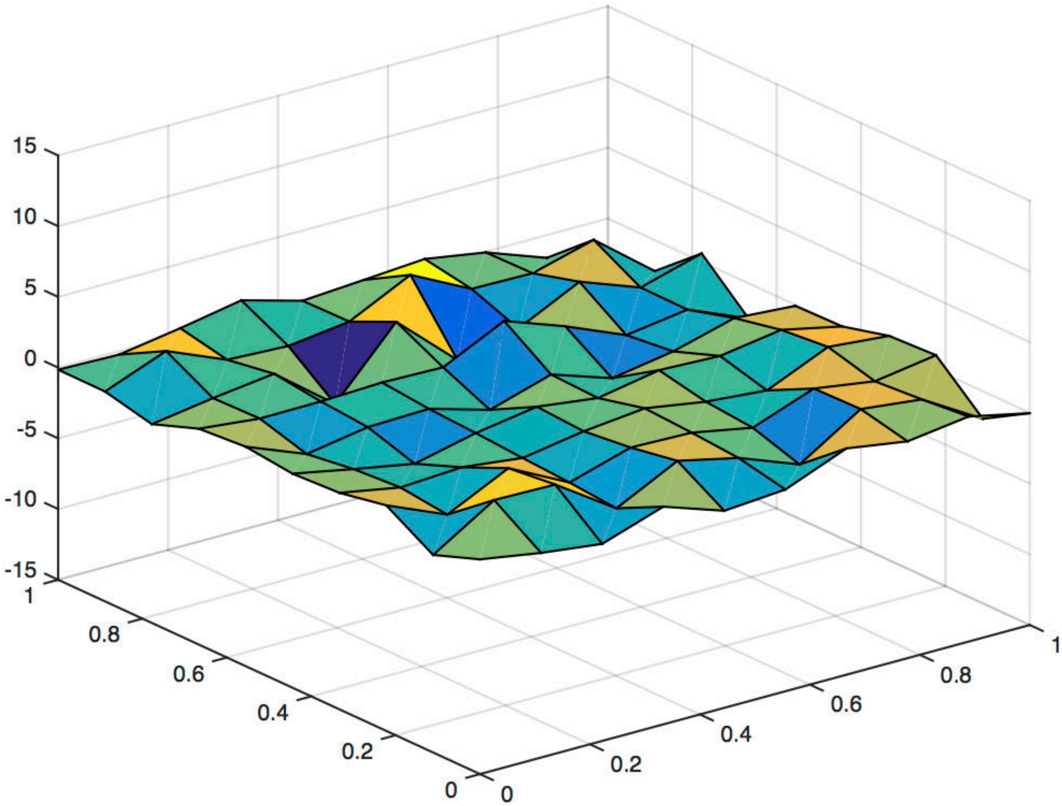
Gaussian: $\mathbb{E}[\xi] = 0, \quad \mathrm{Cov}[\xi(t,x), \xi(t',x')] = \delta(t-t')\,\delta(x-x')$

$$\text{“}\int \xi(t,x)\,\varphi(t,x)\,dt\,dx\text{”} = \langle \xi, \varphi \rangle \sim N(0, \|\varphi\|_{L^2}^2)$$

$$\xi(t,x) = \sum_{k\in\mathbb{N}} \underbrace{Z_k}_{\text{i.i.d. } N(0,1)} \cdot \underbrace{\psi_k(t,x)}_{\text{orthonormal basis of } L^2} \in \mathcal{C}^\alpha = \mathcal{B}^\alpha_{\infty,\infty} \qquad \alpha = -\frac{1+d}{2} - \varepsilon < 0$$

White noise

Heuristically: "i.i.d. for $(t,x) \in \mathbb{R}^{1+d}$"



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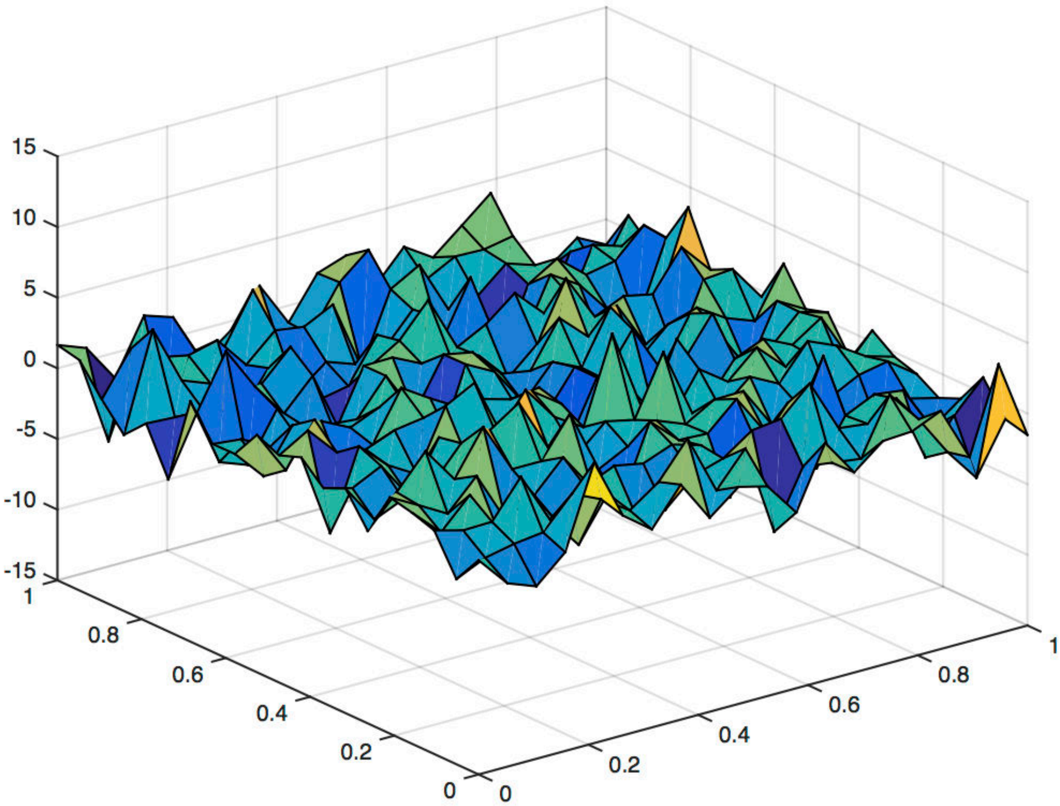

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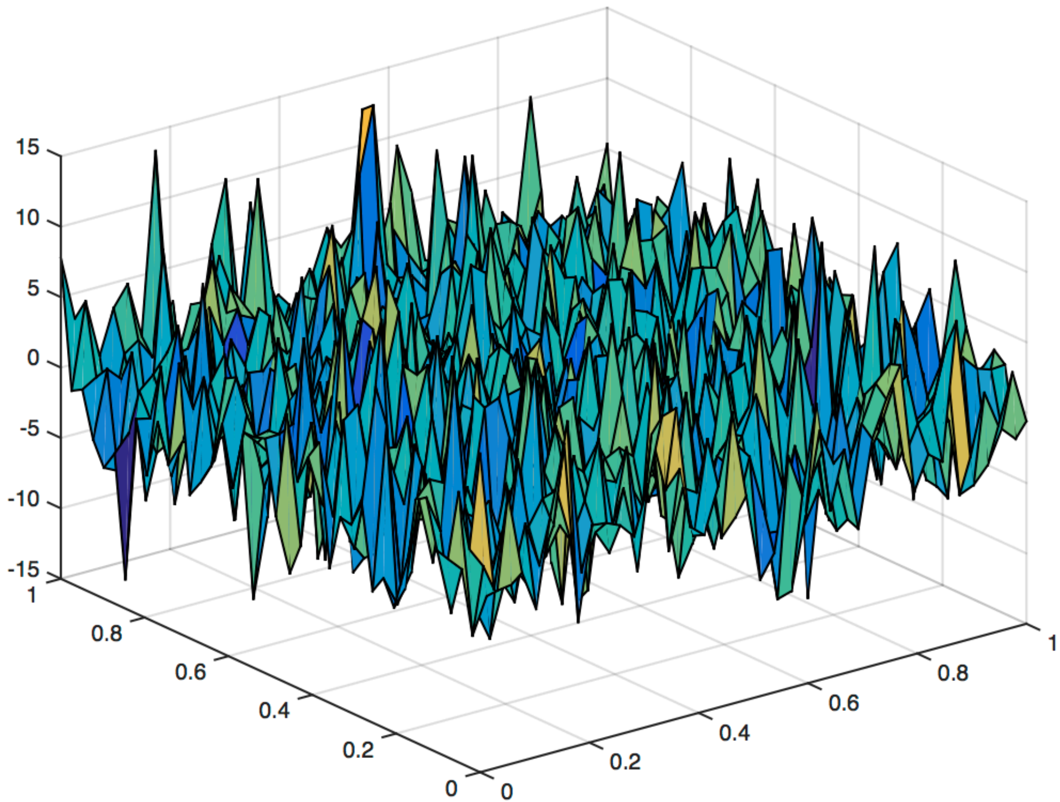


WHITE NOISE

BROWNIAN MOTION

(d=0) White noise on $\mathbb{R}$ $\quad \xi(t) = \frac{d}{dt} B(t)$ $\qquad \mathcal{C}^{-\frac{1}{2}-\varepsilon}$

White noise $\xi(t,x)$ is irregular $\qquad \mathcal{C}^{-\frac{1+d}{2}-\varepsilon}$

$\Downarrow$

$u(t,x)$ and $h(t,x)$ expected to be $\begin{cases} \text{non-smooth functions} & (d=1) \\ \text{genuine distributions} & (d \geq 2) \end{cases}$

$\mathcal{C}_t^{\frac{2-d}{4}-\varepsilon} \times \mathcal{C}_x^{\frac{2-d}{2}-\varepsilon}$

How to define singular products? $\begin{cases} \xi(t,x)\, u(t,x) & (\text{SHE}) \\ |\nabla h(t,x)|^2 & (\text{KPZ}) \end{cases}$

SHE AND KPZ : DIFFICULT, YET INTERESTING!

They are both ill-defined PDEs, due to singular products.

There is no classical Banach space of functions/distributions s.t.

- singular products are well-defined & continuous operators;
- the PDE can be solved as a fixed point via contraction.

Yet, we can regularize (or discretize) the noise $\xi_\varepsilon(t,x)$.

Do the corresponding solutions converge as $\varepsilon \downarrow 0$?

$$u_\varepsilon(t,x) \longrightarrow u(t,x) \qquad h_\varepsilon(t,x) \longrightarrow h(t,x) \qquad ?$$

THE CASE d=1

Breakthroughs obtained in the 2010s for sub-critical PDEs

⇝ including SHE and KPZ for d=1

- REGULARITY STRUCTURES [Hairer]
- PARACONTROLLED CALCULUS [Gubinelli, Imkeller, Perkowski]
- ENERGY SOLUTIONS [Goncalves, Jara]
- RENORMALIZATION [Kupiainen]

SHE and KPZ for d=1 well understood in a robust way

"PATHWISE"

THE CASE d=1

[Hairer 13, 14] [Hairer, Pardoux 15]

- SHE solution $u(t,x)$ for $d=1$ well-posed by Ito-Walsh integration.

- If $\xi_\varepsilon = \xi * \rho_\varepsilon$, to have $u_\varepsilon \to u$ we need to renormalize SHE :

$$\partial_t u_\varepsilon(t,x) = \Delta u_\varepsilon(t,x) + \beta\, \xi_\varepsilon(t,x)\, u_\varepsilon(t,x) \underbrace{- C_\rho\, u_\varepsilon(t,x)}_{\text{extra term!}}$$

- If we define $h_\varepsilon := \log u_\varepsilon \to h := \log u$, we have

$$\partial_t h_\varepsilon(t,x) = \Delta h_\varepsilon(t,x) + |\nabla h_\varepsilon(t,x)|^2 + \beta\, \xi_\varepsilon(t,x) \underbrace{- C_\varepsilon}_{\to -\infty\ !}$$

II. Main Results

THE CRITICAL DIMENSION $d=2$

Formally: if $u(t,x)$ solves SHE, then $\tilde{u}(t,x) := u(\delta^2 t, \delta x)$

$$\partial_t \tilde{u}(t,x) = \Delta \tilde{u}(t,x) + \beta\, \delta^{\frac{2-d}{2}}\, \tilde{\xi}(t,x)\, \tilde{u}(t,x)$$

As $\delta \downarrow 0$, the noise term $\begin{cases} \text{vanishes} & (d<2) \\ \text{stays constant} & (d=2) \\ \text{diverges} & (d>2) \end{cases}$

$d=2$ is the CRITICAL DIMENSION for SHE: no solution theory!

REGULARIZATION / DISCRETIZATION

DISCRETIZED STOCHASTIC HEAT EQUATION

Discretized SHE: (t,x) in the lattice $\Pi_N = \frac{\mathbb{N}}{N} \times \frac{\mathbb{Z}^2}{\sqrt{N}}$ $\quad (N \in \mathbb{N})$

$$\underbrace{\partial_t^N u_N(t,x)}_{\text{TIME DIFFERENCE}} = \underbrace{\frac{1}{4}\Delta^N u_N(t,x)}_{\text{LATTICE LAPLACIAN}} + \beta N \underbrace{\xi_N(t+\tfrac{1}{N},x)}_{\text{I.I.D. ZERO MEAN, UNIT VARIANCE (suitable choice)}} \underbrace{\langle u_N(t,x) \rangle}_{\text{SPACE AVERAGE}} \qquad \text{(D-SHE)}$$

$$N \cdot \left\{ u(t+\tfrac{1}{N},x) - u(t,x) \right\} \qquad \frac{N}{4} \sum_{x' \sim x} \left\{ u(t,x') - u(t,x) \right\} \qquad \frac{1}{4} \sum_{x' \sim x} u(t,x')$$

$$x' = x \pm \frac{e_i}{\sqrt{N}}$$

Discretized Stochastic Heat Equation

Can we hope that, as $N \to \infty$, $u_N(t,x)$ has a limit "$u(t,x)$" ?

- View u_N as a (random) distribution, in fact measure on $\mathbb{R}^2$:

$$u_N(t,x)\,dx \longrightarrow \text{"}u(t,dx)\text{"} \quad ?$$

piecewise constant Lebesgue measure

- Rescale the coupling constant $\beta = \beta_N \to 0$ in a precise way

$$\beta = \frac{\sqrt{\pi}}{\sqrt{\log N}}\left(1 + \frac{\vartheta}{\log N}\right) \qquad \text{for fixed } \vartheta \in \mathbb{R}$$

MAIN RESULT [CSZ 21]

Fix $\vartheta \in \mathbb{R}$ and $\beta = \beta_N$ as in ⊛. Let $u_N(t,x)$ solve (D-SHE).

Then, as $N \to \infty$, we have the convergence in f.d.d. of

$$\left(u_N(t,x)\, dx \right)_{t \geq 0}$$

to a non-trivial limit: (as random measures on $\mathbb{R}^2$)

$$\mathcal{U}^\vartheta = \left(\mathcal{U}_t^\vartheta(dx) \right)_{t \geq 0}$$

which we call the CRITICAL 2D STOCHASTIC HEAT FLOW.

The critical 2d Stochastic Heat Flow

We have built a candidate solution of the CRITICAL 2d SHE

$$\mathcal{U}^{\vartheta} = \left(\mathcal{U}^{\vartheta}_t(dx)\right)_{t\geq 0} = \begin{cases} \text{limit of discretized solutions (D-SHE)} \\ \text{with critical rescaling } \star \text{ of } \beta = \beta_N(\vartheta) \end{cases}$$

(with initial condition $u(0,\cdot) \equiv 1$)

We can actually build a two-parameter process

$$\mathcal{U}^{\vartheta} = \left(\mathcal{U}^{\vartheta}_{s,t}(dy, dx)\right)_{0\leq s\leq t<\infty}$$

where $\mathcal{U}^{\vartheta}_{s,t}(\varphi, dx)$ corresponds to the initial condition $u(s,\cdot) = \varphi(\cdot)$

The critical 2d Stochastic Heat Flow

Despite the fact that $\beta_N \to 0$, the limit $\mathcal{U}^\vartheta$ is random!

$$\mathbb{E}\left[\mathcal{U}_t^\vartheta(dx)\right] = dx$$

explicit & non trivial Kernel

$$\mathbb{E}\left[\mathcal{U}_t^\vartheta(dx)\,\mathcal{U}_t^\vartheta(dy)\right] = K_t^\vartheta(x,y)\,dx\,dy$$

[Bertini, Cancrini 98]

Several features are Known:

- $\mathcal{U}_{at}^\vartheta(d(\sqrt{a}\,x)) \overset{d}{=} a\,\mathcal{U}_t^{\vartheta+\log(a)}(dx)$
- Formulas for higher moments [Gu, Quastel, Tsai 21]

The critical 2d Stochastic Heat Flow

Finally, we recently proved that [CSZ 22+]

$\mathcal{U}_t^{\vartheta}(dx)$ is not a GAUSSIAN MULTIPLICATIVE CHAOS

$:e^{X(dx)}: dx$ generalized Gaussian field

- The Stochastic Heat Flow is a new class of random measures
- It suggests that the "solution" of the critical 2d KPZ – yet to be constructed! – should be a NON GAUSSIAN PROCESS

we cannot take $\log \mathcal{U}_t^{\vartheta}(dx)$

III. Ideas and Techniques

A LINK WITH DIRECTED POLYMERS

Discretized SHE: for $(t,x) = \left(\frac{n}{N}, \frac{z}{\sqrt{N}}\right)$ in $\mathbb{T}_N = \frac{\mathbb{N}}{N} \times \frac{\mathbb{Z}^2}{\sqrt{N}}$

$$\partial_t^N u_N(t,x) = \frac{1}{4}\Delta^N u_N(t,x) + \beta N\, \xi_N(t+\tfrac{1}{N}, x)\, \langle u_N(t,x) \rangle \qquad \text{(D-SHE)}$$

Convenient choice of discretized noise:

$$\xi_N(t,x) \longleftrightarrow \xi_N(n,z) = \frac{e^{\beta\omega(n,z) - \frac{\beta^2}{2}} - 1}{\beta}$$

I.I.D. $\mathcal{N}(0,1)$ (say)

Then $u_N(t,x)$ admits a Feynman-Kac representation formula:

A LINK WITH DIRECTED POLYMERS

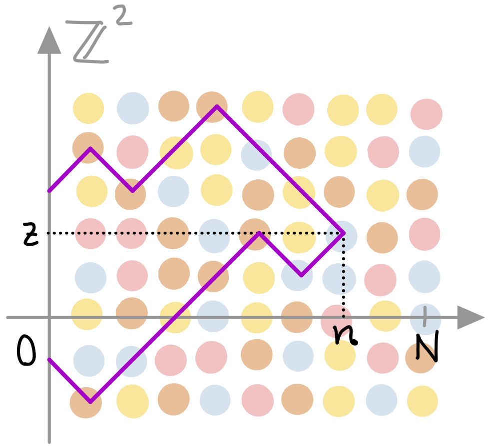
$$u_N(t,x) = Z_N(n,z) = E\left[e^{\sum_{i=0}^{n-1} \beta\,\omega(n-i,S_i) - \frac{\beta^2}{2}} \,\middle|\, S_0 = z \right]$$

(S_i) SIMPLE RANDOM WALK ON $\mathbb{Z}^2$

Partition function of the

DIRECTED POLYMER

IN RANDOM ENVIRONMENT



SECOND MOMENT AND CRITICAL SCALING OF β

$$\mathbb{E}\left[u_N(1,x) \cdot u_N(1,x') \right] = \mathrm{E}\left[e^{\beta^2 \underbrace{\sum_{i=0}^{N} \mathbb{1}_{\{S_i = S'_i\}}}_{\mathcal{L}_N}} \,\middle|\, S_0 = z,\ S'_0 = z' \right]$$

"REPLICA OVERLAP"

Classical result: $\dfrac{\pi}{\log N} \mathcal{L}_N \xrightarrow[N\to\infty]{d} Y \sim \mathrm{Exp}(1)$ [Erdös-Taylor 60]

This explains the CRITICAL SCALING of $\beta = \beta_N$

$$\beta \sim \hat{\beta} \frac{\sqrt{\pi}}{\sqrt{\log N}} \qquad \text{with} \qquad \hat{\beta} = \hat{\beta}_c = 1 + O\left(\frac{1}{\log N}\right)$$

Polynomial Chaos and phase transition

$$U_N(t,x) = 1 + \sum_{k \geq 1} \beta_N^k \sum_{\substack{0 < n_1 < \dots < n_k \leq Nt \\ z_1, \dots, z_k \in \mathbb{Z}^2}} q\big((n_1,z_1), \dots, (n_k,z_k)\big) \cdot \prod_{i=1}^k \xi_N(n_i, z_i)$$

$$P(S_{n_1} = z_1, \dots, S_{n_k} = z_k)$$

- Phase transition at $\hat{\beta} = 1$

$$U_N(t,x) \xrightarrow[N\to\infty]{d} \begin{cases} \exp\{\text{Gaussian}\} & \hat{\beta} < 1 \\ 0 & \hat{\beta} \geq 1 \end{cases}$$

[CSZ 17b]

The Stochastic Heat Flow

- Existence of subsequential limits is easy: (tightness)

$$u_N(t,x)\,dx \xrightarrow{d} \mathcal{U}_t^{\vartheta}(dx)$$

- Non-triviality of the limit is harder... [CSZ 19b]

  ... Uniqueness of the limit is very difficult! [CSZ 21]

- Formulas for all moments of $\mathcal{U}_t^{\vartheta}$ are available... [GQT 21]

  ... but moments grow too fast to determine the law.

THE STOCHASTIC HEAT FLOW

We do not have a characterization of the limit

⇝ We prove uniqueness by a Cauchy argument:

$$u_N(t,x)\,dx \overset{d}{\approx} u_M(t,x)\,dx \qquad \text{for large } N, M$$

↑

We prove closeness in law by COARSE-GRAINING techniques, exploiting self-similarity of the model & moment bounds.

Generalized HLS inequality

Fix a dimension $d \in \mathbb{N}$ and conjugate exponents $p, q \in (1, \infty)$.

For any $f, g : \mathbb{R}^{2d} \to \mathbb{R}$ $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$

$$\int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \frac{f(x,x') \cdot g(y,y')}{\left(|x-y| + |x'-y'| + |x-y'|\right)^{2d}} \, dx\, dx'\, dy\, dy' \leq C \, \|f\|_{L^p} \, \|g\|_{L^q}$$

"critical" Hardy-Littlewood-Sobolev exponent

Generalizes an inequality by [dell'Antonio, Figari, Teta, AIHP 94]

IV. Conclusions and Perspectives

Conclusions

We introduced the CRITICAL 2D STOCHASTIC HEAT FLOW as a scaling limit of directed polymer partition functions

It is a universal process of random measures on $\mathbb{R}^2$: a natural candidate for the solution of the critical 2d Stochastic Heat Equation.

It has many explicit features _

PERSPECTIVES

Many interesting questions are still open:

- SINGULARITY W.R.T. LEBESGUE MEASURE
- FLOW PROPERTY
- CHARACTERIZING PROPERTIES
- TAKING LOG $\rightsquigarrow$ KPZ

Grazie!

Moment Formulas

$$\mathbb{E}\Big[\,\mathcal{U}^{\vartheta}_{t}(dx)\cdot\mathcal{U}^{\vartheta}_{t}(dy)\cdot\mathcal{U}^{\vartheta}_{t}(dz)\,\Big] = \underbrace{K^{(3)}(x,y,z)}_{\lim\limits_{N\to\infty}\mathbb{E}\left[U_N(t,x)\cdot U_N(t,y)\cdot U_N(t,z)\right]}\,dx\,dy\,dz$$

$$K^{(3)}(z_1,z_2,z_3) = \sum_{m\geq 2}\ \int\cdots\int_{\substack{0<a_1<b_1<\dots<a_m<b_m<t\\ x_1,\,y_1,\,\dots,\,x_m,\,y_m\in\mathbb{R}^2}} d\vec{a}\,d\vec{b}\,d\vec{x}\,d\vec{y}\ \ g^{(m)}_{\vec{z}}(\vec{a},\vec{b},\vec{x},\vec{y})$$

MOMENT FORMULAS

$g^{(4)} =$