On the 2d Stochastic Heat Equation and delta Bose gas

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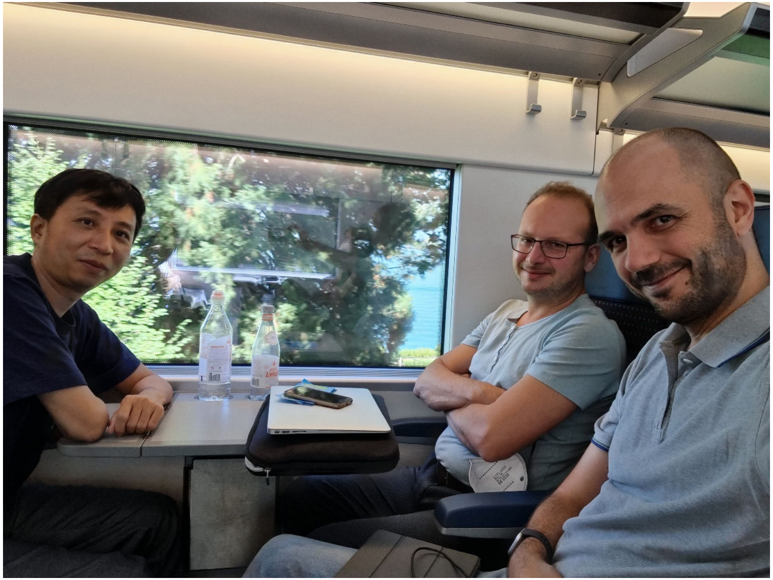
Based on joint works with



Rongfeng Sun and Nikos Zygouras

REFERENCES

[CSZ 21] F. Caravenna, R. Sun, N. Zygouras
THE CRITICAL 2D STOCHASTIC HEAT FLOW, arXiv (2021)

[CSZ 22] " " " " IS NOT A GMC, arXiv (2022)

[CSZ 20] Ann. Probab. 48 (2020)

[CSZ 19b] Commun. Math. Phys. 372 (2019)

[CSZ 19a] Electron. J. Probab. 24 (2019)

[CSZ 17b] Ann. Appl. Probab. 27 (2017)

[CSZ 17a] J. Eur. Math. Soc. 19 (2017)

I. Introduction

THE STOCHASTIC HEAT EQUATION

$$(\text{SHE}) \quad \begin{cases} \partial_t u(t,x) = \Delta u(t,x) + \beta\, \xi(t,x)\, u(t,x) & t>0,\ x\in\mathbb{R}^d \\ u(0,x) \equiv 1 \end{cases}$$

- $\xi(t,x)$ "space-time white noise" (δ-correlated Gaussian)
- $\beta > 0$ coupling constant

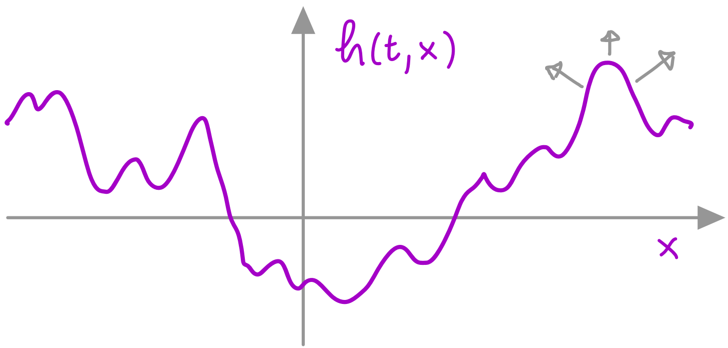
GOAL: Construct the solution $u(t,x)$ for $d=2$

THE KARDAR-PARISI-ZHANG EQUATION

[PRL 1986]

Formally $h(t,x) := \log U(t,x)$ solves

$$(\text{KPZ}) \qquad \partial_t h(t,x) = \underbrace{\Delta h(t,x)}_{\text{SMOOTHING}} + \underbrace{|\nabla h(t,x)|^2}_{\perp \text{ GROWTH}} + \underbrace{\beta\, \xi(t,x)}_{\text{NOISE}}$$



SHE can help us make sense of KPZ

Singularity

(SHE) and (KPZ) are ill-defined due to singular products

$$\xi(t,x)\, u(t,x) \qquad\qquad |\nabla h(t,x)|^2$$

$\xi(t,x)$ is a distribution $\rightsquigarrow$ $u(t,x)$ and $h(t,x)$ expected to be:

$\mathcal{C}_{\mathrm{PAR}}^{-\frac{d}{2}-1-}$

- non-smooth functions $(d=1)$ $\quad \mathcal{C}_{\mathrm{PAR}}^{\frac{1}{2}-}$
- genuine distributions $(d\geq 2)$ $\quad \mathcal{C}_{\mathrm{PAR}}^{\frac{2-d}{2}-}$

Henceforth we focus on (SHE)

The case $d=1$

(1980s) $u(t,x)$ well-posed by stochastic integration (Ito-Walsh)

(2010s) Robust solution theories for "sub-critical" singular PDEs

- REGULARITY STRUCTURES [Hairer]
- PARACONTROLLED CALCULUS [Gubinelli, Imkeller, Perkowski]
- ENERGY SOLUTIONS [Goncalves, Jara]
- RENORMALIZATION [Kupiainen]

All of this breaks down in higher dimensions $d \geqslant 2$

THE ROLE OF DIMENSION

Space-time blow up: $\tilde{u}(t,x) := u(\delta^2 t, \delta x)$

solves $$\partial_t \tilde{u}(t,x) = \Delta \tilde{u}(t,x) + \beta\, \delta^{\frac{2-d}{2}}\, \tilde{\xi}(t,x)\, \tilde{u}(t,x)$$

As $\delta \downarrow 0$ the noise formally $\begin{cases} \text{vanishes} & (d<2) \\ \text{stays constant} & (d=2) \\ \text{diverges} & (d>2) \end{cases}$

$d=2$ is CRITICAL DIMENSION for SHE

REGULARIZING SHE VIA DISCRETIZATION

$(N \gg 1)$

We fix $d=2$. Restrict (t,x) in the lattice $\mathbb{T}_N = \frac{\mathbb{N}}{N} \times \frac{\mathbb{Z}^2}{\sqrt{N}}$:

$$\partial_t^N u_N(t,x) = \Delta^N u_N(t,x) + \beta \cdot N \cdot \xi_N(t+\tfrac{1}{N}, x) \langle u_N(t,x) \rangle$$

DISCR. DERIVATIVE

$N\left\{u(t+\frac{1}{N},x) - u(t,x)\right\}$

DISCR. LAPLACIAN

$\frac{N}{4} \sum_{x' \sim x} \left\{u(t,x') - u(t,x)\right\}$

I.I.D. RVs

$\mathbb{E}[\xi] = 0 \quad \mathbb{E}[\xi^2] = 1$

$\frac{1}{4} \sum_{x' \sim x} u(t,x')$

Well-defined solution $u_N(t,x) \geq 0$

$\left[u_N(0,\cdot) \equiv 1\right]$

CONVERGENCE ?

Can we hope for $U_N \xrightarrow[N\to\infty]{} \mathcal{U}$? (with non-trivial limit)

1. Convergence as (random) distributions : $\varphi \in C_c(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2} \varphi(x)\, U_N(t,x)\, dx \xrightarrow[N\to\infty]{d} \int_{\mathbb{R}^2} \varphi(x)\, \mathcal{U}(t,dx) \ ?$$

i.e. $U_N(t,x)\,dx \xrightarrow{d} \mathcal{U}(t,dx)$ as (random) measures on $\mathbb{R}^2$?

② Rescale the coupling constant

$$\beta = \beta_N \sim \frac{\hat\beta}{\sqrt{\log N}} \xrightarrow[N\to\infty]{} 0$$

$$\mathbb{E}\left[\int \varphi(x)\, U_N(t,x)\, dx\right] \equiv \int \varphi(x)\, dx$$

$$\mathbb{V}\mathrm{AR}\left[\int \varphi(x)\, U_N(t,x)\, dx\right] \longrightarrow \begin{cases} 0 & \text{if } \hat\beta < \sqrt{\pi} \\ \sigma^2(\varphi) & \text{if } \hat\beta = \sqrt{\pi} \\ \infty & \text{if } \hat\beta > \sqrt{\pi} \end{cases}$$

PHASE TRANSITION

For $\hat\beta < \sqrt{\pi}$: $U_N(t,x)\, dx \xrightarrow{d} dx$ = Lebesgue measure ("trivial")

For $\hat\beta = \sqrt{\pi}$ do we have $U_N(t,x)\, dx \xrightarrow{d} \mathcal{U}(t, dx)$?

(non trivial)

II. Main Results

THEOREM [CSZ 21]

Rescale $\beta \sim \frac{\sqrt{\pi}}{\sqrt{\log N}}$, more precisely

$$\text{✪} \qquad \beta = \frac{\sqrt{\pi}}{\sqrt{\log N}}\left(1 + \frac{\vartheta}{\log N}\right) \qquad \vartheta \in \mathbb{R}$$

Then $u_N(t,x)$ converges to a non-trivial limit:

$$\big(u_N(t,x)\,dx\big)_{t\geq 0} \xrightarrow[N\to\infty]{\text{f.d.d.}} \mathscr{U}^{\vartheta} = \big(\mathscr{U}^{\vartheta}_t(dx)\big)_{t\geq 0}$$

We call $\mathscr{U}^{\vartheta}$ the critical 2d STOCHASTIC HEAT FLOW (SHF)

We have built a candidate solution of the Critical 2d SHE :

$$\beta \sim \frac{\sqrt{\pi}}{\sqrt{\log N}}$$

the SHF $\quad \mathcal{U}^{\vartheta} = \left(\mathcal{U}^{\vartheta}_t(dx)\right)_{t \geq 0}$

with initial condition $\mathcal{U}^{\vartheta}_0(dx) \equiv dx \qquad (u(0,\cdot) \equiv 1)$

Remark. We actually build a two-parameter process

$$\mathcal{U}^{\vartheta} = \left(\mathcal{U}^{\vartheta}_{s,t}(dy, dx)\right)_{0 \leq s \leq t < \infty}$$

$\mathcal{U}^{\vartheta}_{s,t}(\psi, dx)$ corresponds to the initial condition $u(s,\cdot) = \psi(\cdot)$

SOME FEATURES OF THE SHF

- $\mathbb{E}\left[\mathcal{U}_t^{\vartheta}(dx)\right] = dx$

- $\mathbb{E}\left[\mathcal{U}_t^{\vartheta}(dx)\,\mathcal{U}_t^{\vartheta}(dy)\right] = K_t^{\vartheta}(x,y)\,dx\,dy$ [Bertini, Cancrini 98]

  $\sim \log\frac{1}{|x-y|} \Rightarrow \mathcal{U}^{\vartheta}$ is random

- $\mathcal{U}_{at}^{\vartheta}(d(\sqrt{a}\,x)) \overset{d}{=} a\,\mathcal{U}_t^{\vartheta+\log(a)}(dx)$

- Formulas for higher moments [Gu, Quastel, Tsai 21] [CSZ 19]

CORRELATION FUNCTION

$$\beta = \frac{\sqrt{\pi}}{\sqrt{\log N}}\left(1 + \frac{\vartheta}{\log N}\right)$$

$$K_N^\beta(t,x,x') := \mathbb{E}\left[U_N(t,x)\cdot U_N(t,x')\right] \xrightarrow[N\to\infty]{} K_t^\vartheta(x,x')$$

It solves the 2-body delta-Bose gas discretized in $\frac{\mathbb{Z}^2}{\sqrt{N}}$:

$$\partial_t^N K_N^\beta = -\mathcal{H}_N^\beta K_N^\beta \qquad \text{where} \qquad \mathcal{H}_N^\beta = -\Delta^N - \beta\cdot N\cdot \mathbb{1}_{\{x=x'\}}$$

$$K_t^\vartheta = e^{-t\mathcal{H}^\vartheta}$$

self-adj. ext. of $\mathcal{H}^\beta = -\Delta - \beta\,\delta(x-x')$

EXPLICIT FORMULA

[Albeverio, Gesztesy, Høegh-Krohn, Holden 87]

HIGHER ORDER CORRELATIONS

$$K_N^{\beta}(t,x_1,\dots,x_n) := \mathbb{E}\left[U_N(t,x_1)\cdots U_N(t,x_n)\right] \xrightarrow[N\to\infty]{} K_t^{\vartheta}(x_1,\dots,x_n)$$

It solves the n-body delta-Bose gas discretized in $\frac{\mathbb{Z}^2}{\sqrt{N}}$:

$$\partial_t^N K_N^{\beta} = -\mathcal{H}_N^{\beta} K_N^{\beta} \qquad \text{where} \qquad \mathcal{H}_N^{\beta} = -\Delta^N - \beta \cdot N \cdot \sum_{1\leq i<j\leq n} \mathbb{1}_{\{x_i = x_j\}}$$

$$K_t^{\vartheta} = e^{-t\mathcal{H}^{\vartheta}}$$ self-adj. ext. of $\mathcal{H}^{\beta} = -\Delta - \beta \sum_{1\leq i<j\leq n} \delta(x_i - x_j)$

EXPLICIT FORMULAS

[dell'Antonio, Figari, Teta 94] [Dimock, Rajeev 04] [Gu, Quastel, Tsai 19]

Gaussian Multiplicative Chaos

Gaussian field $X \sim \mathcal{N}(0, K)$:

$$\mathbb{E}\left[X(dx) \; X(dy) \right] = K(x,y) \; dx \; dy$$

Gaussian Multiplicative Chaos (GMC) (random measure)

$$\mathcal{M}(dx) = \text{“} e^{X(x) - \frac{1}{2} K(x,x)} \; dx \text{”}$$

$$\mathbb{E}\left[\mathcal{M}(dx)\right] = dx \qquad \mathbb{E}\left[\mathcal{M}(dx) \; \mathcal{M}(dy)\right] = e^{K(x,y)} \; dx \; dy$$

GMCs are "canonical": many explicit features

THEOREM The SHF $\mathscr{U}_t^{\vartheta}(dx)$ is NOT a GMC [CSZ22]

Recall: formally $h(t,x) = \log u(t,x)$ solves (KPZ)

CONJECTURE The critical 2d KPZ equation should have a NON GAUSSIAN SOLUTION $\mathscr{H}_t(dx)$

(KPZ) solution yet to be constructed! Cannot take $\log \mathscr{U}_t^{\vartheta}(dx)$

III. Ideas and Techniques

FEYNMAN-KAC FORMULA

$$(t,x) = \left(\frac{n}{N}, \frac{z}{\sqrt{N}}\right) \in \frac{\mathbb{N}}{N} \times \frac{\mathbb{Z}^2}{\sqrt{N}}$$

$$u_N(t,x) = Z_N(n,z) = E\left[e^{\sum_{i=0}^{n-1} \beta\,\omega(n-i,S_i) - \lambda(\beta)} \,\middle|\, S_0 = z \right]$$

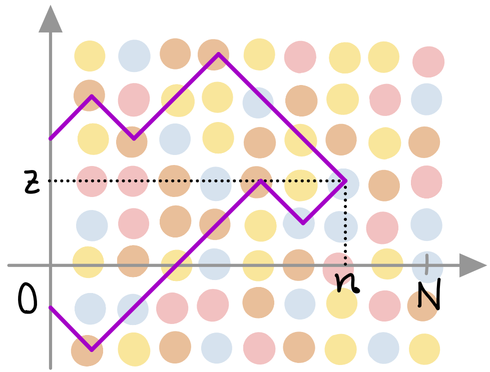
S SIMPLE RANDOM WALK ON $\mathbb{Z}^2$

$$\left(\beta\,\xi = e^{\beta\omega - \lambda(\beta)} - 1\right)$$

Partition function of the

DIRECTED POLYMER

IN RANDOM ENVIRONMENT



MAIN RESULT: STRATEGY OF THE PROOF

$$u_N(t,x)\,dx \xrightarrow{d} \mathcal{U}_t^{\vartheta}(dx)$$

- Existence of subsequential limits is easy [Bertini-Cancrini 98]
- Non-triviality of the limit is harder [CSZ 19b]
- Uniqueness is very difficult [CSZ 21]

(Moments grow too fast to determine the distribution)

HOW TO PROVE UNIQUENESS ?

We use a Cauchy argument:

$$U_N(t,x)\,dx \overset{d}{\approx} U_M(t,x)\,dx \qquad \text{for large } N, M$$

exploiting self-similarity of the model

A. COARSE-GRAINING

B. RENEWAL STRUCTURE

C. LINDEBERG PRINCIPLE

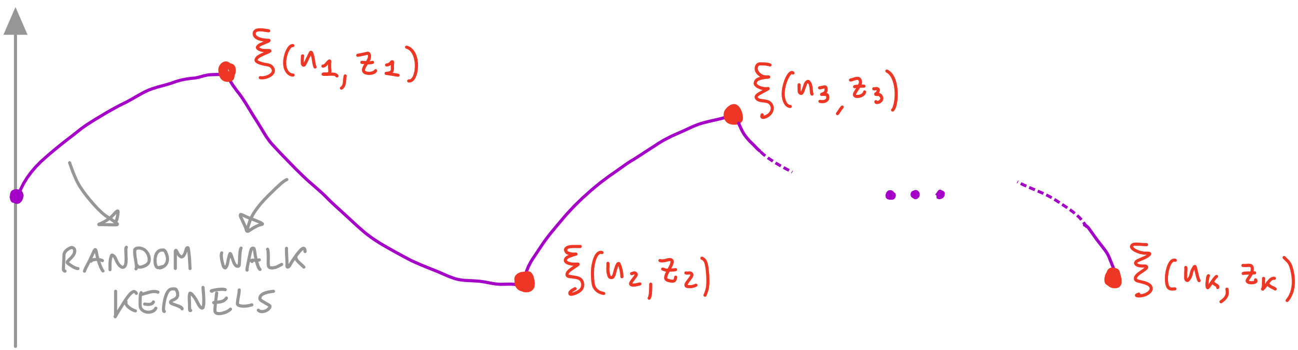
D. FUNCTIONAL INEQUALITIES

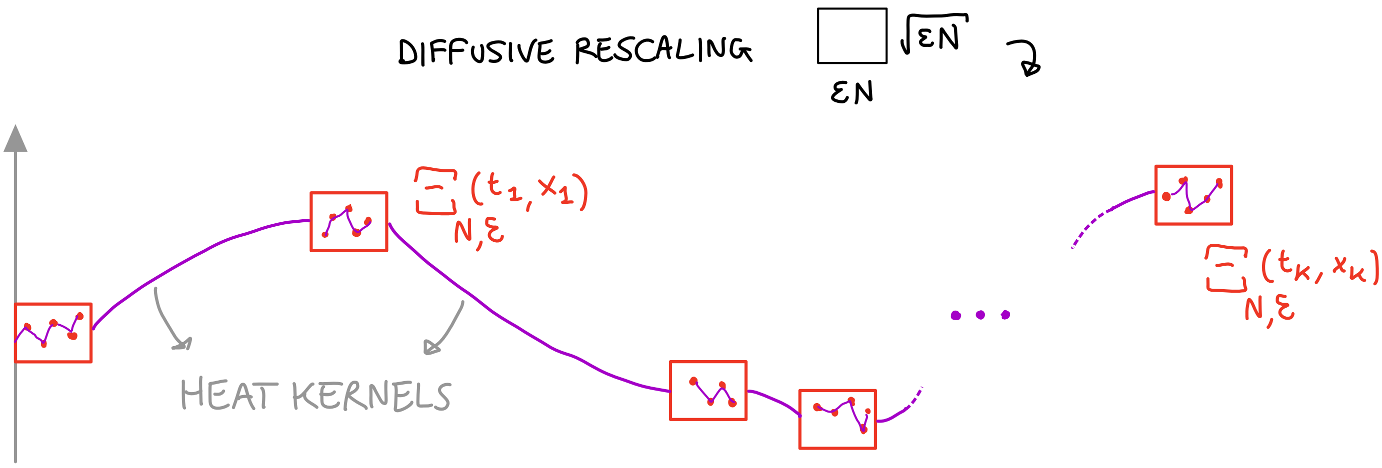
A. Coarse-Graining

$$U_N(t,x) = 1 + \sum_{k\geq 1} \beta^k \sum_{(n_1,z_1),\dots,(n_k,z_k)} q\big((n_1,z_1),\dots,(n_k,z_k)\big) \cdot \prod_{i=1}^{k} \xi(n_i,z_i)$$

(polynomial chaos)

$$P(S_{n_1}=z_1, \dots, S_{n_k}=z_k)$$





Sharp L^2 approximation via a coarse-grained model

$$U_N(t,x)\,dx \approx \mathcal{Z}_{\varepsilon}^{CG}\left(t, dx \,\middle|\, \Xi_{N,\varepsilon}\right) \qquad (\text{as } \varepsilon \downarrow 0)$$

MULTI-LINEAR POLYNOMIAL

"COARSE-GRAINED" NOISE

B. RENEWAL STRUCTURE

Probabilistic interpretation of 2^{ND} moment calculations

$$\mathbb{E}\left[u_N(t,x) \cdot u_N(t,x') \right] = \sum \cdots \; q\big((n_1,z_1),\dots,(n_k,z_k)\big)^2 \cdots$$

$$\xrightarrow[N\to\infty]{} K_t^{\vartheta}(x,x') = 2\pi \int_0^{-l} ds \; g_s(x-x') \int_s^t e^{\vartheta u} \, P(Y_u \le t) \, du$$

[CSZ 19a]

HEAT KERNEL

"DICKMAN SUBORDINATOR"

c. LINDEBERG PRINCIPLE

The distribution of coarse-grained model $\mathcal{Z}^{CG}_{\varepsilon}(t, dx \mid \Xi)$ is insensitive to the distribution of Ξ

(as $\varepsilon \downarrow 0$, provided 1^{ST} & 2^{ND} moments are fixed) [Röllin 2013]

⇝ We can switch $\Xi_{N,\varepsilon}$ to $\Xi_{M,\varepsilon}$ and get our goal

$$U_N(t,x)\,dx \overset{d}{\approx} U_M(t,x)\,dx$$

D. Functional inequalities

Inequalities for Green's function of multiple random walks

"CRITICAL" HARDY-LITTLEWOOD-SOBOLEV INEQUALITY

$$\int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \frac{f(x,x') \cdot g(y,y')}{\left(|x-y| + |x'-y'| + |x-y'|\right)^{2d}} \, dx \, dx' \, dy \, dy' \leq C \, \|f\|_{L^p} \, \|g\|_{L^q}$$

Generalizes an inequality by [dell'Antonio, Figari, Teta 94]

IV. Conclusions and Perspectives

Conclusions

We introduced the CRITICAL 2D STOCHASTIC HEAT FLOW $\mathscr{U}_t^{\vartheta}(dx)$

as the scaling limit of solutions of discretized SHE

$\longleftrightarrow$ directed polymer partition functions

- Universal process of random measures on $\mathbb{R}^2$ ($\neq$ GMC)
- Natural candidate solution for critical 2d SHE

Many explicit features ...

... and several interesting open questions:

- SINGULARITY W.R.T. LEBESGUE MEASURE ✓
- FLOW PROPERTY
- CHARACTERIZING PROPERTIES & UNIVERSALITY
- TAKING LOG $\rightsquigarrow$ KPZ

Statistical Mechanics $\longleftrightarrow$ Singular Stochastic PDEs

RELATED WORKS

- ANISOTROPIC KPZ : $\Delta h = (\partial_x^2 + \partial_y^2) h \rightsquigarrow (\partial_x^2 - \partial_y^2) h$

[Erhard, Cannizzaro, Toninelli]

- SHE WITH LÉVY NOISE : $\mathbb{P}(|\xi| > t) \sim \frac{c}{t^\alpha} \quad 0 < \alpha < 2$

[Berger, Chong, Lacoin]

An announcement

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Courses by

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Grazie